DETERMINANT EXPRESSIONS FOR HYPERELLIPTIC FUNCTIONS
(with an Appendix by Shigeki Matsutani)

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Abstract In this paper we give an elegant generalization of the formula of Frobenius-Stickelberger from elliptic curve theory to all hyperelliptic curves. A formula of Kiepert type is also obtained by a limiting process from this generalization. In the Appendix a determinant expression of D.G. Cantor is also derived.

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Introduction.

Let $\sigma(u)$ and $\wp(u)$ be the usual functions from the classical theory of elliptic functions. The following two formulae were found in the nineteenth-century. The first one is

$$(0.1)\qquad (-1)^{(n-1)(n-2)/2}1!2!\cdots(n-1)!\frac{\sigma(u^{(1)}+u^{(2)}+\cdots+u^{(n)})\prod_{i<j}\sigma(u^{(i)}-u^{(j)})}{\sigma(u^{(1)})^n\sigma(u^{(2)})^n\cdots\sigma(u^{(n)})^n} = \begin{vmatrix} 1 & \wp(u^{(1)}) & \wp'(u^{(1)}) & \wp''(u^{(1)}) & \cdots & \wp^{(n-2)}(u^{(1)}) \\ 1 & \wp(u^{(2)}) & \wp'(u^{(2)}) & \wp''(u^{(2)}) & \cdots & \wp^{(n-2)}(u^{(2)}) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(u^{(n)}) & \wp'(u^{(n)}) & \wp''(u^{(n)}) & \cdots & \wp^{(n-2)}(u^{(n)}) \end{vmatrix}.$$

This formula appeared in the paper of Frobenius and Stickelberger [**11**]. The second one is

$$(0.2)\qquad (-1)^{n-1}(1!2!\cdots(n-1)!)^2\frac{\sigma(nu)}{\sigma(u)^{n^2}} = \begin{vmatrix} \wp' & \wp'' & \cdots & \wp^{(n-1)} \\ \wp'' & \wp''' & \cdots & \wp^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \wp^{(n-1)} & \wp^{(n)} & \cdots & \wp^{(2n-3)} \end{vmatrix}(u).$$

Although this formula can be obtained by a limiting process from (0.1), it was found before (0.1) by [**14**].

If we set $y(u)=\frac{1}{2}\wp'(u)$ and $x(u)=\wp(u)$, then we have the equation $y(u)^2=x(u)^3+\cdots$, that is a defining equation of the elliptic curve to which the functions $\wp(u)$ and $\sigma(u)$ are attached. Here the complex number u and the coordinate $(x(u),y(u))$ correspond by the identity

$$u=\int_{\infty}^{(x(u),y(u))}\frac{dx}{2y}$$

with an appropriate choice of path of the integral. Then (0.1) and (0.2) are easily rewritten as

$$\tag{0.3} \begin{aligned} &\frac{\sigma(u^{(1)}+u^{(2)}+\cdots+u^{(n)})\prod_{i<j}\sigma(u^{(i)}-u^{(j)})}{\sigma(u^{(1)})^n\sigma(u^{(2)})^n\cdots\sigma(u^{(n)})^n} \\ &= \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & yx(u^{(1)}) & x^3(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & yx(u^{(2)}) & x^3(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & yx(u^{(n)}) & x^3(u^{(n)}) & \cdots \end{vmatrix} \end{aligned}$$

and

$$\tag{0.4} \begin{aligned} &1!2!\cdots(n-1)!\frac{\sigma(nu)}{\sigma(u)^{n^2}} \\ &= \begin{vmatrix} x' & y' & (x^2)' & (yx)' & (x^3)' & \cdots \\ x'' & y'' & (x^2)'' & (yx)'' & (x^3)'' & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ x^{(n-1)} & y^{(n-1)} & (x^2)^{(n-1)} & (yx)^{(n-1)} & (x^3)^{(n-1)} & \cdots \end{vmatrix}(u), \end{aligned}$$

respectively.

The author recently gave a generalization of the formulae (0.3) and (0.4) to the case of genus two in [**19**] and to the case of genus three in [**20**]. The aim of this paper is to generalize (0.3), (0.4) and the results in [**19**], [**20**] to all hyperelliptic curves (see Theorem 7.2 and Theorem 8.3). The idea of our generalization arises from the unique paper [**12**] of D. Grant. It can be summarized in a phrase, "*Think not on the Jacobian but on the curve itself.*"

Fay's famous formula (44) in p.33 of [**10**] which generalizes an addition formula on the Jacobian variety, known as Schottky-Klein, is another generalization of (0.3). The author does not know whether Fay's formula is able to yield a generalization of (0.4). Our formula is quite elegant in comparison with Fay's one and naturally gives a generalization of (0.4). Though no explicit connection to Fay's formula with ours is known at present, recently the paper [**9**] appeared, and this paper seems to investigate this problem.

We now present the minimal fundamentals needed to explain our results. Let $f(x)$ be a monic polynomial in x of degree $2g+1$ with g a positive integer. Assume that $f(x)=0$ has no multiple roots. Let C be the hyperelliptic curve defined by $y^2=f(x)$. Then C is of genus g and is ramified at infinity. We denote by ∞ the unique point at infinity on C. Let $\mathbf{C}^g$ be the coordinate space of all vectors of integrals

$$\Big(\int_\infty^{P_1}+\cdots+\int_\infty^{P_g}\Big)\Big(\frac{1}{2y},\ \frac{x}{2y},\ \cdots,\ \frac{x^{g-1}}{2y}\Big)dx$$

of the first kind for $P_j\in C$. Let $\Lambda\subset\mathbf{C}^g$ be the lattice of their periods. So $\mathbf{C}^g/\Lambda$ is the Jacobian variety of C. We denote the canonical map by $\kappa:\mathbf{C}^g\to\mathbf{C}^g/\Lambda$. We have an embedding $\iota: C\hookrightarrow\mathbf{C}^g/\Lambda$ defined by $P\mapsto(\int_\infty^P\frac{dx}{2y},\ \int_\infty^P\frac{xdx}{2y},\ \cdots,\ \int_\infty^P\frac{x^{g-1}dx}{2y})$ mod Λ.

Therefore $\iota(\infty)=(0,0,\cdots,0)\in\mathbf{C}^g/\Lambda$. We regard an algebraic function on C, which we call a *hyperelliptic function* in this article, as a function on a universal Abelian covering $\kappa^{-1}\iota(C)$ ($\subset\mathbf{C}^g$) of C. If $u=(u_1,\ \cdots,\ u_g)$ is in $\kappa^{-1}\iota(C)$, we denote by $(x(u),y(u))$ the coordinate of the corresponding point on C by

$$u=\int_\infty^{(x(u),y(u))}\Big(\frac{1}{2y},\ \frac{x}{2y},\ \cdots,\ \frac{x^{g-1}}{2y}\Big)dx$$

with appropriate choice of a path for the integrals. Needless to say, we have $(x(0,0,\cdots,0),$ $y(0,0,\cdots,0))=\infty$.

Our new point of view is characterized by the following three featuring of the formulae (0.3) and (0.4). Firstly, the sequence of functions of u whose values at $u=u^{(j)}$ are displayed in the j^{th} row of the determinant of (0.3) is just a sequence of the monomials of $x(u)$ and $y(u)$ displayed according to the order of their poles at $u=0$. Secondly, the two sides of (0.3) as a function of $u=u^{(j)}$ and those of (0.4) should be regarded as functions defined on the universal (Abelian) covering space $\mathbf{C}$ *not* of the Jacobian variety but *of the elliptic curve*. Thirdly, the expression of the left hand side of (0.4) states that the function on the two sides of (0.4) is characterized as an elliptic function such that its zeroes are exactly the points different from ∞ whose n-plication is just on the standard theta divisor in the Jacobian of the curve, and such that its only pole is at ∞. In the case of the elliptic curve above, the standard theta divisor is just the point at infinity.

Surprisingly enough, these three featuring can be used to derive a natural generalization of these formulae for hyperelliptic curves. More concretely, for $n\geqq g$ our generalization of (0.4) is obtained by replacing the sequence giving the rows of the right hand side by the sequence

$$1,\ x(u),\ x^2(u),\ \cdots,\ x^g(u),\ y(u),\ x^{g+1}(u),\ yx(u),\ \cdots.$$

Here $u=(u_1,u_2,\cdots,u_g)$ is on $\kappa^{-1}\iota(C)$, with the monomials of $x(u)$ and $y(u)$ displayed according to the order of their poles at $u=(0,0,\cdots,0)$, replacing the derivatives with respect to $u\in\mathbf{C}$ by those with respect to u_1 along $\kappa^{-1}\iota(C)$. The left hand side of (0.4) is replaced by

$$\pm 1!2!\cdots(n-1)!\sigma(nu)/\sigma_\sharp(u)^{n^2}, \tag{0.5}$$

where $n\geqq g$, $\sigma(u)=\sigma(u_1,u_2,\cdots,u_g)$ is a well-tuned Riemann theta series, which is a natural generalization of the classical $\sigma(u)$. The function $\sigma_\sharp$ is defined in the table below:

genus g	1	2	3	4	5	6	7	8	$\cdots$
$\sigma_\sharp$	σ	σ_2	σ_2	σ_{24}	σ_{24}	σ_{246}	σ_{246}	σ_{2468}	$\cdots$

where $\sigma_{ij\cdots\ell}(u)=\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}\cdots\frac{\partial}{\partial u_\ell}\sigma(u)$. The function (0.5) is a natural generalization of the *n-division polynomial* of an elliptic curve, as mentioned in Remark 8.4 below. For the case $n\leqq g$, we need a slight modification as in Theorems 7.2(1) and 8.3(1). As a function on $\kappa^{-1}\iota(C)$, $\sigma_\sharp(u)$ has its zeroes only at the points $\kappa^{-1}\iota(\infty)$ (Proposition 6.5(1)). This property is exactly the same as for the classical $\sigma(u)$. The hyperelliptic function (0.5) can be regarded as a function on $\mathbf{C}^g$ via theta functions. Although this function on $\mathbf{C}^g$ is no longer a function on the Jacobian, it is indeed expressed simply in terms of theta functions and is treated in a very similar way to that of elliptic functions.

The most difficult problem was to find the left hand side of our expected generalization of (0.3). For simplicity we assume $n \geqq g$. The answer is remarkably elegant and is

$$\frac{\sigma(u^{(1)}+u^{(2)}+\cdots+u^{(n)})\prod_{i<j}\sigma_\flat(u^{(i)}-u^{(j)})}{\sigma_\sharp(u^{(1)})^n\sigma_\sharp(u^{(2)})^n\cdots\sigma_\sharp(u^{(n)})^n},$$

where $u^{(j)} = (u_1^{(j)}, u_2^{(j)}, \cdots, u_g^{(j)})$ $(j = 1, \cdots, n)$ are variables on $\kappa^{-1}\iota(C)$ and $\sigma_\flat(u)$ is defined as in the table below:

genus g	1	2	3	4	5	6	7	8	$\cdots$
$\sigma_\flat$	σ	σ	σ_3	σ_3	σ_{35}	σ_{35}	σ_{357}	σ_{357}	$\cdots$

Once we found this, we could prove the formula by, roughly speaking, comparing the divisors of the two sides. As the formula (0.4) is obtained by a limiting process from (0.3), our generalization of (0.4) is obtained by a similar limiting process from the generalization of (0.3).

Cantor [**8**] gave another determinant expression of the function that is characterized in the third featuring explained above, for any hyperelliptic curve. The expression of Cantor should be seen as a generalization of a formula due to Brioschi (see [**4**, p.770, ℓ.3]).

Concerning the paper [**19**], Matsutani pointed out that (0.4) can be generalized to all hyperelliptic curves, and he proved that the resulting formula is equivalent to Cantor's one. He kindly permitted the author to include his proof as an Appendix in this paper.

Matsutani's observation stimulated the author to start working on an extension of (0.3) for all hyperelliptic curves. The method of this paper is entirely different from that of [**19**] and [**20**]. It gives probably one of the simplest approaches to these extensions, and is based on the paper of Buchstaber, Enolskii and Leykin [**6**]. At the beginning of this research the author computed several cases by the low blow method as in [**19**] and [**20**]. While Theorem 7.2 was still a conjecture Professor V.Z. Enolskii suggested to the author that to prove the conjecture it would be important to investigate the leading terms of the sigma function as in [**6**].

Now we outline the idea of the proof. When the curve C, defined by $y^2 = f(x)$, deforms to a singular curve $y^2 = x^{2g+1}$ the canonical limit of the function $\sigma(u)$ is known to be a Schur polynomial from the theory of representations of symmetric groups. The paper [**6**] treated this fact quite explicitly. Such a limit polynomial is called the *Schur-Weierstrass polynomial* in that paper. For our argument, we need a slight extension of this fact (see Section 4). To prove our formula of Frobenius-Stickelberger type by induction on the number of variables $u^{(j)}$ we need relations to connect each factor of the numerator to a factor of the denominator in the left hand side of Theorem 7.2. So, after proving such a connection with the Schur-Weierstrass polynomial as explained in Section 2, we will lift the connection to the case of the sigma function as in Section 6. For this, we need additional facts on the vanishing of some derivatives of the sigma function, as is described in Section 5.

The results from Section 1 to Section 6 are easily generalized to quite wide family of algebraic curves. Such curves are called (n,s)-*curves* in [**6**] and [**7**]. Unfortunately the standard theta divisor or every standard theta subvariety, i.e. an image of the symmetric product of some copies of the curve, in the case of such a general curve, is *not symmetric* in the Jacobian. Here the word "standard" means that the embedding of the curve into its Jacobian variety sends the point at infinity to the origin. Hence the sigma function of such a general curve has no involution and our naive generalization ended in failure.

There are also various generalizations of (0.1) (or (0.3)) in the case of genus two different from our line of approach. If the reader is interested in them, he should consult Introduction of [**19**].

Finally, The authors would like deeply to express thanks to the referee who read carefully the first version of this paper which was written hardly to read. If this final version is much easier to read, all things depend on the referee's considerations.

Convention.

We use the following notation throughout the paper. We denote, as usual, by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. In an expression for the Laurent expansion of a function, the symbol $(d^\circ(z_1, z_2, \cdots, z_m) \geqq n)$ stands for the terms of total degree at least n with respect to the variables z_1, z_2, $\cdots$, z_m. This notation *never* means that the terms are monomials only of z_1, $\cdots$, z_m. When the variable or the least total degree clear from the context, we simply denote them by $(d^\circ \geqq n)$ or the dots "$\cdots$".

We will often omit zero entries from a matrix. For a simplicity we will occasionally denote an unspecified matrix entry with an asterisk.

For cross-references in this paper, we indicate a formula as (1.2), and each of Lemmas, Propositions, Theorems and Remarks also as 1.2.

Contents.

1. The Schur-Weierstrass Polynomial.

We begin with a review of fundamentals on Schur-Weierstrass polynomials. Our main references are [**15**] and [**6**].

Let g be a fixed positive integer, and $u_g^{(1)}, \cdots, u_g^{(g)}$ be indeterminates. We fix n ($0 \leqq n \leqq g$) and we denote by $\boldsymbol{u}_g$ the set of variables $u_g^{(1)}, \cdots, u_g^{(n)}$. For each $k \geqq 0$ we denote by $(-1)^k U_k^{[n]}(\boldsymbol{u}_g)$ the k^{th} *complete symmetric function*, namely the sum of all monomials of total degree k of the variables $u_g^{(1)}, \cdots, u_g^{(n)}$. We will emphasis by the superscript $[n]$ that $U_k^{[n]}(\boldsymbol{u}_g)$ is a function of a set of n variables $\boldsymbol{u}_g$. For $k < 0$, we regard $U_k^{[n]}(\boldsymbol{u}_g)$ as 0.

We now consider the determinant

$$|U_{g-2i+j+1}^{[g]}(\boldsymbol{u}_g)|_{1 \leqq i,j \leqq g}.$$

If we write simply $U_k = U_k^{[g]}(\boldsymbol{u}_g)$ and let $U_k = 0$ if $k < 0$, then this is explicitly of the form

$$\text{(1.1a)} \qquad \begin{vmatrix} U_g & U_{g+1} & U_{g+2} & \cdots & U_{2g-2} & U_{2g-1} \\ U_{g-2} & U_{g-1} & U_g & \cdots & U_{2g-4} & U_{2g-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_1 & U_2 & U_3 & \cdots & * & * \\ & U_0 & U_1 & \cdots & * & * \\ & & & \ddots & \vdots & \vdots \\ & & & & U_0 & U_1 \end{vmatrix}$$

for odd g, or

$$\text{(1.1b)} \qquad \begin{vmatrix} U_g & U_{g+1} & U_{g+2} & U_{g+3} & \cdots & U_{2g-2} & U_{2g-1} \\ U_{g-2} & U_{g-1} & U_g & U_{g+1} & \cdots & U_{2g-4} & U_{2g-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_0 & U_1 & U_2 & U_3 & \cdots & * & * \\ & & U_0 & U_1 & \cdots & * & * \\ & & & & \ddots & \vdots & \vdots \\ & & & & & U_0 & U_1 \end{vmatrix}$$

for even g.

In the sequel we denote

$$\text{(1.2)} \qquad \begin{aligned} p_j &:= (u_g^{(1)})^j + \cdots + (u_g^{(g)})^j, \\ u_j^{(i)} &:= \tfrac{1}{2(g-j)+1}(u_g^{(i)})^{2(g-j)+1}, \\ u^{(i)} &:= (u_1^{(i)}, \cdots, u_g^{(i)}), \\ u_j &:= u_j^{(1)} + u_j^{(2)} + \cdots + u_j^{(g)} = \tfrac{1}{2(g-j)+1} p_{2(g-j)+1}, \\ u &:= u^{(1)} + u^{(2)} + \cdots + u^{(g)} = (u_1, u_2, \cdots, u_g). \end{aligned}$$

We explain our $|U_{g-2i+j+1}^{[g]}(\boldsymbol{u}_g)|_{1 \leqq i,j \leqq g}$ is no other than $S_{2,2g+1}$ in [**6**]. We introduce new variables $s_1, s_2, \cdots, s_{2g-1}$ satisfying

$$\text{(1.3)} \qquad p_j = -{s_1}^j - {s_2}^j - \cdots - {s_{2g-1}}^j, \quad (1 \leqq j \leqq 2g-1).$$

In fact, such s_j's exist by the following reason. Denoting by $\varepsilon_k(\boldsymbol{s})$ with $\boldsymbol{s}=(s_1,s_2,\cdots,s_{2g-1})$ the elementary symmetric function of degree k of s_j's, the numbers s_1, s_2, $\cdots$, s_{2g-1} satisfying (1.3) are solutions of

$$\varepsilon_k(\boldsymbol{s})=\frac{1}{k!}\begin{vmatrix} -p_1 & 1 & & & \\ -p_2 & -p_1 & 2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ -p_{k-1} & -p_{k-2} & -p_{k-3} & \cdots & k-1 \\ -p_k & -p_{k-1} & -p_{k-2} & \cdots & -p_1 \end{vmatrix}=(-1)^k U_k^{[g]}(\boldsymbol{u}_g)$$

for $k=1,\,2,\,\cdots,\,2g-1$. This fact is given by [**15**, p.29, $\ell.-4$ and p.28, $\ell.13$]. The fundamental theorem of the algebraic equations shows that such a system of equations always has solutions. Hence, we see that $|U_k^{[g]}(\boldsymbol{u}_g)|$ coincides with the Schur-Weierstrass polynomial $S_{2,2g+1}(-p_1,-p_3,\cdots,-p_{2g-1})$ of Theorem 4.3 in [**6**],

Remark 1.4. Here note that our notation of u_j's are slightly different from z_j's of [**6**]. The correspondence is given by $z_j=-(2j-1)u_{g-j+1}$. Moreover it would be helpful for the reader to understand the relation of setting of [**6**] and ours that we compare with the integral (3.7) below and the definition (5.3) of [**6**] that has a multiplicative constant and is of opposite direction of integral.

Besides $|U_{g-2i+j+1}^{[g]}(\boldsymbol{u}_g)|$ is obviously a polynomial of $u_g^{(1)}$, $\cdots$, $u_g^{(g)}$, we can prove the following.

Proposition 1.5. *The polynomial* $|U_{g-2i+j+1}^{[g]}(\boldsymbol{u}_g)|_{1\leqq i,j\leqq g}$ *above is completely determined by the* g *values of* $-p_1$, $-p_3$, $\cdots$, $-p_{2g-1}$ *defined above, so that, of* u_1, u_2, $\cdots$, u_g.

Proof. See Theorem 4.1 in [**6**, p.86]. □

So we may write in this paper that

$$S(u):=|U_{g-2i+j+1}^{[g]}(\boldsymbol{u}_g)|_{1\leqq i,j\leqq g}.$$

This polynomial $S(u)$ is called the *Schur-Weierstrass polynomial.*

We introduce a weight which is defined by taking the weight of u_j being $2(g-j)+1$. We call this the *Sato weight.* It is easy to see that $S(u)$ is of homogeneous with respect to Sato weight, and has the Sato weight $\frac{1}{2}g(g+1)$.

Let m be a fixed positive integer and ξ_1, $\cdots$, ξ_m be indeterminates. We denote $(-1)^k$ times the sum of monomials of total degree k of ξ_1, $\cdots$, ξ_m by $U_k^{[m]}(\xi)$ as above, where ξ means the set of ξ_1, $\cdots$, ξ_m.

Definition 1.6. Let m, ξ_i, and $U_k^{[m]}(\xi)$ are as above. A matrix whose every row is successive $(m+1)$ terms of the sequence

$$\cdots,\,0,\,0,\,1,\,U_1^{[m]}(\xi),\,U_2^{[m]}(\xi),\,\cdots$$

except the set of terms

$$0,\,\cdots,\,0,\,0,\,1$$

is called a *fundamental matrix without a simple row* with respect to ξ_1, $\cdots$, ξ_m.

The following is used several times in Section 2.

Lemma 1.7. *Let m, ξ_i, and $U_k^{[m]}(\xi)$ are as above. Let M be a fundamental matrix without a simple row with respect to $\xi_1, \cdots, \xi_m$. We denote by $\varepsilon_j(\xi)$, the elementary symmetric function of $\xi_1, \cdots, \xi_m$ of degree j. Then we have*

$$M \begin{bmatrix} \varepsilon_m(\xi) \\ \varepsilon_{m-1}(\xi) \\ \vdots \\ \varepsilon_1(\xi) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

Proof. See [**15**, p.21, (2.6′)]. □

Although the following will be not used in this paper explicitly, it is deeply related to the Riemann singularity theorem which is mentioned in Section 5. So we give it here.

Lemma 1.8. *As a polynomial in $u_g^{(1)}, \cdots, u_g^{(g-1)}, u_g^{(g)}$, we have*

$$S(u^{(1)} + \cdots + u^{(g-1)} + u^{(g)}) = 0$$

identically when $u_g^{(g)} = 0$.

Proof. This formula follows from 1.7 by setting $m = g - 1$ and M to be the matrix whose determinant expresses $S(u^{(1)} + \cdots + u^{(g-1)})$. □

2. Derivatives of the Schur-Weierstrass Polynomial.

We will discuss some derivatives of the Schur-Weierstrass polynomial, in order to investigate the corresponding derivatives of the sigma function in Section 6.

Definition 2.1. For an integer n with $1 \leqq n \leqq g$, we denote by $\natural^n$ the set of positive integers i such that $n+1 \leqq i \leqq g$ with $i \equiv n+1 \mod 2$. Namely, $\natural^n$ is the set $\{n+1,\ n+3,\ \cdots,\ g-3,\ g-1\}$ or $\{n+1,\ n+3,\ \cdots,\ g-2,\ g\}$ according as $n \equiv g \mod 2$ or not.

Definition 2.2. We denote by $S_{\natural^n}(u)$ the derivative

$$\Big(\prod_{i\in\natural^n}\frac{\partial}{\partial u_i}\Big)S(u).$$

We define in particular $\sharp = \natural^1$ and $\flat = \natural^2$, so that $S_\sharp(u) = S_{\natural^1}(u)$ and $S_\flat(u) = S_{\natural^2}(u)$.

In this Section we define, as in Section 1, $p_k := \sum_{i=1}^{g}(u_g^{(i)})^k$ and $u_j = \frac{1}{2g-2j+1}p_{2g-2j+1}$ for $j = 1, \cdots, g$.

Lemma 2.3. *If we regard a polynomial of $U_1^{[g]}(\boldsymbol{u}_g)$, $U_2^{[g]}(\boldsymbol{u}_g)$, $\cdots$ as a polynomial in p_1, $\cdots$, p_{2g-1}, we then have*

$$k\frac{\partial}{\partial p_k} = (-1)^k\sum_{r\geqq 0}U_r\frac{\partial}{\partial U_{k+r}},$$

where we simply write $U_j = U_j^{[g]}(\boldsymbol{u}_g)$.

Proof. See [**15**, p.76]. □

Now we continue to write $U_j = U_j^{[g]}(\boldsymbol{u}_g)$. The formula states that $(-1)^k k(\partial/\partial p_k)S(u)$ is the sum of the determinants obtained by "shifting by k " one of the rows to the right direction of the matrix of the determinant expression of $S(u)$.

Proposition 2.4. *Let n be an integer such that $1 \leqq n \leqq g-1$ and v_g be a scalar variable. Let*

$$\begin{aligned} v &= \left(\tfrac{1}{2g-1}{v_g}^{2g-1}, \cdots, \tfrac{1}{3}{v_g}^3, v_g\right), \\ u^{(j)} &= \left(\tfrac{1}{2g-1}(u_g^{(j)})^{2g-1}, \cdots, \tfrac{1}{3}(u_g^{(j)})^3, u_g^{(j)}\right). \end{aligned}$$

Then

(1) $S_\sharp(v) = -(-1)^{(g-1)(g-2)(g-3)/2}{v_g}^g$,

(2) $S_{\natural^{n+1}}(u^{(1)} + \cdots + u^{(n)} + v)$

$$= (-1)^{(g-n)(g-n-1)/2}S_{\natural^n}(u^{(1)} + \cdots + u^{(n)}){v_g}^{g-n} + (d^\circ(v_g) \geqq g-n+2),$$

(3) $S_\flat(2v) = -(-1)^{g(g-1)(g-2)/2}2{v_g}^{2g-1}$.

We firstly prove (2), secondly (1), and finally (3).

Proof of (2). While we consider four cases according to the parity of g and n, all the cases are similarly proved. We see that

$$\natural^{n+1}=\begin{cases}\{n+2,\ n+4,\ \cdots,\ g-2,\ g\} & \text{if } n\equiv g \mod 2,\\ \{n+2,\ n+4,\ \cdots,\ g-3,\ g-1\} & \text{if } n\not\equiv g \mod 2.\end{cases}$$

We denote the number of this set by ν:

$$\nu=\begin{cases}(g-n)/2 & \text{if } n\equiv g \mod 2,\\ (g-n-1)/2 & \text{if } n\not\equiv g \mod 2.\end{cases}$$

The expansion that we concern is the derivative $S_{\natural^{n+1}}(u)$ of $S(u)$ which is a function of g variables $u=(u_1,\cdots,u_g)$ with substituted later by $u=u^{(1)}+\cdots+u^{(n)}+v$. We will apply

$$\begin{aligned} D_{n+1}&=\prod_{i\in\natural^{n+1}}\frac{\partial}{\partial u_i}\\ &=\begin{cases}(2g-2n-3)\dfrac{\partial}{\partial p_{2g-2n-3}}(2g-2n-7)\dfrac{\partial}{\partial p_{2g-2n-7}}\cdots 5\dfrac{\partial}{\partial p_5}1\dfrac{\partial}{\partial p_1}\\ \hfill(\text{if } n\equiv g \mod 2),\\ (2g-2n-3)\dfrac{\partial}{\partial p_{2g-2n-3}}(2g-2n-7)\dfrac{\partial}{\partial p_{2g-2n-7}}\cdots 7\dfrac{\partial}{\partial p_7}3\dfrac{\partial}{\partial p_3}\\ \hfill(\text{if } n\not\equiv g \mod 2)\end{cases}\end{aligned} \tag{2.5}$$

to $S(u)$. The formula in 2.3 and expressing $S(u)$ as the determinant $|U_{g-2i+j+1}|_{1\leqq i,j\leqq g}$ with $U_k=U_k^{[g]}(u^{(1)},\cdots,u^{(g)})$ show explicitly the result given by applying D_{n+1} to $S(u)$. Namely, each of the factors of D_{n+1} in (2.5) is simply the sum of certain *shifting to the right* of every rows of $S(u)=|U_{g-2i+j+1}|$. To explain more concretely, we regard the each row of $S(u)$ to be a partial sequence consisting successive g terms of the two-sided infinite sequence

$$\cdots,\ 0,\ 0,\ \cdots,\ 0,\ U_0,\ U_1,\ U_2,\ \cdots.$$

We denote by sh_i^j the operation shifting by j terms to the right on the i^{th} row. For example, if the i^{th} row is $(U_2,U_3,U_4,U_5,\cdots,U_{g+1})$ then sh_i^3 transforms this row to $(0,U_0,U_1,U_2,\cdots,U_{g-2})$. Then we have

$$\begin{aligned} S_{\natural^{n+1}}(u)&=D_{n+1}S(u)\\ &=\gamma'_{n+1}\cdot\begin{cases}\displaystyle\sum_{1\leqq i_1,\cdots,i_\nu\leqq g}\mathrm{sh}_{i_1}^{4\nu-1}\mathrm{sh}_{i_2}^{4\nu-5}\cdots\mathrm{sh}_{i_{\nu-1}}^{5}\mathrm{sh}_{i_\nu}^{1}|U_{g-2i+j+1}| & \text{if } n\equiv g \mod 2,\\ \displaystyle\sum_{1\leqq i_1,\cdots,i_\nu\leqq g}\mathrm{sh}_{i_1}^{4\nu-1}\mathrm{sh}_{i_2}^{4\nu-5}\cdots\mathrm{sh}_{i_{\nu-1}}^{7}\mathrm{sh}_{i_\nu}^{3}|U_{g-2i+j+1}| & \text{if } n\not\equiv g \mod 2.\end{cases}\end{aligned} \tag{2.6}$$

Here γ'_{n+1} is a signature $\pm$ coming from the top of the formula in 2.3, and is given by

$$\gamma'_{n+1}=\begin{cases}(-1)^{(g-n+1)(g-n)/2} & \text{if } n\equiv g \mod 2,\\ (-1)^{(g-n)(g-n-1)/2} & \text{if } n\not\equiv g \mod 2.\end{cases}$$

We claim that if we put $u = u^{(1)} + \cdots + u^{(n)} + v$ in (2.6), namely if we set $U_{g-2i+j+1}$ to be $U^{[n+1]}_{g-2i+j+1} = U^{[n+1]}_{g-2i+j+1}(u^{(1)}_g, \cdots, u^{(n)}_g, v_g)$ then *all the terms in* (2.6) *vanish except only one certain term.* The unique non-vanishing term is obtained from

$$\gamma'_{n+1}|M| := \gamma'_{n+1} \cdot \begin{cases} \mathrm{sh}^{4\nu-1}_{n+1}\mathrm{sh}^{4\nu-5}_{n+2}\cdots\mathrm{sh}^{5}_{n+\nu-1}\mathrm{sh}^{1}_{n+\nu}|U_{g-2i+j+1}| & \text{if } \ n \equiv g \mod 2, \\ \mathrm{sh}^{4\nu-1}_{n+1}\mathrm{sh}^{4\nu-5}_{n+2}\cdots\mathrm{sh}^{7}_{n+\nu-1}\mathrm{sh}^{3}_{n+\nu}|U_{g-2i+j+1}| & \text{if } \ n \not\equiv g \mod 2 \end{cases}$$

by putting $u = u^{(1)} + \cdots + u^{(n)}$. We denote it by $\gamma'_{n+1}|M^{[n+1]}|$. Here the determinant $|M|$ is expressed as follows: if $n \equiv g$ mod 2 then

$$\begin{vmatrix}
U_g & U_{g+1} & \cdots & U_{g+n} & * & * & * & * & * & \cdots & * & * & * & * \\
U_{g-2} & U_{g-1} & \cdots & U_{g+n-2} & * & * & * & * & * & \cdots & * & * & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
U_{g-2n+2} & U_{g-2n+3} & \cdots & U_{g-n+2} & * & * & * & * & * & \cdots & * & * & * & * \\
U_{g-2n} & U_{g-2n+1} & \cdots & U_{g-n} & * & * & * & * & * & \cdots & * & * & * & * \\
 & & & & & & & & & & & & & U_0 \\
 & & & & & & & & & & & U_0 & U_1 & U_2 \\
 & & & & & & & & & \iddots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & U_0 & U_1 & \cdots & * & * & * & * \\
 & & & & & U_0 & U_1 & U_2 & U_3 & \cdots & * & * & * & * \\
 & & & & U_0 & U_1 & U_2 & U_3 & U_4 & \cdots & * & * & * & * \\
 & & & & & & U_0 & U_1 & U_2 & \cdots & * & * & * & * \\
 & & & & & & & & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & & & & U_0 & U_1 & U_2 & U_3 \\
 & & & & & & & & & & & & U_0 & U_1
\end{vmatrix},$$

and if $n \not\equiv g$ mod 2 then

$$\begin{vmatrix}
U_g & U_{g+1} & \cdots & U_{g+n} & * & * & * & * & * & \cdots & * & * & * & * \\
U_{g-2} & U_{g-1} & \cdots & U_{g+n-2} & * & * & * & * & * & \cdots & * & * & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
U_{g-2n+2} & U_{g-2n+3} & \cdots & U_{g-n+2} & * & * & * & * & * & \cdots & * & * & * & * \\
U_{g-2n} & U_{g-2n+1} & \cdots & U_{g-n} & * & * & * & * & * & \cdots & * & * & * & * \\
 & & & & & & & & & & & & & U_0 \\
 & & & & & & & & & & & U_0 & U_1 & U_2 \\
 & & & & & & & & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & U_0 & U_1 & U_2 & \cdots & * & * & * & * \\
 & & & & U_0 & U_1 & U_2 & U_3 & U_4 & \cdots & * & * & * & * \\
 & & & & & U_0 & U_1 & U_2 & U_3 & \cdots & * & * & * & * \\
 & & & & & & & U_0 & U_1 & \cdots & * & * & * & * \\
 & & & & & & & & & \ddots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & & & & U_0 & U_1 & U_2 & U_3 \\
 & & & & & & & & & & & & U_0 & U_1
\end{vmatrix}.$$

Both of the cases, each determinant is obviously transformed to

$$= \gamma''_{n+1} \left| \begin{matrix} U_g & U_{g+1} & \cdots & U_{g+n} & * & * & \cdots & * \\ U_{g-2} & U_{g-1} & \cdots & U_{g+n-2} & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ U_{g-2n+2} & U_{g-2n+3} & \cdots & U_{g-n+2} & * & * & \cdots & * \\ U_{g-2n} & U_{g-2n+1} & \cdots & U_{g-n} & * & * & \cdots & * \\ & & & & U_0 & * & \cdots & * \\ & & & & & U_0 & \cdots & * \\ & & & & & & \ddots & \vdots \\ & & & & & & & U_0 \end{matrix} \right|,$$

where γ''_{n+1} is the signature coming from exchanges of rows and is given by

$$\gamma''_{n+1} = \begin{cases} 1 & \text{if } g-(n+1) \not\equiv 2 \mod 4, \\ -1 & \text{if } g-(n+1) \equiv 2 \mod 4. \end{cases}$$

We will show the claim above by the following three steps.

Step 1. If the index $(i_1, \cdots, i_\nu)$ corresponding to a term in (2.6) contains a repetition, then such the term vanishes when we put $u = u^{(1)} + \cdots + u^{(n)}$, by the following reason. Since j of sh_i^j is anytime odd, any twice of shift in a row is a shift by even terms. By looking at the expression (1.1a) and (1.1b) of $|U_{g-2i+j+1}|$, we see that the shifted row coincides with another row or all the terms in the row are 0, and such the determinant vanishes before getting the remaining shifts and substituting $u = u^{(1)} + \cdots + u^{(n)}$. Hence, in this case, no non-zero term appears.

Step 2. In this step, any matrix $[a_{ij}]$ of size $g \times g$ which we will consider its determinant is regarded to be divided into four blocks according as $i \leqq (n+1)$ or not, and as $j \leqq (n+1)$ or not. We take arbitrary term of the sum (2.6) (which was already suffered all shifts). We reorder its rows such that any column is ordered of indexes increasing. We denote the obtained determinant by $|M_1|$. Moreover we denote by $|M_1^{[n+1]}|$ the determinant obtained from $|M_1|$ by putting $u = u^{(1)} + \cdots + u^{(n)} + v$. Suppose the region of $|M_1^{[n+1]}|$ consisting the $(n+1)^{\text{st}}$ row and the below contains at least one *multi-step* as

$$|M_1^{[n+1]}| = \left| \begin{matrix} * & & & & & * & & \\ & & \ddots & \vdots & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} \\ & & \cdots & U_{k-1}^{[n+1]} & U_k^{[n+1]} & U_{k+1}^{[n+1]} & U_{k+2}^{[n+1]} & \cdots \\ & & & & & U_0^{[n+1]} & U_1^{[n+1]} & \cdots \\ & & & & & & & \ddots \end{matrix} \right| \qquad (k \geqq 1).$$

└── taking $(n+2)$ columns ──┘

Then the successive $(n+2)$ columns from the column that contains the multi-step above to the left direction form a matrix without simple rows and are linearly dependent by 1.7 with respect to $(n+1)$ variables $u_g^{(1)}, \cdots, u_g^{(n)}, v_g$. Hence $|M_1^{[n+1]}| = 0$. Therefore we may consider determinants such that all their diagonals in the right-lower block are $U_0(=1)$.
Step 3. Since the totality of the shifts in every terms in (2.6) are always same, the Step 2 states that we may consider the only terms in (2.6) such that the sum of the indices of the entries of the first column in the upper left block is equal to such the sum for $|M|$. Since we can shift only to the right, the first row of $|M|$ is no other than the first row of $S(u)$. Hence we may suppose that the sum of the indices from the $(2,1)$-entry to the $(n+1,1)$-entry in the first column is equal to such the sum for $|M|$. Then the second row must be just the second row of $|M|$. Repeating such the consideration, we arrive that no row in the upper blocks suffered any shift sh_i^j. Hence we see that $\gamma'_{n+1}|M^{[n+1]}|$ is unique non-zero term in (2.6).

Summing up, we see that all the terms in (2.6) except $\gamma'_{n+1}|M^{[n+1]}|$ vanish, namely we have

$$\begin{aligned}&S_{\natural^{n+1}}(u^{(1)}+\cdots+u^{(n)}+v)\\ &=\gamma_{n+1}\begin{vmatrix} U_g^{[n+1]} & U_{g+1}^{[n+1]} & \cdots & U_{g+n-1}^{[n+1]} & U_{g+n}^{[n+1]} & * & * & \cdots & * \\ U_{g-2}^{[n+1]} & U_{g-1}^{[n+1]} & \cdots & U_{g+n-3}^{[n+1]} & U_{g+n-2}^{[n+1]} & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ U_{g-2n+2}^{[n+1]} & U_{g-2n+3}^{[n+1]} & \cdots & U_{g-n+1}^{[n+1]} & U_{g-n+2}^{[n+1]} & * & * & \cdots & * \\ U_{g-2n}^{[n+1]} & U_{g-2n+1}^{[n+1]} & \cdots & U_{g-n-1}^{[n+1]} & U_{g-n}^{[n+1]} & * & * & \cdots & * \\ & & & & & 1 & * & \cdots & * \\ & & & & & & 1 & \cdots & * \\ & & & & & & & \ddots & \vdots \\ & & & & & & & & 1 \end{vmatrix},\end{aligned} \tag{2.7}$$

where $U_k^{[n+1]} = U_k^{[n+1]}(u_1^{(g)}, \cdots, u_g^{(g)}, v_g)$ and

$$\gamma_{n+1} = \gamma'_{n+1}\gamma''_{n+1} = \begin{cases} (+1)(+1) = 1 & \text{if } g-n \equiv 0 \mod 4, \\ (+1)(+1) = 1 & \text{if } g-n \equiv 1 \mod 4, \\ (-1)(+1) = -1 & \text{if } g-n \equiv 2 \mod 4, \\ (-1)(-1) = 1 & \text{if } g-n \equiv 3 \mod 4. \end{cases}$$

Here we have used that $U_0 = 1$. It is easily checked that $\gamma_{n+1} = (-1)^{(g-n)(g-n-1)(g-n-3)}$. Similarly, we have

$$S_{\natural^n}(u^{(1)}+\cdots+u^{(n)}) = \gamma_n \begin{vmatrix} U_g^{[n]} & U_{g+1}^{[n]} & \cdots & U_{g+n-1}^{[n]} & * & * & \cdots & * \\ U_{g-2}^{[n]} & U_{g-1}^{[n]} & \cdots & U_{g+n-3}^{[n]} & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ U_{g-2n-2}^{[n]} & U_{g-2n-1}^{[n]} & \cdots & U_{g-n+1}^{[n]} & * & * & \cdots & * \\ & & & & 1 & * & \cdots & * \\ & & & & & 1 & \cdots & * \\ & & & & & & \ddots & \vdots \\ & & & & & & & 1 \end{vmatrix}, \tag{2.8}$$

where $U_k^{[n]} = U_k^{[n]}(u_1^{(g)}, \cdots, u_g^{(g)})$. For the determinant (2.7), we subtract v_g times the n^{th} column from the $(n+1)^{\text{st}}$ column. In the next time, we subtract v_g times the $(n-1)^{\text{st}}$ column from the n^{th} column. Repeating such the transformations for (2.7), we see that (2.7) is equal to

$$(2.9)\qquad \gamma_{n+1}\begin{vmatrix} U_g^{[n]} & U_{g+1}^{[n]} & \cdots & U_{g+n-1}^{[n]} & (-1)^{g+n}{v_g}^{g+n} & * & * & \cdots & * \\ U_{g-2}^{[n]} & U_{g-1}^{[n]} & \cdots & U_{g+n-3}^{[n]} & (-1)^{g+n-2}{v_g}^{g+n-2} & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ U_{g-2n}^{[n]} & U_{g-2n+1}^{[n]} & \cdots & U_{g-n+1}^{[n]} & (-1)^{g-n+2}{v_g}^{g-n+2} & * & * & \cdots & * \\ U_{g-2n-2}^{[n]} & U_{g-2n-1}^{[n]} & \cdots & U_{g-n-1}^{[n]} & (-1)^{g-n}{v_g}^{g-n} & * & * & \cdots & * \\ & & & & & 1 & * & \cdots & * \\ & & & & & & 1 & \cdots & * \\ & & & & & & & \ddots & \vdots \\ & & & & & & & & 1 \end{vmatrix}.$$

Expanding this by its $(n+1)^{\text{st}}$ column and looking at (2.8) yield the expansion of (2) up to a signature. The signature is obviously given by

$$\gamma_{n+1}/\gamma_n = \begin{cases} (+1)/(+1) = 1 & \text{if } \ g-n \equiv 0 \mod 4, \\ (+1)/(-1) = -1 & \text{if } \ g-n \equiv 1 \mod 4, \\ (-1)/(+1) = -1 & \text{if } \ g-n \equiv 2 \mod 4, \\ (+1)/(+1) = 1 & \text{if } \ g-n \equiv 3 \mod 4 \end{cases}$$

times the signature of the $(n+1, n+1)$-entry of (2.9), namely the signature is

$$\gamma_{n+1}/\gamma_n \cdot (-1)^{g-n} = (-1)^{(g-n)(g-n-3)/2}(-1)^{g-n} = (-1)^{(g-n)(g-n-1)/2}.$$

Thus we have proved (2).

Proof of (1). By (2.8), we have for $v = (\frac{1}{2g-1}{v_g}^{2g-1}, \cdots, \frac{1}{3}{v_g}^3, v_g)$ that

$$S_\sharp(v) = \gamma_1(-1)^g {v_g}^g = (-1)^{g(g-1)(g-3)/2}(-1)^g {v_g}^g = -(-1)^{(g-1)(g-2)(g-3)/2}{v_g}^g.$$

Proof of (3). For $v = (\frac{1}{2g-1}{v_g}^{2g-1}, \cdots, \frac{1}{3}{v_g}^3, v_g)$, we have by (2.8) that

$$\begin{aligned} S_\flat(2v) &= \gamma_2\left(U_g^{[2]}(v,v)\cdot U_{g-1}^{[2]}(v,v) - U_{g-2}^{[2]}(v,v)\cdot U_{g+1}^{[2]}(v,v)\right) \\ &= (-1)^{(g-1)(g-2)g/2}(-1)^{2g-1}\left((g+1){v_g}^g\cdot g{v_g}^{g-1} - (g-1){v_g}^{g-2}\cdot(g+2){v_g}^{g+1}\right) \\ &= -(-1)^{g(g-1)(g-2)/2}\left((g+1)g-(g-1)(g+2)\right){v_g}^{2g-1} \\ &= -(-1)^{g(g-1)(g-2)/2} 2{v_g}^{2g-1}. \end{aligned}$$

Now all the statements have been proved completely. □

3. Hyperelliptic Functions.

In this section we recall fundamentals of the theory of hyperelliptic functions.

Let C be a smooth projective model of a curve of genus $g > 0$ defined over $\mathbf{C}$ whose affine equation is given by $y^2 = f(x)$, where

$$f(x) = \lambda_0 x^{2g+1} + \lambda_1 x^{2g} + \cdots + \lambda_{2g} x + \lambda_{2g+1}.$$

In this paper, we always have the agreement $\lambda_0 = 1$. We will use, however, the letter λ_0 too when this notation makes an equation of homogeneous weight easy to read.

We denote by ∞ the point of C at infinity. It is known that the set of

$$\omega_j := \frac{x^{j-1}dx}{2y} \quad (j = 1, \cdots, g)$$

forms a basis of the space of the differential forms of the first kind. As usual we let $[\omega' \ \omega'']$ be the period matrix for a suitable choice of the basis of the fundamental group of C. Then the modulus of C is given by $Z := {\omega'}^{-1}\omega''$. The lattice of periods is denoted by Λ, that is

$$\Lambda := \omega' \, {}^t[\mathbf{Z} \quad \mathbf{Z} \quad \cdots \quad \mathbf{Z}] + \omega'' \, {}^t[\mathbf{Z} \quad \mathbf{Z} \quad \cdots \quad \mathbf{Z}] \ (\subset \mathbf{C}^g).$$

Let

$$\eta_j := \frac{1}{2y} \sum_{k=j}^{2g-j} (k+1-j) \lambda_{2g-k-j} x^k dx \quad (j = 1, \cdots, g),$$

which are differential forms of the second kind without poles except at ∞ (see [**2**, p.195, Ex. i] or [**3**, p.314]). We introduce the matrices of periods $[\eta' \ \eta'']$ with respect to $\eta_1, \cdots, \eta_g$ for the basis of the fundamental group of C chosen as above. We let

$$\delta'' := {}^t\left[\frac{1}{2} \quad \frac{1}{2} \quad \cdots \quad \frac{1}{2}\right], \quad \delta' := {}^t\left[\frac{g}{2} \quad \frac{g-1}{2} \quad \cdots \quad \frac{1}{2}\right] \text{ and } \delta := \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix}.$$

For a and b in $\left(\frac{1}{2}\mathbf{Z}\right)^g$, we let

$$\begin{aligned} \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z) &= \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z; Z) \\ &= \sum_{n \in \mathbf{Z}^g} \exp\left[2\pi i \left\{\frac{1}{2} \, {}^t(n+a) Z (n+a) + {}^t(n+a)(z+b)\right\}\right]. \end{aligned}$$

Then the hyperelliptic sigma function on $\mathbf{C}^g$ associated with C is defined by

$$\tilde{\sigma}(u) = \exp(-\frac{1}{2} u \eta' {\omega'}^{-1} \, {}^t u) \vartheta[\delta] ({\omega'}^{-1} \, {}^t u; \ Z) \tag{3.1}$$

up to a multiplicative constant, where $u = (u_1, u_2, \cdots, u_g)$. We shall fix this constant later.

Definition 3.2. We define the *Sato weight* by taking such the weight of u_j being $2(g-j)+1$. When we consider several variables $u^{(1)}, \cdots, u^{(n)}$ on $\kappa^{-1}\iota(C)$ for some n, we also regard the Sato weight of each $u_j^{(i)}$ to be $2(g-j)+1$. Moreover we define the Sato weight of λ_j to be $-2j$.

To fix the multiplicative constant above, we recall the following.

Lemma 3.3. (1) *The power-series expansion of $\widetilde{\sigma}(u)$ with respect to $u_1, u_2, \cdots, u_g$ has polynomials coefficients in $\lambda_0, \lambda_1, \cdots, \lambda_{2g+1}$, and is homogeneous in the Sato weight.*
(2) *The terms of least total degree of the power-series expansion of the function $\widetilde{\sigma}(u)$ with respect to the variables $u_1, \cdots, u_g$ is either a non-zero constant multiple of the Hankel type determinant*

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_{(g+1)/2} \\ u_2 & u_3 & \cdots & u_{(g+3)/2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{(g+1)/2} & u_{(g+3)/2} & \cdots & u_g \end{vmatrix}$$

if g is odd, or of

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_{g/2} \\ u_2 & u_3 & \cdots & u_{(g+2)/2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{g/2} & u_{(g+2)/2} & \cdots & u_{g-1} \end{vmatrix}$$

if g is even.

Proof. While the statement (1) is shown in the proof of Corollary 1 of [**7**], it is also proved as follows. This proof is given by the referee. For any non-zero constant α, the transform given by $x \mapsto \alpha^2 x$, $y \mapsto \alpha^{2g+1} y$, $\lambda_j \mapsto \alpha^{2j}\lambda_j$ defines an isomorphisim of the curve C that maps $\frac{x^j dx}{2y} \mapsto \alpha^{2(j-g)-1}\frac{x^j dx}{2y}$, and hence $u_j \mapsto \alpha^{2(j-g)-1}u_j$. Since the situation on the zeroes and the periodicity of $\sigma(u)$ are invariant under this transform, and $\sigma(u)$ is determined up to a constant multiple by its zeroes and periodicity, its expansion should be homogeneous with respect to the Sato weight. The statement (2) is proved in Proposition 2.2 in [**5**, p.32] or [**3**, pp.359-360]. □

In this paper we let $\sigma(u)$ be the function such that it is a constant multiple of $\widetilde{\sigma}(u)$ and the terms of least total degree of its power-series expansion at $u=(0,0,\cdots,0)$ are just the Hankel type determinant above.

For $u \in \mathbf{C}^g$ we conventionally denote by u' and u'' the elements of $\mathbf{R}^g$ such that $u = \omega' u' + \omega'' u''$, where $[\omega',\ \omega'']$ are the period matrix above. We define a $\mathbf{C}$-valued $\mathbf{R}$-bilinear form $L(\quad,\quad)$ by

$$L(u,v) = u\,{}^t(\eta' v' + \eta'' v'')$$

for u, $v \in \mathbf{C}^g$. For ℓ in Λ, the lattice of periods as defined in Section 1, let

$$\chi(\ell) = \exp[2\pi i({}^t\ell'\delta'' - {}^t\ell''\delta') - \pi i\,{}^t\ell'\ell''].$$

Lemma 3.4. *The function $\sigma(u)$ is an odd function if $g \equiv 1$ or 2 modulo 4, and an even function if $g \equiv 3$ or 0 modulo 4.*

Proof. See [**17**, p.3.97 and p.3.100]. □

Lemma 3.5. (The translational relation) *The function $\sigma(u)$ satisfies*

$$\sigma(u+\ell) = \chi(\ell)\sigma(u)\exp L(u+\tfrac{1}{2}\ell, \ell)$$

for all $u \in \mathbf{C}^g$ and $\ell \in \Lambda$.

For a proof of this formula we refer to the reader to [**2**, p.286].

Remark 3.6. The Riemann form of $\sigma(u)$, which is defined by $E(u,v) = L(u,v) - L(v,u)$, $(u,\, v \in \mathbf{C}^g)$ is simply written as $E(u,v) = 2\pi i(u'\,{}^t v'' - u''\,{}^t v')$ (see Lemma 3.1.2(2) in [**18**, p.396]). Hence, $E(\quad,\quad)$ is an $i\mathbf{R}$-valued form and $2\pi i\mathbf{Z}$-valued on $\Lambda \times \Lambda$. In particular the pfaffian of $E(\quad,\quad)$ is 1. This simple expression is one of the convenient properties for distinguishing $\sigma(u)$ from the theta series without multiplication by an exponential prefactor in (3.1).

Let J be the Jacobian variety of the curve C. We identify J with the Picard group $\mathrm{Pic}^\circ(C)$ of the linearly equivalent classes of divisors of degree zero of C. Let $\mathrm{Sym}^g(C)$ be the g^{th} symmetric product of C. Then we have a birational map

$$\begin{aligned} \mathrm{Sym}^g(C) &\to \mathrm{Pic}^\circ(C) = J \\ (P_1, \cdots, P_g) &\mapsto \text{the class of } P_1 + \cdots + P_g - g\cdot\infty. \end{aligned}$$

As an analytic manifold, J is identified with $\mathbf{C}^g/\Lambda$. We denote by κ the canonical map $\mathbf{C}^g \to \mathbf{C}^g/\Lambda = J$. We embed C into J by $\iota : Q \mapsto Q - \infty$. For each $n = 1, \cdots, g-1$ let $\Theta^{[n]}$ be the subvariety of J determined by the set of classes of the form $P_1+\cdots+P_n - n\cdot\infty$. This is called the *standard theta subvariety* of dimension n. Obviously $\Lambda = \kappa^{-1}\iota(\infty)$ and $\Theta^{[1]} = \iota(C)$.

Analytically, each point $(P_1, \cdots, P_g)$ of $\mathrm{Sym}^g(C)$ is canonically mapped to

$$u = (u_1, \cdots, u_g) = \left(\int_\infty^{P_1} + \cdots + \int_\infty^{P_g}\right)(\omega_1,\ \cdots,\ \omega_g), \tag{3.7}$$

and $\sigma(u)$ is regarded as a function on the universal covering space $\mathbf{C}^g$ of J with the canonical map κ above and the natural coordinate u of $\mathbf{C}^g$.

If $u = (u_1, \cdots, u_g)$ is in $\kappa^{-1}\iota(C)$, we denote by $(x(u), y(u))$ the coordinate of the point on C corresponding to u, so that $u = \int_\infty^{(x(u),y(u))}(\frac{1}{2y},\ \frac{x}{2y},\ \cdots,\ \frac{x^{g-1}}{2y})dx$ with appropriate choice for a path of the integrals. Then we have that $x(-u) = x(u)$, $y(-u) = -y(u)$, and $(x(0,0,\cdots,0), y(0,0,\cdots,0)) = \infty$. We frequently use the following lemma in the rest of the paper.

Lemma 3.8. *Suppose $u \in \kappa^{-1}\iota(C)$. The Laurent expansions of $x(u)$ and $y(u)$ at $u = (0, \cdots, 0)$ on the pull-back $\kappa^{-1}\iota(C)$ of C to $\mathbf{C}^g$ are*

$$x(u) = \frac{1}{{u_g}^2} + (d^\circ(u_g) \geqq 0), \quad y(u) = -\frac{1}{{u_g}^{2g+1}} + (d^\circ(u_g) \geqq -2g+1)$$

with their coefficients in $\mathbf{Q}[\lambda_0, \lambda_1, \cdots, \lambda_{2g+1}]$. Moreover $x(u)$ and $y(u)$ are homogeneous in the Sato weight -2 and $-(2g+1)$, respectively.

Proof. We take $t = \dfrac{1}{\sqrt{x}}$ as a local parameter at ∞ along $\iota(C)$. If u is in $\kappa^{-1}\iota(C)$ and sufficiently near $(0, 0, \cdots, 0)$, we are agree to that t, $u = (u_1, \cdots, u_g)$ and (x, y) are coordinates of the same point on C. Then

$$\begin{aligned} u_g &= \int_\infty^{(x,y)} \frac{x^{g-1}dx}{2y} \\ &= \int_\infty^{(x,y)} \frac{x^{-3/2}dx}{2\sqrt{1 + \lambda_1 \frac{1}{x} + \cdots + \lambda_{2g+1}\frac{1}{x^{2g+1}}}} \\ &= \int_0^t \frac{t^3 \cdot \left(-\frac{2}{t^3}\right) dt}{2 + (d^\circ \geqq 1)} \\ &= -\,t + (d^\circ(t) \geqq 2). \end{aligned}$$

Hence $x(u) = \frac{1}{u_g^2} + (d^\circ(u_g) \geqq -1)$ and our assertion is proved, because $x(-u) = x(u)$ and $y(-u) = -y(u)$. The rest of the statements are obvious from the calculation above. □

Lemma 3.9. *If $u = (u_1, u_2, \cdots, u_g)$ is a variable on $\kappa^{-1}(\Theta^{[1]})$, then*

$$\begin{aligned} u_1 &= \tfrac{1}{2g-1}{u_g}^{2g-1} + (d^\circ(u_g) \geqq 2g), \\ u_2 &= \tfrac{1}{2g-3}{u_g}^{2g-3} + (d^\circ(u_g) \geqq 2g-2), \\ &\cdots\cdots\cdots \\ u_{g-1} &= \tfrac{1}{3}{u_g}^3 + (d^\circ(u_g) \geqq 4) \end{aligned}$$

with the coefficients in $\mathbf{Q}[\lambda_0, \lambda_1, \cdots, \lambda_{2g+1}]$, and these expansions are homogeneous with respect to the Sato weight.

Proof. The assertions are easily obtained by similar calculations as in the proof of 3.8. □

Remark 3.10. The second set of equalities in (1.2) is canonical limit of the equalities in 3.9 when we let all the coefficients λ_1, $\cdots$, λ_{2g+1} tend to 0, because of the homogeneity in 3.9.

4. The Schur-Weierstrass Polynomial and the Sigma Function.

The polynomial $S(u)$ closely relates to the function $\sigma(u)$ as follows. We denote

$$S_{i_1 i_2 \cdots i_n}(u) = \frac{\partial}{\partial u_{i_1}} \frac{\partial}{\partial u_{i_2}} \cdots \frac{\partial}{\partial u_{i_n}} S(u), \quad \sigma_{i_1 i_2 \cdots i_n}(u) = \frac{\partial}{\partial u_{i_1}} \frac{\partial}{\partial u_{i_2}} \cdots \frac{\partial}{\partial u_{i_n}} \sigma(u).$$

Proposition 4.1. *The function $\sigma(u)$ has the power-series expansion*

$$\sigma(u) = (-1)^{g(g-1)(g-3)/2} S(u) + (d^\circ(\lambda_1, \lambda_2, \cdots, \lambda_{2g+1}) \geqq 1) \in \mathbf{Q}[\lambda_1, \cdots, \lambda][[u_1, \cdots, u_g]]$$

at $u = (0, 0, \cdots, 0)$, and is homogeneous with respect to the Sato weight.

Proof. It is easy to see from the proof of the Corollary 1 of [**7**] that the power-series expansion of $\sigma(u)$ with respect to u_1, u_2, $\cdots$, u_g belongs to $\mathbf{Q}[\lambda_0, \lambda_1, \cdots, \lambda_{2g+1}][[u_1, u_2, \cdots, u_g]]$, namely, is expanded over the rational numbers. If g is odd then

$$S_{13\cdots g}(u) = \left| \begin{array}{cccccccc} & & & & & & & U_0 \\ & & & & & U_0 & U_1 & U_2 \\ & & & & \cdot^{\cdot^{\cdot}} & \vdots & \vdots & \vdots \\ & & U_0 & U_1 & \cdots & * & * & * \\ U_0 & U_1 & U_2 & U_3 & \cdots & * & * & * \\ & U_0 & U_1 & U_2 & \cdots & * & * & * \\ & & & U_0 & \cdots & * & * & * \\ & & & & & \vdots & \vdots & \vdots \\ & & & & & & U_0 & U_1 \end{array} \right| = 1,$$

and $\sigma_{13\cdots g}(0, 0, \cdots, 0) = 1$ by 3.3. If g is even then

$$S_{13\cdots(g-1)}(u) = \left| \begin{array}{cccccccc} & & & & & & & U_0 \\ & & & & & U_0 & U_1 & U_2 \\ & & & & \cdot^{\cdot^{\cdot}} & \vdots & \vdots & \vdots \\ & & & U_0 & \cdots & * & * & * \\ & U_0 & U_1 & U_2 & \cdots & * & * & * \\ U_0 & U_1 & U_2 & U_3 & \cdots & * & * & * \\ & & U_0 & U_1 & \cdots & * & * & * \\ & & & & & \vdots & \vdots & \vdots \\ & & & & & & U_0 & U_1 \end{array} \right| = \begin{cases} 1 & (g \equiv 0 \mod 4), \\ -1 & (g \equiv 2 \mod 4); \end{cases}$$

and $\sigma_{13\cdots(g-1)}(0, 0, \cdots, 0) = 1$ by 3.3 again. Hence the form of expansion of $\sigma(u)$ follows straight from Theorem 6.3 of [**6**]. The last statement is a repetition of 3.3(1). □

Corollary 4.2. *The derivatives of $\sigma(u)$ and $S(u)$ are related as follows :*

$$\sigma_{i_1 i_2 \cdots i_n}(u) = (-1)^{g(g-1)(g-3)/2} S_{i_1 i_2 \cdots i_n}(u) + (d^\circ(\lambda_1, \lambda_2, \cdots, \lambda_{2g+1}) \geqq 1).$$

This series is homogeneous in the Sato weight.

Proof. This is obvious from 4.1. □

5. The vanishing Structure of the Sigma Function and of Its Derivatives.

We investigate vanishing structure of $\sigma(u)$ and of its *derivatives* by using Riemann singularity theorem and by calculations using the Brill-Noether matrices. The following is fundamental for us.

Proposition 5.1. (Riemann singularity theorem) *For a given $u \in \kappa^{-1}(\Theta^{[g-1]})$, we denote a divisor on the curve C which corresponds the point u modulo Λ by $P_1 + \cdots + P_{g-1} - (g-1)\cdot\infty$. Then*

$$\dim \Gamma(C, \mathcal{O}(P_1 + \cdots + P_{g-1})) = r + 1$$

if and only if both of the following hold :
(1) *$\sigma_{i_1 i_2 \cdots i_h}(u) = 0$ for any $h \leqq r$ and for any $i_1, \cdots, i_h \in \{1, 2, \cdots, g\}$; and*
(2) *There exists an $(r+1)$-tuple $\{i_1, i_2, \cdots, i_{r+1}\}$ such that $\sigma_{i_1 i_2 \cdots i_{r+1}}(u) \neq 0$.*

Proof. By (3.1) we easily restate the fact stated in [**1**, pp.226-227] into as above. □

To compute the dimension of the 0^{th} cohomology group above we recall the Brill-Noether matrix defined as follows. We fix a local parameter of every point of C. To make clear the following argument we define the local parameter t at each point P by

$$t = \begin{cases} y & \text{if } y(P) = 0, \\ x - x(P) & \text{if } y(P) \neq 0 \text{ and } P \neq \infty, \\ \frac{1}{\sqrt{x}} & \text{if } P = \infty. \end{cases}$$

We denote by Ω^1 the sheaf of differential forms of the first kind. For a point P of C, let t be the local parameter defined above. We denote by P_t the point of C at which the value of the local parameter is t. Then we define for $\mu \in \Gamma(C, \Omega^1)$

$$\delta^\ell \mu(P) = \frac{d^\ell}{dt^\ell} \int_\infty^{P_t} \mu \bigg|_{t=0}.$$

Since μ is a holomorphic form, $\delta^\ell \mu(P)$ takes a finite value at every point P. Let $D := \sum_{j=1}^k n_j P_j$ with different P_js be an effective divisor. A matrix with $\deg D := \sum n_j$ rows and g columns is called the *Brill-Noether matrix* for D if its $(n_1 + \cdots + n_{j-1} + \ell, i)$-entry is $\delta^\ell \omega_i(P_j)$, where $1 \leqq \ell \leqq n_j$ and $\omega_i = \frac{x^{i-1}}{2y} dx$. We denote by $B(D)$ the Brill-Noether matrix for D. Our computation starts with the following.

Proposition 5.2. *Let D be an effective divisor of C, Then*

$$\dim \Gamma(C, \mathcal{O}(D)) = \deg D + 1 - \operatorname{rank} B(D).$$

Proof. For $\mu \in \Gamma(C, \Omega^1)$, we can find uniquely the set of elements $c_1, \cdots, c_g \in \mathbf{C}$ such that $\mu = c_1\omega_1 + \cdots + c_g\omega_g$. Let $D = \sum_{j=1}^k n_j P_j$. In this situation, the three statements
(1) $\mu \in \Gamma(C, \Omega^1(-D))$,
(2) $\delta^\ell \mu(P_j) = 0$ for all j and ℓ with $1 \leqq j \leqq k$ and $1 \leqq \ell \leqq n_j$, and

(3) $B(D)\begin{bmatrix} c_1 \\ \vdots \\ c_g \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$
are equivalent. So $\dim \Gamma(C, \Omega^1(-D)) = g - \operatorname{rank} B(D)$. The Riemann-Roch theorem states

$$\dim \Gamma(C, \mathcal{O}(D)) = \deg D - g + 1 + \dim \Gamma(C, \Omega^1(-D)).$$

Hence $\dim \Gamma(C, \mathcal{O}(D)) = \deg D + 1 - \operatorname{rank} B(D)$. □

To compute the rank of $B(D)$ we need only considering the case where D is of the form

$$D = P_1 + P_2 + \cdots + P_n + (g-n-1)\cdot\infty, \quad (P_i \neq P_j,\ \overline{P_i} \text{ for any } i \neq j, \text{ and } P_j \neq \infty)$$

where $\overline{P_i}$ is the hyperelliptic involution of P_i. Then the matrix $B(D)$ is given by

$$\left[\begin{array}{cccc|cccccc}
\frac{1}{2y}(P_1) & \frac{x}{2y}(P_1) & \cdots & \frac{x^{n-1}}{2y}(P_1) & \frac{x^n}{2y}(P_1) & \cdots & \frac{x^{g-3}}{2y}(P_1) & \frac{x^{g-2}}{2y}(P_1) & \frac{x^{g-1}}{2y}(P_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{1}{2y}(P_n) & \frac{x}{2y}(P_n) & \cdots & \frac{x^{n-1}}{2y}(P_n) & \frac{x^n}{2y}(P_n) & \cdots & \frac{x^{g-3}}{2y}(P_n) & \frac{x^{g-2}}{2y}(P_n) & \frac{x^{g-1}}{2y}(P_n) \\
\hline
 & & & & 0 & \cdots & 0 & 0 & 1 \\
 & & & & 0 & \cdots & 0 & 0 & \\
 & & & & 0 & \cdots & 0 & 1 & \\
 & & & & 0 & \cdots & 0 & & \\
 & & & & 0 & \cdots & 1 & & \\
 & & & & \vdots & \iddots & & &
\end{array}\right].$$

Here the right lower block was calculated by similar way to 3.8 and 3.9. Therefore the rank of $B(D)$ is $n + (g-n-1)/2$ or $n + (g-n)/2$ according as $g-n$ is odd or even. Summing up the considerations above, we have

$$\begin{aligned}\dim \Gamma(C, \mathcal{O}(P_1 + P_2 + \cdots + P_n + (g-n-1)\infty)) &= (g-1) + 1 - (n + \lfloor (g-n)/2 \rfloor) \\ &= \lfloor (g-n-1)/2 \rfloor + 1.\end{aligned}$$

Again we denote by u the point in $\kappa^{-1}(\Theta^{[g-1]})$ corresponding to $P_1 + P_2 + \cdots + P_n + (g - n-1)\infty - (g-1)\infty$. Proposition 5.1 yields that if $h \leqq \lfloor (g-n-1)/2 \rfloor$, then $\sigma_{i_1 i_2 \cdots i_h}(u) = 0$ for all $i_1, \cdots, i_h$ and $\sigma_{j_1 j_2 \cdots j_{\lfloor (g-n-1)/2 \rfloor + 1}}(u) \neq 0$ for some $j_1, j_2, \cdots, j_{\lfloor (g-n-1)/2 \rfloor + 1}$. Here we record the first fact as follows.

Lemma 5.3. *We fix the genus g. Let n ($\leqq g-1$) and h be integers such that $0 \leqq h \leqq \lfloor (g-n-1)/2 \rfloor$. Let $i_1, i_2, \cdots, i_h$ be arbitrary h elements in $\{1, 2, \cdots, g\}$. Then the function $u \mapsto \sigma_{i_1 i_2 \cdots i_h}(u)$ on $\kappa^{-1}(\Theta^{[n]})$ is identically zero.*

6. Special Derivatives of the Sigma function.

We will introduce some special derivatives of the sigma function. These are important to state our Frobenius-Stickelberger type formula.

Definition 6.1. Let $\natural^n$ be the set defined in 2.1. Then we define a derivative $\sigma_{\natural^n}(u)$ of $\sigma(u)$ by

$$\sigma_{\natural^n}(u) = \Bigg(\prod_{i\in\natural^n}\frac{\partial}{\partial u_i}\Bigg)\sigma(u)$$

In particular we define

$$\sigma_\sharp(u) = \sigma_{\natural^1}(u), \quad \sigma_\flat(u) = \sigma_{\natural^2}(u).$$

These functions are given in the following table.

genus	$\sigma_\sharp$	$\sigma_\flat$	$\sigma_{\natural^3}$	$\sigma_{\natural^4}$	$\sigma_{\natural^5}$	$\sigma_{\natural^6}$	$\sigma_{\natural^7}$	$\sigma_{\natural^8}$	$\sigma_{\natural^9}$	$\sigma_{\natural^{10}}$	$\cdots$
1	σ	σ	σ	σ	σ	σ	σ	σ	σ	σ	$\cdots$
2	σ_2	σ	σ	σ	σ	σ	σ	σ	σ	σ	$\cdots$
3	σ_2	σ_3	σ	σ	σ	σ	σ	σ	σ	σ	$\cdots$
4	σ_{24}	σ_3	σ_4	σ	σ	σ	σ	σ	σ	σ	$\cdots$
5	σ_{24}	σ_{35}	σ_4	σ_5	σ	σ	σ	σ	σ	σ	$\cdots$
6	σ_{246}	σ_{35}	σ_{46}	σ_5	σ_6	σ	σ	σ	σ	σ	$\cdots$
7	σ_{246}	σ_{357}	σ_{46}	σ_{57}	σ_6	σ_7	σ	σ	σ	σ	$\cdots$
8	σ_{2468}	σ_{357}	σ_{468}	σ_{57}	σ_{68}	σ_7	σ_8	σ	σ	σ	$\cdots$
9	σ_{2468}	σ_{3579}	σ_{468}	σ_{579}	σ_{68}	σ_{79}	σ_8	σ_9	σ	σ	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table of $\sigma_{\natural^n}(u)$

We are going to prepare tools for investigating the zeroes of these derivatives.

Lemma 6.2. *Suppose* $0 \leqq n \leqq g-1$. *Let* $\check{\natural}^n$ *be a proper subset of* $\natural^n$, *and let*

$$\sigma_{\check{\natural}^n}(u) = \Bigg(\prod_{i\in\check{\natural}^n}\frac{\partial}{\partial u_i}\Bigg)\sigma(u)$$

Then the function $u \mapsto \sigma_{\check{\natural}^n}(u)$ *on* $\kappa^{-1}(\Theta^{[n]})$ *is identically zero.*

Proof. Since the number of elements in the set $\natural^n$ is $\lfloor (g-n+1)/2 \rfloor$, we see the number of elements in $\check{\natural}^n$ is less than or equal to $\lfloor (g-n-1)/2 \rfloor$. Hence we have the assertion from 5.3. □

Lemma 6.3. (The translational relation) *Let* n *be an integer such that* $1 \leqq n \leqq g-1$. *Assume* u *belongs to* $\kappa^{-1}(\Theta^{[n]})$. *Then we have*

$$\sigma_{\natural^n}(u+\ell) = \chi(\ell)\sigma_{\natural^n}(u)\exp L(u+\tfrac{1}{2}\ell, \ell)$$

for all $\ell \in \Lambda$.

Proof. This follows from 6.2 by using 3.5 and 6.1. □

Proposition 6.4. *Let n be an integer such that $1 \leqq n \leqq g-1$. The space spanned by the functions $u \mapsto \varphi(u)$ on $\kappa^{-1}(\Theta^{[n]})$ vanishing only on $\kappa^{-1}(\Theta^{[n-1]})$ and satisfying the equation*

$$\varphi(u+\ell) = \chi(\ell)\varphi(u)\exp L(u+\tfrac{1}{2}\ell,\ell)$$

for all $\ell \in \Lambda$ is one dimensional.

Proof. Let $\varphi_1(u)$ and $\varphi_2(u)$ be non-trivial functions on $\kappa^{-1}(\Theta^{[n]})$ with the stated properties. Since $\Theta^{[n-1]}$ is a prime divisor of the variety $\Theta^{[n]}$, the equations assumed for these functions imply that we may assume that the vanishing order of $\varphi_2(u)$ on $\kappa^{-1}(\Theta^{[n-1]})$ is less than or equal to that of $\varphi_1(u)$. Then the function φ_1/φ_2 is holomorphic on $\kappa^{-1}(\Theta^{[n]})$. Classically, this is a situation where we might use a special case of Hartogs' analytic continuation theorem. On the other hand, we have

$$\frac{\varphi_1}{\varphi_2}(u+\ell) = \frac{\varphi_1}{\varphi_2}(u) \quad \text{for all } u \in \kappa^{-1}(\Theta^{[n]}) \text{ and } \ell \in \Lambda,$$

by the supposed equations. Therefore φ_1/φ_2 can be regarded as a holomorphic function on $\Theta^{[n]}$. Hence this is a constant function, by Liouville's theorem. □

Proposition 6.5. *Let v is a variable on $\kappa^{-1}(\Theta^{[1]})$. Then we have the following.*
(1) *The function $v \mapsto \sigma_\sharp(v)$ has a zero of order g at $v = (0,0,\cdots,0)$ modulo Λ and no zero elsewhere. This function has an expansion of the form*

$$\sigma_\sharp(v) = (-1)^{(g-2)(g-3)/2}{v_g}^g + (d^\circ(v_g) \geqq g+2).$$

(2) *Let n be an integer such that $1 \leqq n \leqq g-1$. Suppose v, $u^{(1)}$, $u^{(2)}$, $\cdots$, $u^{(n)}$ belong to $\kappa^{-1}(\Theta^{[1]})$. If $u^{(1)}+\cdots+u^{(n)} \notin \kappa^{-1}(\Theta^{[n-1]})$, then the function $v \mapsto \sigma_{\natural^{n+1}}(u^{(1)}+\cdots+u^{(n)}+v)$ has zeroes of order 1 at $v = -u^{(1)}$, $\cdots$, $-u^{(n)}$, a zero of order $g-n$ at $v = (0,0,\cdots,0)$ modulo Λ and no other zero elsewhere. This function has an expansion of the form*

$$\sigma_{\natural^{n+1}}(u^{(1)}+\cdots+u^{(n)}+v) = (-1)^{(g-n)(g-n-1)/2}\sigma_{\natural^n}(u^{(1)}+\cdots+u^{(n)}){v_g}^{g-n}+(d^\circ(v_g) \geqq g-n+1).$$

(3) *If $u \notin \kappa^{-1}(\Theta^{[g-1]})$ then $\sigma(u) \neq 0$.*

Proof. The statement (3) is well-known (see Theorem 5.3 in [**19**, p.3.80]). We prove (1) and (2). The usual argument by integration of the logarithm of

$$\begin{aligned}&\sigma_{\natural^{n+1}}(u^{(1)}+\cdots+u^{(n)}+v+\ell)\\ &\qquad = \chi(\ell)\sigma_{\natural^{n+1}}(u^{(1)}+\cdots+u^{(n)}+v)\exp L(u^{(1)}+\cdots+u^{(n)}+v+\tfrac{1}{2}\ell,\ell)\end{aligned}$$

of 6.3 along the boundary of a polygon representation of the Riemann surface of C shows either that the functions $v \mapsto \sigma_\sharp(v)$ and $v \mapsto \sigma_{\natural^{n+1}}(u^{(1)}+\cdots+u^{(n)}+v)$ above have exactly g zeroes modulo Λ or they vanish identically (see [**13**, p.147] for details). These functions, however, do not vanish identically because of 2.4 and 4.2. The other statements of (1) follow from 4.2 and 2.4(1).

We prove the rest of statement (2). Since the number of the elements in $\natural^{n+1}$ is $\lfloor (g-n)/2 \rfloor$, 5.3 shows that the function $v \mapsto \sigma_{\natural^{n+1}}(u^{(1)}+\cdots+u^{(n)}+v)$ has zeroes at least at $v=-u^{(1)}$, $\cdots$, $-u^{(n)}$. For $v \in \kappa^{-1}(\Theta^{[1]})$, we let

$$\sigma_{\natural^{n+1}}(u^{(1)}+\cdots+u^{(n)}+v)=\sum_{j=0}^{\infty}\varphi^{(j)}(u^{(1)}+\cdots+u^{(n)}){v_g}^j,$$

where $\varphi^{(j)}$ are certain functions on $\kappa^{-1}(\Theta^{[n]})$. Let the Sato weight of $\sigma_{\natural^{n+1}}(u^{(1)}+\cdots+u^{(n)}+v)$ to be m, so that $m=ng-\frac{1}{2}n(n+1)$. Let j_0 be the minimal j's such that $\varphi^{(j)}(u^{(1)}+\cdots+u^{(n)})$ is not identically 0 as a function of the variables $u^{(1)}$, $\cdots$, $u^{(n)}$, So, we can write

$$\sigma_{\natural^{n+1}}(u^{(1)}+\cdots+u^{(n)}+v)=\varphi^{(j_0)}(u^{(1)}+\cdots+u^{(n)}){v_g}^{j_0}+\cdots.$$

Then the Sato weight of $\varphi^{(j_0)}(u^{(1)}+\cdots+u^{(n)})$ is $m-j_0$. We claim that the function $u \mapsto \varphi^{(j_0)}(u)$ on $\kappa^{-1}(\Theta^{[n]})$ satisfies the equation in 6.4. To prove it, let us take $\ell \in \Lambda$ and $u \in \kappa^{-1}(\Theta^{[n]})$. Then, since $u+\ell \in \kappa^{-1}(\Theta^{[n]})$, we have

$$\sigma_{\natural^{n+1}}(u+v+\ell)=\varphi^{(j_0)}(u+\ell){v_g}^{j_0}+\cdots.$$

On the other hand, the translational relation for $\sigma_{\natural^{n+1}}$ gives

$$\begin{aligned}\sigma_{\natural^{n+1}}(u+v+\ell)&=\chi(\ell)\sigma_{\natural^{n+1}}(u+v)\exp L(u+v+\tfrac{1}{2}\ell,\ell)\\&=\chi(\ell)\big(\varphi^{(j_0)}(u){v_g}^{j_0}+\cdots\big)\exp L(u+v+\tfrac{1}{2}\ell,\ell)\\&=\chi(\ell)\big(\varphi^{(j_0)}(u){v_g}^{j_0}+\cdots\big)\big\{\exp L(u+\tfrac{1}{2}\ell,\ell)+(d^\circ(v_g)\geqq 1)\big\}\\&=\chi(\ell)\varphi^{(j_0)}(u)\exp L(u+\tfrac{1}{2}\ell,\ell){v_g}^{j_0}+\big(d^\circ(v_g)\geqq (j_0+1)\big).\end{aligned}$$

Comparing these two above, we see that

$$\varphi^{(j_0)}(u+\ell)=\chi(\ell)\varphi^{(j_0)}(u)\exp L(u+\tfrac{1}{2}\ell,\ell).$$

So, we have shown the claim above. Proposition 6.4 yields that $\varphi^{(j_0)}(u)$ is, as functions on $\kappa^{-1}(\Theta^{[n]})$, equal to $\sigma_{\natural^n}(u)$ up to a multiplicative constant. The Sato weight of $\sigma_{\natural^n}(u)$ is $(n-1)g-\frac{1}{2}(n-1)n=m-(g-n)$. By 4.1, 2.4(2), and 2.4(1), we see that the expansion of $\sigma_{\natural^n}(u)$ contains a term whose coefficient is 1 or -1. Therefore, the multiplicative constant above must be a polynomial of $\lambda_0(=1)$, $\cdots$, λ_{2g+1}. Hence the Sato weight of $\varphi^{(j_0)}(u)$ is larger than or equal to $m-(g-n)$, namely $j_0 \geqq g-n$. Therefore we have identically

$$\varphi^{(j)}(u)=0 \quad \text{for } j=0,\ \cdots,\ g-n-1.$$

Because we already found at least n zeroes of $v \mapsto \sigma_{\natural^{n+1}}(u^{(1)}+\cdots+u^{(n)}+v)$ for $v \neq 0$, it is impossible that $\varphi^{(g-n)}(u^{(1)}+\cdots+u^{(n)})=0$ identically. So we see $j_0=g-n$ and have

$$\sigma_{\natural^{n+1}}(u+v)=\varphi^{(g-n)}(u){v_g}^{g-n}+(d^\circ(v_g)\geqq g)$$

for the non-trivial function $\varphi^{(g-n)}(u)$. Thus, we see that

$$\varphi^{(g-n)}(u)=(-1)^{(g-n)(g-n-1)/2}\sigma_{\natural^n}(u)$$

for $u \in \kappa^{-1}(\Theta^{[n]})$ by 2.4(2) and 4.2. Now all the other statements are clear. □

Lemma 6.6. *Let* $u \in \kappa^{-1}(\Theta^{[1]})$*. Then*

$$\sigma_\flat(2u) = (-1)^{g-1} 2{u_g}^{2g-1} + (d^\circ(u_g) \geqq 2g+1)\,.$$

Proof. The statement follows from 2.4(3), 4.2 and 3.4. □

Lemma 6.7. *Let* $u \in \kappa^{-1}(\Theta^{[1]})$*. Then*

$$\frac{\sigma_\flat(2u)}{\sigma_\sharp(u)^4} = (-1)^g 2y(u).$$

Proof. Lemma 6.3 shows that the left hand side is periodic with respect to Λ, and is an odd function by 3.4. We have by 6.6 that

$$\begin{aligned}
\frac{\sigma_\flat(2u)}{\sigma_\sharp(u)^4} &= \frac{(-1)^{g-1} 2{u_g}^{2g-1} + (d^\circ(u_g) \geqq 2g+1)}{\left({u_g}^g + (d^\circ(u_g) \geqq g+2)\right)^4} \\
&= (-1)^{g-1} \frac{2}{{u_g}^{2g+1}} + \cdots \\
&= (-1)^g 2y(u).
\end{aligned}$$

□

7. Frobenius-Stickerberger Type Formulae.

The initial case of our Frobenius-Stickelberger type formulae is as follows.

Lemma 7.1. *Suppose that u and v are in $\kappa^{-1}(\Theta^{[1]})$. We have*

$$(-1)^{g+1}\frac{\sigma_\flat(u+v)\sigma_\flat(u-v)}{\sigma_\sharp(u)^2\sigma_\sharp(v)^2} = -x(u)+x(v)\left(=\begin{vmatrix}1 & x(u)\\ 1 & x(v)\end{vmatrix}\right).$$

Proof. As a function of u (or v), we see that the left hand side is periodic with respect to Λ, by 6.3. Moreover we see that the left hand side has only pole at $u=(0,0,\cdots,0)$ modulo Λ by 6.5(1). Proposition 6.5 also shows the Laurent expansion of the left hand side is of the form

$$\frac{(\sigma_\sharp(v){u_g}^{g-1}+\cdots)(\sigma_\sharp(-v){u_g}^{g-1}+\cdots)}{({u_g}^g+\cdots)^2\sigma_\sharp(v)^2} = (-1)^g\frac{1}{{u_g}^2}+\cdots = (-1)^g x(u)+\cdots.$$

Here we have used the fact that $\sigma_\sharp(-v)=(-1)^g\sigma_\sharp(v)$ which follows from 3.4. Since both sides has the same zeroes at $u=v$ and $u=-v$, they coincide. □

The general case of our Frobenius-Stickelberger type formula is as follows.

Theorem 7.2. *Let n be a fixed integer. Suppose $u^{(1)}$, $\cdots$, $u^{(n)}$ are variables on $\kappa^{-1}(\Theta^{[1]})$.*
(1) *If $2\leqq n\leqq g$, then we have*

$$(-1)^{g+\frac{1}{2}n(n+1)}\frac{\sigma_{\natural^n}(u^{(1)}+\cdots+u^{(n)})\prod_{i<j}\sigma_\flat(u^{(i)}-u^{(j)})}{\sigma_\sharp(u^{(1)})^n\cdots\sigma_\sharp(u^{(n)})^n}
= \begin{vmatrix}1 & x(u^{(1)}) & x^2(u^{(1)}) & \cdots & x^{n-1}(u^{(1)})\\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & \cdots & x^{n-1}(u^{(2)})\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^{n-1}(u^{(n)})\end{vmatrix}.$$

(2) *If $n\geqq g+1$, then we have*

$$(-1)^{\frac{1}{2}(2n-g)(g-1)}\frac{\sigma(u^{(1)}+\cdots+u^{(n)})\prod_{i<j}\sigma_\flat(u^{(i)}-u^{(j)})}{\sigma_\sharp(u^{(1)})^n\cdots\sigma_\sharp(u^{(n)})^n}
= \begin{vmatrix}1 & x(u^{(1)}) & x^2(u^{(1)}) & \cdots & x^g(u^{(1)}) & y(u^{(1)}) & x^{g+1}(u^{(1)}) & xy(u^{(1)}) & x^{g+2}(u^{(1)}) & \cdots\\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & \cdots & x^g(u^{(2)}) & y(u^{(2)}) & x^{g+1}(u^{(2)}) & xy(u^{(2)}) & x^{g+2}(u^{(2)}) & \cdots\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^g(u^{(n)}) & y(u^{(n)}) & x^{g+1}(u^{(n)}) & xy(u^{(n)}) & x^{g+2}(u^{(n)}) & \cdots\end{vmatrix},$$

where the right hand side is an n by n determinant.

Proof. While this Theorem is proved by 6.5 and the same argument as in [**19**], [**20**], we describe details here because of complexity of the signature of the top in the formulae.

We prove (1) first by three Steps as follows.

Step 1. Lemma 6.3 shows both sides in (1) are periodic functions of $u^{(n)}$ with respect to Λ. Hence we regard them as functions on C and will compare their divisors.
Step 2. We are still regarding both sides as functions of $u^{(n)}$. By continuity, we may assume that $u^{(1)}, \cdots, u^{(n-1)}$ are pairwise different. The right hand side vanishes at $u^{(1)}, \cdots, u^{(n-1)}$ by 6.5(2). The left hand side has zeroes of order 1 at the same points because of the product of $\sigma_\flat$'s. We see that the left hand side has a pole only at $u^{(n)} = (0, 0, \cdots, 0)$ modulo Λ and its order is $(g-n+1)+(g-1)(n-1)-gn = 2(n-1)$ by 6.5(1). Obviously, the right hand side has also a pole only at $u^{(n)} = (0, 0, \cdots, 0)$ modulo Λ. Because the lowest term of the Laurent expansion of it is coming only from (n, n)-entry, the pole is also of order $2(n-1)$. Let $v^{(1)}, \cdots, v^{(n-1)}$ are the other zeroes of the left hand side than $u^{(1)}, \cdots, u^{(n-1)}$. As we also consider their multiplicities, it may happen that some of $v^{(1)}, \cdots, v^{(n-1)}$ coincide with any of $u^{(1)}, \cdots, u^{(n-1)}$. Anyway, Abel-Jacobi theorem shows that

$$u^{(1)} + \cdots + u^{(n-1)} + v^{(1)} + \cdots + v^{(n-1)} \in \Lambda.$$

We denote this element by ℓ. Then 6.3 states that

$$\sigma_{\natural^n}(u^{(1)} + \cdots + u^{(n-1)} + u^{(n)}) = \sigma_{\natural^n}(u^{(n)} - v^{(1)} - \cdots - v^{(n-1)})\chi(\ell)\exp L(u + \tfrac{1}{2}\ell, \ell).$$

Hence this function of $u^{(n)}$ has zeroes at $v^{(1)}, \cdots, v^{(n-1)}$ of order 1 by 6.5(2). Therefore the two sides have the same divisor, and they are equal up to multiplication of a function of $u^{(1)}, \cdots, u^{(n-1)}$.
Step 3. Finally, we check the undetermined function above is 1 by comparing the coefficients of the lowest term in the Laurent expansion with respect to $u^{(n)}$. The expansion of the left hand side is

$$\begin{aligned}
&\Big[\big\{(-1)^{(g-n+1)(g-n)/2}\sigma_{\natural^{n-1}}(u^{(1)} + \cdots + u^{(n-1)}){v_g}^{g-n+1} + \cdots\big\} \\
&\quad \prod_{i<j\leqq n-1}\sigma_\flat(u^{(i)} - u^{(j)}) \prod_{i=1}^{n-1}\big\{(-1)^{(g-1)(g-2)/2}\sigma_\sharp(u^{(i)})(-u_g^{(n)})^{g-1} + \cdots\big\}\Big] \\
&\qquad \Big/ \Big[\sigma_\sharp(u^{(1)})^n \cdots \sigma_\sharp(u^{(n-1)})^n\big\{(-1)^{(g-2)(g-3)/2}(u_g^{(n)})^g + \cdots\big\}^n\Big] + \cdots \\
&= (-1)^{(g-n+1)(g-n)/2+(g-1)(g-2)(n-1)/2+(g-1)-(g-2)(g-3)n/2} \\
&\qquad \frac{\sigma_{\natural^{n-1}}(u^{(1)} + \cdots + u^{(n-1)})\prod_{i<j\leqq n-1}\sigma_\flat(u^{(i)} - u^{(j)})}{\sigma_\sharp(u^{(1)})^{n-1}\cdots\sigma_\sharp(u^{(n-1)})^{n-1}}\frac{1}{(u_g^{(n)})^{2(n-1)}} + \cdots.
\end{aligned}$$

Here we see easily that the index of (-1) is given by

$$\begin{aligned}
&\frac{(g-n+1)(g-n)}{2} + \frac{(g-1)(g-2)(n-1)}{2} + (g-1) - \frac{(g-2)(g-3)n}{2} \\
&\equiv g + \frac{1}{2}(n-1)n \mod 2.
\end{aligned}$$

The expansion

$$\begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & \cdots & x^{n-2}(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & \cdots & x^{n-2}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x(u^{(n-1)}) & x^2(u^{(n-1)}) & \cdots & x^{n-2}(u^{(n-1)}) \end{vmatrix} x^{n-1}(u^{(n)}) + \cdots$$

of the right hand side with $x(u^{(n)}) = 1/(u^{(n)})^2 + \cdots$ and the induction hypothesis shows the coefficients coincide. Thus, we have proved (1).

Now let us prove (2). While the argument is entirely similar to that of (1), we describe it explicitly again in order to make the reader easier to check the signature of the desired formula. Regarding both sides to be functions of $u^{(n)}$ again, the things corresponding to the Steps 1 and 2 above are proved similarly. So, we omit them. To prove the part corresponding to Step 3 above, we note that the leading term of Laurent expansion of the right hand side contains the following signature;

$$\frac{(-1)^{(n+g-1)}}{(u_g^{(n)})^{n+g-1}} \begin{vmatrix} 1 & x(u^{(1)}) & \cdots & x^g(u^{(1)}) & y(u^{(1)}) & x^{g+1}(u^{(1)}) & xy(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & \cdots & x^g(u^{(2)}) & y(u^{(2)}) & x^{g+1}(u^{(2)}) & xy(u^{(2)}) & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n-1)}) & \cdots & x^g(u^{(n-1)}) & y(u^{(n-1)}) & x^{g+1}(u^{(n-1)}) & xy(u^{(n-1)}) & \cdots \end{vmatrix}.$$

Expanding the left hand side gives

$$\begin{aligned} &\Big[\big\{\sigma(u^{(1)} + \cdots + u^{(n-1)}) + \cdots\big\} \\ &\qquad \prod_{i<j\leqq n-1} \sigma_\flat(u^{(i)} - u^{(j)}) \prod_{i=1}^{n-1} \big\{(-1)^{(g-1)(g-2)/2}\sigma_\sharp(u^{(i)})(-u_g^{(n)})^{g-1} + \cdots\big\}\Big] \\ &\qquad\qquad \Big/ \Big[\sigma_\sharp(u^{(1)})^n \cdots \sigma_\sharp(u^{(n-1)})^n \big\{\big((-1)^{(g-2)(g-3)/2}(u_g^{(n)})^g\big)^n + \cdots\big\}\Big] \\ &= (-1)^{(g-1)(g-2)(n-1)/2+(g-1)-(g-2)(g-3)n/2} \\ &\qquad \frac{\sigma_{\natural^{n-1}}(u^{(1)} + \cdots + u^{(n-1)}) \prod_{i<j\leqq n-1} \sigma_\flat(u^{(i)} - u^{(j)})}{\sigma_\sharp(u^{(1)})^{n-1} \cdots \sigma_\sharp(u^{(n-1)})^{n-1}} \frac{1}{(u_g^{(n)})^{n+g-1}} + \cdots. \end{aligned}$$

Because the index of the quotient of signatures of the two leading terms above is given by

$$\begin{aligned} \frac{(g-1)(g-2)(n-1)}{2} + (g-1) - \frac{(g-2)(g-3)n}{2} - (n+g-1) \\ \equiv \frac{(2n-2-g)(g-1)}{2} \mod 2, \end{aligned}$$

we can use the induction hypothesis and have (2). □

8. Kiepert Type Formulae.

The function $\sigma(u)$ directly relates with $x(u)$ as follows.

Lemma 8.1. *Fix j with $0 \leqq j \leqq g$. Let u and v are on $\kappa^{-1}(\Theta^{[1]})$. Then*

$$\lim_{u\to v} \frac{\sigma_\flat(u-v)}{u_j - v_j} = \frac{1}{x^{j-1}(v)}.$$

Proof. Because of 7.1 we have

$$\frac{x(u)-x(v)}{u_j-v_j} = (-1)^g \frac{\sigma_\flat(u+v)}{\sigma_\sharp(u)^2\sigma_\sharp(v)^2}\cdot\frac{\sigma_\flat(u-v)}{u_j-v_j}.$$

Now we let u tend to v. Then the limit of the left hand side is

$$\lim_{u\to v}\frac{x(u)-x(v)}{u_j-v_j} = \frac{dx}{du_j}(v).$$

This is equal to $2y/x^{j-1}(v)$ by the definition. The required formula follows from 6.7. □

Definition 8.2. For $u \in \kappa^{-1}(\Theta^{[1]})$ we denote by $\psi_n(u)$ the function $\sigma_{\natural^n}(nu)/\sigma_\sharp(u)^{n^2}$ if $n \leqq g$, and $\sigma(nu)/\sigma_\sharp(u)^{n^2}$ if $n \geqq g+1$.

This function $\psi_n(u)$ has the following expression, which is a natural generalization of the classical formula of Kiepert [**14**].

Theorem 8.3. (Kiepert type formula) *Let $u \in \kappa^{-1}(\Theta^{[1]})$ and n be a positive integer.*
(1) *If $1 \leqq n \leqq g$ then $\psi_n(u) = (-1)^{g+\frac{1}{2}n(n+1)}(2y(u))^{n(n-1)/2}$.*
(2) *We fix j with $1 \leqq j \leqq g$. If $n \geqq g+1$ then we have*

$$(-1)^{(2n-g)(g-1)/2}(1!2!\cdots(n-1)!)\psi_n(u) = x^{(j-1)n(n-1)/2}(u)\times$$
$$\begin{vmatrix} x' & (x^2)' & \cdots & (x^g)' & y' & (x^{g+1})' & (yx)' & (x^{g+2})' & \cdots \\ x'' & (x^2)'' & \cdots & (x^g)'' & y'' & (x^{g+1})'' & (yx)'' & (x^{g+2})'' & \cdots \\ x''' & (x^2)''' & \cdots & (x^3)''' & y''' & (x^{g+1})''' & (yx)''' & (x^{g+2})''' & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ x^{(n-1)} & (x^2)^{(n-1)} & \cdots & (x^g)^{(n-1)} & y^{(n-1)} & (x^{g+1})^{(n-1)} & (yx)^{(n-1)} & (x^{g+2})^{(n-1)} & \cdots \end{vmatrix}(u).$$

where the size of the matrix is $(n-1)$ by $(n-1)$, the symbols $'$, $''$, $\cdots$, ${}^{(n-1)}$ denote $\frac{d}{du_j}$, $\left(\frac{d}{du_j}\right)^2$, $\cdots$, $\left(\frac{d}{du_j}\right)^{n-1}$.

Proof. If $1 \leqq n \leqq g+1$, the right hand side of 7.2 is a Vandermonde determinant. Hence we have statement (1), by using 8.1. Statement (2) is proved by the same argument as in [**19**], using 7.2 and 8.1. □

Remark 8.4. The polynomials $\psi_n(u)$ are the natural generalization of *division polynomials* of an elliptic curve, and are used to find torsion points on the curve C in the Jacobian variety J. Indeed, for $n \geqq g$, $u \in \kappa^{-1}\iota(C)$ is an n-torsion in J if and only if all of $\psi_{n-g+1}(u)$, $\psi_{n-g+2}(u)$, $\cdots$, $\psi_n(u)$, $\cdots$, $\psi_{n+g-1}(u)$ vanish. Detailed description of this fact is seen in [**8**].

Finally we mention the degree of the polynomials above.

Proposition 8.5. *The number of roots, counting multiplicities, of the equation* $\psi_n(u) = 0$ *is* $\frac{1}{2}n(n-1)(2g+1)$ *if* $1 \leqq n \leqq g-1$ *and* $n^2 g - \frac{1}{2}g(g+1)$ *if* $n \geqq g$.

Proof. The number of the roots is equal to the order of the pole at $u = (0, \cdots, 0)$. If $u \in \kappa^{-1}(\Theta^{[1]})$, then u_g is a local parameter at $u = (0, \cdots, 0)$ because of 3.9. We calculate it for the case $j = g$. We denote u_g by t. If $n \geqq g$, the lowest term of the Laurent expansion of the determinant in 8.3 at $u = (0, 0, \cdots, 0)$ is exactly the same as that of

$$\begin{vmatrix} -\frac{2}{t^3} & -\frac{4}{t^5} & \cdots & -\frac{2g}{t^{2g+1}} & \cdots & -\frac{n+g-1}{t^{n+g}} \\ \frac{4\cdot 3}{t^4} & \frac{6\cdot 5}{t^6} & \cdots & \frac{(2g+2)(2g+1)}{t^{2g+2}} & \cdots & \frac{(n+g+1)(n+g)}{t^{n+g+1}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (-1)^{n-1}\frac{(n+1)\cdots 3}{t^{n+1}} & (-1)^{n-1}\frac{(n+3)\cdots 5}{t^{n+3}} & \cdots & (-1)^{n-1}\frac{(n+2g-1)\cdots(2g-1)}{t^{2g+n-1}} & \cdots & (-1)^{n-1}\frac{(2n+g-1)\cdots(n+g-1)}{t^{2n+g-2}} \end{vmatrix}.$$

This is equal to

$$\begin{vmatrix} \frac{1}{t^3} & & & & & & \\ & \frac{1}{t^6} & & & * & & \\ & & \ddots & & & & \\ & & & \frac{1}{t^{3g}} & & & \\ & & & & \frac{1}{t^{3g+2}} & & \\ & & & & & \ddots & \\ & & & & & & \frac{1}{t^{2n+4g}} \end{vmatrix}$$

times a non-zero constant. This determinant is the $(\frac{3}{2}g(g+1)+(n-g-1)(n+2g))^{\text{th}}$ power of $1/t$. Hence we see that $x^{(g-1)n(n-1)/2}(u) = 1/t^{(g-1)n(n-1)} + \cdots$. So the lowest term of the Laurent expansion of the right hand side of the formula in 8.3 is the $\left(n^2 g - \frac{1}{2}g(g+1)\right)^{\text{st}}$ power of $1/t$. If $n < g$ then the lowest term is easily seen to be the $\frac{1}{2}n(n-1)(2g+1)^{\text{st}}$ power of $1/t$. □

Finally we have a remark. The derivation D_∞ used in Mumford's book [**17**, p.3.155] is essentially no other than our $\partial/\partial u_g$. While we could rewrite our expressions using mainly this D_∞, the author hopes to describe such things in other occasion.

Appendix: Connection of The formulae of Cantor-Brioschi and those of Kiepert type (by S. Matsutani).

In this Appendix we prove a formula of Cantor in [**8**] (Theorem A.1 below) by using 8.3. This is a detailed exposition of the appendix in [**16**]. Since our argument is convertible, 8.3 is proved using the formula of Cantor.

Let $u = (u_1, u_2, \cdots, u_g)$ be the system of variables explained in Section 3. We assume that u belongs to $\kappa^{-1}\iota(C)$. So we may use the notation $x(u)$ and $y(u)$. If $\mu(u)$ is a function on $\kappa^{-1}\iota(C)$ we can regard it locally as a function of u_1. We denote by

$$\mu'(u),\ \mu''(u),\ \cdots,\ \mu^{(\nu)}(u),\ \cdots$$

the functions obtained by applying

$$\tfrac{d}{du_1},\ (\tfrac{d}{du_1})^2,\ \cdots,\ (\tfrac{d}{du_1})^\nu,\ \cdots$$

to the function $\mu(u)$ along $\iota(C)$; and by

$$\dot{\mu}(u),\ \ddot{\mu}(u),\ \cdots,\ \mu^{\langle\nu\rangle}(u),\ \cdots$$

the functions given by applying

$$\tfrac{d}{dx},\ (\tfrac{d}{dx})^2,\ \cdots,\ (\tfrac{d}{dx})^\nu,\ \cdots$$

to $\mu(u)$. Here we regard $\mu(u)$ locally as a function of $x = x(u)$.
Recall that $\psi_n(u)$ was defined in 8.2. A determinant expression for $\psi_n(u)$, due to Cantor, is the following.

Theorem A.1. (Cantor [**8**]) *Suppose $n \geqq g+2$. Let s be the largest integer not exceeding $(n-g)/2$, and $r = n-1-s$. Then*

$$\psi_n(u) = \varepsilon_n \cdot (2y)^{n(n-1)/2} \times \begin{cases} \begin{vmatrix} \dfrac{y^{\langle g+2\rangle}}{(g+2)!} & \dfrac{y^{\langle g+3\rangle}}{(g+3)!} & \cdots & \dfrac{y^{\langle r+1\rangle}}{(r+1)!} \\ \dfrac{y^{\langle g+3\rangle}}{(g+3)!} & \dfrac{y^{\langle g+4\rangle}}{(g+4)!} & \cdots & \dfrac{y^{\langle r+2\rangle}}{(r+2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{y^{\langle r+1\rangle}}{(r+1)!} & \dfrac{y^{\langle r+2\rangle}}{(r+2)!} & \cdots & \dfrac{y^{\langle n-1\rangle}}{(n-1)!} \end{vmatrix} & \textit{if}\ \ n \not\equiv g \mod 2, \\ \\ \begin{vmatrix} \dfrac{y^{\langle g+1\rangle}}{(g+1)!} & \dfrac{y^{\langle g+2\rangle}}{(g+2)!} & \cdots & \dfrac{y^{\langle r+1\rangle}}{(r+1)!} \\ \dfrac{y^{\langle g+2\rangle}}{(g+2)!} & \dfrac{y^{\langle g+3\rangle}}{(g+3)!} & \cdots & \dfrac{y^{\langle r+2\rangle}}{(r+2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{y^{\langle r+1\rangle}}{(r+1)!} & \dfrac{y^{\langle r+2\rangle}}{(r+2)!} & \cdots & \dfrac{y^{\langle n-1\rangle}}{(n-1)!} \end{vmatrix} & \textit{if}\ \ n \equiv g \mod 2, \end{cases}$$

where ε_n is given by the following table :

$g \setminus n \bmod 4$	0	1	2	3
0	-1	-1	1	-1
1	-1	1	1	1
2	1	-1	-1	-1
3	1	1	-1	1

.

Remark A.2. (1) The both matrices above are of size $s \times s$.
(2) The number r and s are just the number of entries of the form $(x^k)'$ with $k \geqq 1$ and of the form $(yx^j)'$ with $j \geqq 0$, respectively, in the first row of the determinant in 8.3(2).
(3) The constant factor is not clear from our definition in 8.2 and [**8**] of $\psi_n(u)$. It is determined when our calculation has been completed.
(4) This formula for the case $g = 1$ is known classically (Brioschi [**4**]).
(5) Since our proof depends on 7.2, it works over only the field of complex numbers.

The following Lemma is easily checked.

Lemma A.3. *Let $m > 0$ be any integer. One has*

$$\left(\frac{d}{du_1}\right)^m = (2y)^m(u)\left(\frac{d}{dx}\right)^m + \sum_{j=1}^{m} a_j^{(m)}(u)\left(\frac{d}{dx}\right)^j.$$

Here $a_j^{(m)}(u)$ are polynomials in $y(u)$, $\frac{dy}{dx}(u)$, $\frac{d^2y}{dx^2}(u)$, $\cdots$, $\frac{d^{m-1}y}{dx^{m-1}}(u)$.

Let $s = n - 1 - r$. Then, by 8.3 for $j = 1$, we have

$$c_n' 1!2! \cdots (n-1)! \psi_n(u)$$
$$= (-1)^{r(r+1)/2} \begin{vmatrix} x' & (x^2)' & \cdots & (x^r)' & y' & (yx)' & \cdots & (yx^{s-1})' \\ x'' & (x^2)'' & \cdots & (x^r)'' & y'' & (yx)'' & \cdots & (yx^{s-1})'' \\ x''' & (x^2)''' & \cdots & (x^r)''' & y''' & (yx)''' & \cdots & (yx^{s-1})''' \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ x^{(n-1)} & (x^2)^{(n-1)} & \cdots & (x^r)^{(n-1)} & y^{(n-1)} & (yx)^{(n-1)} & \cdots & (yx^{s-1})^{(n-1)} \end{vmatrix} (u).$$

Here the determinant is $n-1$ by $n-1$. By A.3 we have that

$$\begin{bmatrix} \frac{d}{du_1} \\ (\frac{d}{du_1})^2 \\ (\frac{d}{du_1})^3 \\ \vdots \\ (\frac{d}{du_1})^{n-1} \end{bmatrix} = \begin{bmatrix} 2y & & & & \\ a_2^{(1)} & (2y)^2 & & & \\ a_3^{(1)} & a_3^{(2)} & (2y)^3 & & \\ \vdots & \vdots & \vdots & \ddots & \\ a_{n-1}^{(1)} & a_{n-1}^{(2)} & a_{n-1}^{(3)} & \cdots & (2y)^{n-1} \end{bmatrix} \begin{bmatrix} \frac{d}{dx} \\ \left(\frac{d}{dx}\right)^2 \\ \left(\frac{d}{dx}\right)^3 \\ \vdots \\ \left(\frac{d}{dx}\right)^{n-1} \end{bmatrix}.$$

Now we consider

$$^t\left[\frac{d}{dx}\mu \quad \left(\frac{d}{dx}\right)^2 \mu \quad \left(\frac{d}{dx}\right)^3 \mu \quad \cdots \quad \left(\frac{d}{dx}\right)^{n-1} \mu \right]$$

for $\mu = x,\, x^2,\, \cdots$ and $y,\, yx,\, yx^2,\, \cdots$. Obviously

$$\begin{bmatrix} \frac{d}{dx} \\ \left(\frac{d}{dx}\right)^2 \\ \left(\frac{d}{dx}\right)^3 \\ \vdots \\ \left(\frac{d}{dx}\right)^{n-1} \end{bmatrix} [\, x \quad x^2 \quad \cdots \quad x^r\,] = \left[\begin{array}{ccccc} 1! & & & * & \\ & 2! & & & \\ & & 3! & & \\ & & & \ddots & \\ & & & & r! \\ \hline & & \mathbf{0} & & \end{array}\right].$$

For $\mu = y,\, yx,\, \cdots,\, yx^{s-1}$, we have

$$\begin{bmatrix} \frac{d}{dx} \\ \left(\frac{d}{dx}\right)^2 \\ \left(\frac{d}{dx}\right)^3 \\ \vdots \\ \left(\frac{d}{dx}\right)^{n-1} \end{bmatrix} [\, y \quad yx \quad yx^2 \quad \cdots \quad yx^{s-1}\,]$$

$$= \begin{bmatrix} \binom{1}{0}\dot{y} & \binom{1}{0}\dot{y}x + \binom{1}{1}y & \binom{1}{0}\dot{y}x^2 + \binom{1}{1}y\cdot 2x & \binom{1}{0}\dot{y}x^3 + \binom{1}{1}y\cdot 3x^2 & \cdots \\ \binom{2}{0}\ddot{y} & \binom{2}{0}\ddot{y}x + \binom{2}{1}\dot{y} & \binom{2}{0}\ddot{y}x^2 + \binom{2}{1}\dot{y}\cdot 2x + \binom{2}{2}y\cdot 2! & \binom{2}{0}\ddot{y}x^3 + \binom{2}{1}\dot{y}\cdot 3x^2 + \binom{2}{2}y\cdot 3\cdot 2x & \cdots \\ \binom{3}{0}\dddot{y} & \binom{3}{0}\dddot{y}x + \binom{3}{1}\ddot{y} & \binom{3}{0}\dddot{y}x^2 + \binom{3}{1}\ddot{y}\cdot 2x + \binom{3}{2}\dot{y}\cdot 2! & \binom{3}{0}\dddot{y}x^3 + \binom{3}{1}\ddot{y}\cdot 3x^2 + \binom{3}{2}\dot{y}\cdot 3\cdot 2x + \binom{3}{3}y\cdot 3! & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \left(\left[\begin{array}{ccccc} \binom{1}{0}\frac{d}{dx} & \binom{1}{1} & & & \\ \binom{2}{0}\left(\frac{d}{dx}\right)^2 & \binom{2}{1}\frac{d}{dx} & \binom{2}{2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{s-1}{0}\left(\frac{d}{dx}\right)^{s-1} & \binom{s-1}{1}\left(\frac{d}{dx}\right)^{s-2} & \binom{s-1}{2}\left(\frac{d}{dx}\right)^{s-3} & \cdots & \binom{s-1}{s-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{r}{0}\left(\frac{d}{dx}\right)^{r} & \binom{r}{1}\left(\frac{d}{dx}\right)^{r-1} & \binom{r}{2}\left(\frac{d}{dx}\right)^{r-2} & \cdots & \binom{r}{s-1}\left(\frac{d}{dx}\right)^{r-s+1} \\ \hline \binom{r+1}{0}\left(\frac{d}{dx}\right)^{r+1} & \binom{r+1}{1}\left(\frac{d}{dx}\right)^{r} & \binom{r+1}{2}\left(\frac{d}{dx}\right)^{r-1} & \cdots & \binom{r+1}{s-1}\left(\frac{d}{dx}\right)^{r-s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n-1}{0}\left(\frac{d}{dx}\right)^{n-1} & \binom{n-1}{1}\left(\frac{d}{dx}\right)^{n-2} & \binom{n-1}{2}\left(\frac{d}{dx}\right)^{n-3} & \cdots & \binom{n-1}{s-1}\left(\frac{d}{dx}\right)^{n-s} \end{array}\right] \right.$$

$$\left. \times \begin{bmatrix} y & yT & y\cdot T^2 & \cdots & y\cdot T^{s-1} \\ & y & y\cdot 2T & \cdots & y\cdot (s-1)T^{s-2} \\ & & y\cdot 2! & \cdots & y\cdot (s-1)(s-2)T^{s-3} \\ & & & \ddots & \vdots \\ & & & & y\cdot (s-1)! \end{bmatrix} \right) \Bigg|_{T=x}.$$

Thus

$$\det\left(\begin{bmatrix} \frac{d}{dx} \\ \left(\frac{d}{dx}\right)^2 \\ \left(\frac{d}{dx}\right)^3 \\ \vdots \\ \left(\frac{d}{dx}\right)^{n-1} \end{bmatrix} [\, x \quad x^2 \quad \cdots \quad x^r \quad y \quad yx \quad \cdots \quad yx^{s-1}\,] \right)$$

is equal to $(1!2!\cdots r!)$ times

$$\det\left(\begin{bmatrix} \binom{r+1}{0}\left(\frac{d}{dx}\right)^{r+1} & \cdots & \binom{r+1}{s-1}\left(\frac{d}{dx}\right)^{r-s+2} \\ \vdots & \ddots & \vdots \\ \binom{n-1}{0}\left(\frac{d}{dx}\right)^{n-1} & \cdots & \binom{n-1}{s-1}\left(\frac{d}{dx}\right)^{n-s} \end{bmatrix} \begin{bmatrix} y & yT & y\cdot T^2 & \cdots & y\cdot T^{s-1} \\ & y & y\cdot 2T & \cdots & y\cdot(s-1)T^{s-2} \\ & & y\cdot 2! & \cdots & y\cdot(s-1)(s-2)T^{s-3} \\ & & & \ddots & \vdots \\ & & & & y\cdot(s-1)! \end{bmatrix}\right)\Bigg|_{T=x}$$

$$= \det\left(\begin{bmatrix} \binom{r+1}{0}\left(\frac{d}{dx}\right)^{r+1} & \cdots & \binom{r+1}{s-1}\left(\frac{d}{dx}\right)^{r-s+2} \\ \vdots & \ddots & \vdots \\ \binom{n-1}{0}\left(\frac{d}{dx}\right)^{n-1} & \cdots & \binom{n-1}{s-1}\left(\frac{d}{dx}\right)^{n-s} \end{bmatrix} \begin{bmatrix} y & & & & \\ & y\cdot 1! & & & \\ & & y\cdot 2! & & \\ & & & \ddots & \\ & & & & y\cdot(s-1)! \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} \binom{r+1}{0}\left(\frac{d}{dx}\right)^{r+1} & 1!\binom{r+1}{1}\left(\frac{d}{dx}\right)^{r} & \cdots & (s-1)!\binom{r+1}{s-1}\left(\frac{d}{dx}\right)^{r-s+2} \\ \binom{r+2}{0}\left(\frac{d}{dx}\right)^{r+2} & 1!\binom{r+2}{1}\left(\frac{d}{dx}\right)^{r+1} & \cdots & (s-1)!\binom{r+2}{s-1}\left(\frac{d}{dx}\right)^{r-s+3} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n-1}{0}\left(\frac{d}{dx}\right)^{n-1} & 1!\binom{n-1}{1}\left(\frac{d}{dx}\right)^{n-2} & \cdots & (s-1)!\binom{n-1}{s-1}\left(\frac{d}{dx}\right)^{n-s} \end{bmatrix} \begin{bmatrix} y & & & & \\ & y & & & \\ & & y & & \\ & & & \ddots & \\ & & & & y \end{bmatrix}\right).$$

By dividing the first row by $(r+1)!$, the second row by $(r+2)!$, and so on, we see that the above is equal to

$$(r+1)!(r+2)!\cdots(n-1)!\begin{vmatrix} \dfrac{y^{\langle r+1\rangle}}{(r+1)!} & \dfrac{y^{\langle r\rangle}}{r!} & \cdots & \dfrac{y^{\langle r-s+2\rangle}}{(r-s+2)!} \\ \dfrac{y^{\langle r+2\rangle}}{(r+2)!} & \dfrac{y^{\langle r+1\rangle}}{(r+1)!} & \cdots & \dfrac{y^{\langle r-s+3\rangle}}{(r-s+3)!} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{y^{\langle n-1\rangle}}{(n-1)!} & \dfrac{y^{\langle n-2\rangle}}{(n-2)!} & \cdots & \dfrac{y^{\langle n-s\rangle}}{(n-s)!} \end{vmatrix}.$$

If we reorder the columns (or the rows) of this determinant, we have the right hand side of A.1 with the signature $(-1)^{s(s-1)/2}$. Then the total signature is given by

$$\begin{aligned} &(-1)^{(2n-g)(g-1)/2}\cdot(-1)^{s(s-1)/2+r(r-1)/2} \\ &\quad= (-1)^{(2n-g)(g-1)/2}\cdot(-1)^{-s(s-1)/2+r(r-1)/2} \\ &\quad= (-1)^{(2n-g)(g-1)/2}\cdot(-1)^{(r-s)(r+s)/2-(r-s)/2} \\ &\quad= (-1)^{(2n-g)(g-1)/2}\cdot(-1)^{(r-s)(n-2)/2} \\ &\quad= (-1)^{(2n-g)(g-1)/2}\cdot\begin{cases} (-1)^{g(n-2)/2} & \text{if } n\not\equiv g \mod 2, \\ (-1)^{(g-1)(n-2)/2} & \text{if } n\equiv g \mod 2. \end{cases} \end{aligned}$$

Computing this for n and g modulo 4 yields the value of ε_n in the table in A.1, and the desired formula have been obtained. □

References

1. E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, *Geometry of algebraic curves, Vol.1, Grundlehren der mathematischen Wissenschaften 267*, Springer, Berlin, 1984.
2. H.F. Baker, *Abelian functions — Abel's theorem and the allied theory including the theory of the theta functions —*, Cambridge Univ. Press, 1897; reprint, 1995.
3. H.F. Baker, On the hyperelliptic sigma functions, *Amer. J. of Math.* **20** (1898), 301–384.
4. F. Brioschi, Sur quelques formules pour la multiplication des fonctions elliptiques, *C. R. Acad. Sci. Paris* **59** (1864), 769–775.
5. V.M. Buchstaber, V.Z. Enolskii and D.V. Leykin, Kleinian functions, hyperelliptic Jacobians and applications, *Reviews in Math. and Math. Physics* **10** (1997), 1–125.
6. V.M. Buchstaber, V.Z. Enolskii, and D.V. Leykin, Rational analogs of Abelian functions, *Functional Anal. Appl.* **33** (1999), 83–94.
7. V.M. Buchstaber, D.V. Leykin, and V.Z. Enolskii, σ-function of (n, s)-curves, *Comm. Moscow Math. Surv.* **54** (1999), 628-629.
8. D.G. Cantor, On the analogue of the division polynomials for hyperelliptic curves, *J. reine angew. Math.* **447** (1994), 91–145.
9. J.C. Eilbeck, V.Z. Enolskii and E.P. Previato, On a generalized Frobenius-Stickelberger addition formula, *Lett. in Math. Phys.* **63** (2003), 5–17.
10. J. Fay, *Theta functions on Riemann surfaces, Lecture Notes in Math., 352*, Springer-Verlag, 1973.
11. F.G. Frobenius and L. Stickelberger, Zur Theorie der elliptischen Functionen, *J. reine angew. Math.* **83** (1877), 175–179.
12. D. Grant, A generalization of a formula of Eisenstein, *Proc. London Math. Soc.* **62** (1991), 121–132.
13. R.C. Gunning, *Riemann surfaces and generalized theta functions, Ergebnisse der Mathematik und ihrer Grenzgebiete, 91*, Springer-Verlag, 1976.
14. L. Kiepert, Wirkliche Ausführung der ganzzahligen Multiplikation der elliptichen Funktionen, *J. reine angew. Math.* **76** (1873), 21–33.
15. I.G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, Oxford, 1995.
16. S. Matsutani, Elliptic and hyperelliptic solutions of discrete Painlevé I and its extensions to higher order difference equations, *Phys. Lett. A* **300** (2002), 233-242.
17. D. Mumford, *Tata lectures on theta II, Progress in Mathematics 43*, Birkhäuser, Boston, 1984.
18. Y. Ônishi, Complex multiplication formulae for hyperelliptic curves of genus three, *Tokyo J. Math.* **21** (1998), 381–431 (A list of corrections is available from the author's Web page).
19. Y. Ônishi, Determinant expressions for Abelian functions in genus two, *Glasgow Math. J.* **44** (2002), 353-364.
20. Y. Ônishi, Determinant expressions for hyperelliptic functions in genus three, *Tokyo J. Math.* **27** (2004), 299-312.

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