Current Status on the Theory of Abelian Functions as compared with That of Elliptic Functions

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Definition of plane telescopic curves ((e, q)-curve)

$\mathfrak{X}$: a compact Riemann surface.

$\infty \in \mathfrak{X}$: a fixed point.

Weierstrass non gaps at ∞

$e := \min(\mathrm{Weier}(\infty) \setminus \{0\})$.

$q := \min\{ m \in \mathrm{Weier}(\infty) \setminus \{0\} \mid \gcd(e, m) = 1 \}$ $\quad \boxed{\gcd(e, q) = 1}$

Then there are functions x and y on $\mathfrak{X}$ s.t. $\quad t$: local param. at ∞

$\mathrm{div}(x) > -e \cdot \infty$, $x = \frac{1}{t^e} + \cdots$, and $\mathrm{div}(y) > -q \cdot \infty$, $y = \frac{1}{t^q} + \cdots$

and satisfying

$$\boxed{(*)\ y^e - p_q(x) y^{e-1} - p_{2q}(x) y^{e-2} - \cdots - p_{eq-q}(x) y - p_{eq}(x) = 0,}$$

極を打ち消す様にしていくと得られる。

where $p_k(x)$ is a polynomial of x with $\deg p_k(x) \leq \left\lfloor \frac{k}{e} \right\rfloor$ and

$$\boxed{p_{eq}(x) = x^q + \cdots .}$$

If we take a curve of this type, the theory of ellipt. fcts. is generalized uniquely, in the naive sense.

Moreover (*) can be regarded as a model of $\mathfrak{X}$. (possibly has singularities.) We call it an $\boxed{(e, q)\text{-curve}}$.

Examples of (e, q)-curves $\boxed{g} = \frac{(e-1)(q-1)}{2} =$ "the genus if smooth"

$$\boxed{(*)\ y^e - p_q(x)\,y^{e-1} - p_{2q}(x)\,y^{e-2} - \cdots - p_{eq-q}(x)\,y - p_{eq}(x) = 0,} \quad \deg p_k(x) \leq \left\lfloor \frac{k}{e} \right\rfloor$$

$$\left(y \rightsquigarrow y + \frac{1}{e}\,p_q(x),\ x \rightsquigarrow x - \frac{1}{q}\mu_e \right) \qquad p_{eq}(x) = x^q + \cdots .$$

e	q	g	f(x, y)
2	3	1	$y^2 - (x^3 + \mu_4 x + \mu_6)$ $(\longleftrightarrow \wp'^2 = 4\wp^3 - \boxed{g_2}\wp - \boxed{g_3})$
2	5	2	$y^2 - (x^5 + \mu_4 x^3 + \mu_6 x^2 + \mu_8 x + \mu_{10})$
3	4	3	$y^3 - (\mu_2 x^2 + \mu_5 x + \mu_8)y - (x^4 + \mu_6 x^2 + \mu_9 x + \mu_{12})$
•	•	•	
•	•	•	
•	•	•	

Note that $\#\{\mu_j\} = 2g$ (for these cases).

We denote $\boxed{\mathcal{C}}$ the curve over $\operatorname{Spec} \mathbb{Q}[\mu]$.
with regarding $\{\mu_j\}$ indeterminates.

(Sometime we switch μ_j's to being constants in $\mathbb{C}$.)

Some advantage of plane telescopic curves (example)

[Yfan Yang] "Defining equations for modular curves."

He choose a model of $X_0(N)$ as a plane telescopic curve which may have singularities.

Then, Weil parametrizations (modular correspondences) are expressed very simply by using the coordinates of such models.

$f =$ "LHS" $-$ "RHS"

Example.

$$\begin{tikzcd} & \boxed{X_0(37):\ Y^3+7(X-37)Y^2-7X(X-37)Y = X^2(X-36)(X-37)} \arrow[dl, "\gamma_1"'] \arrow[dr, "\gamma_2"] & \\ \boxed{X_0(37)/w_{37}:\ y^2-y = x^3+x} & & \boxed{X_0(37)/h:\ y^2+y = x^3+x^2-23x-50} \end{tikzcd}$$

(genus $= 2$)

$$\gamma_1^*\left(\frac{dx}{2y+1}\right) = \frac{(X+Y)\,dX}{f_Y} \qquad \gamma_2^*\left(\frac{dx}{2y+1}\right) = \frac{(X-Y)\,dX}{f_Y}$$

$\exists ?$ reduction theory: to describe Abelian fcts. for the "upper curve" via those of "lower curves"

Examples on connection between Abelian fcts. and L-fcts.

$$\left(B_2 = \frac{1}{6}\right.$$

① Solution to the Basler(Basel) problem

$$\sum_{l\in\pi\mathbb{Z}}{}'\frac{1}{(u-l)^2}+\frac{1}{u^2}=\frac{1}{\sin^2 u}=\frac{1}{u^2}+\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}B_{2n}}{2n}\frac{u^{2n-2}}{(2n-2)!}$$

$$\rightsquigarrow \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}\ \text{etc.}$$

← Can you show this without trigonom. fcts.?

②

$$\sum_{l\in\boxed{\Lambda}}{}'\left(\frac{1}{(u-l)^2}-\frac{1}{l^2}\right)+\frac{1}{u^2}=\wp(u)=\frac{1}{u^2}+\overset{\frac{2^4E_4}{4}}{\frac{g_2}{5}}\cdot\frac{u^2}{2!}+\overset{\frac{2^6E_6}{6}=0}{\frac{6g_3}{7}}\cdot\frac{u^4}{4!}+\overset{\frac{2^8E_8}{8}}{\frac{3g_2^2}{5}}\cdot\frac{u^6}{6!}+\frac{24g_2g_3}{11}\cdot\frac{u^8}{8!}+\cdots$$

$\cdots \leftarrow \left(\text{if } \Lambda=\omega_1\mathbb{Z}[i]\right)$

$\boxed{\Lambda}$ – the lattice of periods

$$\sum_{\lambda\in\mathbb{Z}[i]}{}'\frac{1}{\lambda^{4n}}\left(=4\sum_{\lambda\in\mathbb{Z}[i]/\langle i\rangle}{}'\frac{\bar{\lambda}^{4n}}{(N\lambda)^{4n}}\right)=\boxed{\omega_1}^{4n}\cdot\frac{2^{4n}E_{4n}}{(4n)!}$$

$(\Lambda=\omega_1\mathbb{Z}[i])$

E_{4n} ← We call them Bernoulli–Hurwitz (BH) numbers.

ω_1 – the period of $\frac{dx}{2y}$ on $y^2=x^4-x$

↑ Special values of a Hecke's L-fct.

$\rightsquigarrow$ Kummer congr. $\rightsquigarrow$ p-adic Riemann ζ, p-adic Hecke L-fcts.

② For instance, let $e=2$ (i.e. $y^2 = x^{2g+1} + \mu_4 x^{2g-1} + \cdots + \mu_{4g+2}$),

and $u := u_1 := \int_{\infty}^{(x,y)} \frac{x^{g-1}}{2y}\,dx$.

generalization of B-H numbers

inverse fct.

Then $$x = x(u) = \frac{1}{u^2} + \sum_{n=1}^{\infty} \frac{C_{2n}}{2n} \cdot \frac{u^{2n-2}}{(2n-2)!} \quad \left(\text{define } \boxed{C_{2n}}\right)$$

The radius of convergence of this series is positive.

$y^2 = x^5 - 1$

Similarly, we have generalized BH-numbers for each (e, g)-curve!

They satisfy von Staudt-Clausen and Kummer congr. as well.

However, the $\boxed{\text{modulus } p^{\square}}$ for the Kummer congr. seems to become smaller as going away from CM-type.

$(p^a \to p^{\lfloor \frac{a}{2} \rfloor})$

(having larger symmetry)

(seems to be no research even for $g=1$ case.)

☆ The core of investigation on Abelian fcts is just to research on the σ-fct. attached to $\mathcal{C}$.

Definition of the σ-fct. for (2,3)-curve

No ambiguity on the sign

$$\sigma(u) = \boxed{-\frac{\omega'}{2\pi}\,\eta(\omega''/\omega')^{-3}}\,\exp\left(-\frac{1}{2}\,u\,\eta'\omega'^{-1}u\right)\underline{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(u\,|\,\omega''/\omega')}$$

unique odd theta

modular invariant

$$\left(=\sqrt{\frac{2\pi}{\omega'}}\cdot\Delta^{-\frac{1}{8}}\right)$$

$$\mathcal{C}:\ y^2 = x^3 - \frac{g_2}{4}x - \frac{g_3}{4} \qquad \underline{\left(\wp'^2 = 4\wp^3 - g_2\wp - g_3\right)}$$

$$(\omega', \omega''),\ (\eta', \eta'')$$
$$\downarrow$$
$$(d\omega' + c\omega'',\ b\omega' + a\omega'')$$
$$(d\eta' + c\eta'',\ b\eta' + a\eta'')$$

with $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$

$$\left(\tau = \omega''/\omega' \mapsto \frac{a\tau + b}{c\tau + d}\right)$$

$$H_1(\mathcal{C}, \mathbb{Z}) = \mathbb{Z}\alpha_1 + \mathbb{Z}\beta_1$$

$$\begin{bmatrix} \int_{\alpha_1}\frac{dx}{2y} & \int_{\beta_1}\frac{dx}{2y} \\ \int_{\alpha_1}\frac{x\,dx}{2y} & \int_{\beta_1}\frac{x\,dx}{2y} \end{bmatrix} = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix} =: \Omega.$$

$$\boxed{\wp(u) = -\frac{d^2}{du^2}\log\sigma(u).}$$

Note that $\sigma(u)$ is a “chunk of periods”.

Selected formulae and properties on σ- and ℘-fcts.

relation between ℘(u) and σ(u).

① $$\wp(u) = -\frac{d^2}{du^2}\log\boxed{\sigma(u)} \longleftrightarrow \sigma(u) = u\cdot\exp\left(\int_0^u\!\!\int_0^u\left(\wp(u)-\frac{1}{u^2}\right)du\,du\right)$$

② $\sigma(u)=0 \Longleftrightarrow u\in\Lambda.$

③ The addition formula.

$$c_n\cdot\frac{\sigma(u^{(1)}+\cdots+u^{(n)})\prod_{i<j}\sigma(u^{(i)}-u^{(j)})}{\prod_i \sigma(u^{(i)})^n} = \begin{vmatrix} 1 & \wp(u^{(1)}) & \wp'(u^{(1)}) & \wp''(u^{(1)}) & \cdots \\ 1 & \wp(u^{(2)}) & \wp'(u^{(2)}) & \wp''(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & \wp(u^{(n)}) & \wp'(u^{(n)}) & \wp''(u^{(n)}) & \cdots \end{vmatrix}$$

(Frobenius-Stickelberger)

where $c_n = (-1)^{\binom{n-1}{2}}1!\cdot 2!\cdots(n-1)!$ $\left(\text{For } n=2,\ \frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2}\right)$

④ $\psi_n(u) := \frac{\sigma(nu)}{\sigma(u)^{n^2}}$: n-division polynomial (a polynomial of ℘(u) if n : odd
or that × ℘′(u) if n : even)

power series expansion

⑤ $$\sigma(u) = u + \frac{g_2}{2}\cdot\frac{u^5}{5!} + 6g_3\cdot\frac{u^7}{7!} + \frac{9g_2^2}{4}\cdot\frac{u^9}{9!} + 18g_2g_3\cdot\frac{u^{11}}{11!} + \cdots \in \mathbb{Z}\left[\frac{g_2}{2}, g_3\right]\langle\langle u\rangle\rangle^{\mathrm{wt}=1}$$

translational formula

⑥ $$\sigma(u+\ell) = \sigma(u)\cdot\chi(\ell)\cdot\exp\overbrace{L\left(u+\tfrac{1}{2}\ell,\ \ell\right)}$$

$:= \left(u+\frac{1}{2}\ell\right)(\ell'\eta'+\eta''\ell'')$ for

$\ell = \ell'\omega'+\ell''\omega'' \in \Lambda$ with

$\chi(\ell)=\pm 1$

Weierstrass' infinit product

⑦ $$\sigma(u) = u\prod_{\ell\in\Lambda}{}'\left(1-\frac{u}{\ell}\right)\cdot\exp\left(\frac{u}{\ell}+\frac{u^2}{2\ell^2}\right)$$

$$\left[\int_{(\alpha_1,\beta_1)}\begin{pmatrix}1\\ x\end{pmatrix}\frac{dx}{2y}\right] =: \begin{bmatrix}\omega' & \omega''\\ \eta' & \eta''\end{bmatrix} =: \Omega$$

Analytic Construction of $\sigma(u)$.

$$\Omega = \left[\int_{\{\alpha_i\ \beta_i\}} \begin{bmatrix} \vec{\omega} \\ \vec{\eta} \end{bmatrix} \right] = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix} (2g \times 2g)$$

$$\Delta = \mathrm{disc}(\mathscr{C}) = e^{g-1} q^{e-1} \det(\boxed{T})$$

Certain $g \times g$ matrix

We define later (?)

This part counters divergence of ϑ-fct.

The generalized Legendre relation:

$${}^t\omega'' \eta' - {}^t\omega' \eta'' = 2\pi i I$$

(Riemann's theta fct.)

$$\boxed{\sigma(u) = \left(\frac{2\pi}{|\omega'|}\right)^{g/2} \Delta^{-\frac{1}{8}} \exp\left(-\frac{1}{2}\,{}^t u\, \eta' \omega'^{-1} u\right) \vartheta\begin{bmatrix} \delta'' \\ \delta' \end{bmatrix}(u \,|\, \omega'' \omega'^{-1})}$$

$u = (u_{w_g}, \cdots, u_1) \in \mathbb{C}^g$

(rationalize "coefficients")

These parts make $\sigma(u)$ modular invariant.

Comes from the Riemann constant vector.

By this $\sigma(u) = 0 \iff u \in \kappa^{-1}(\Theta^{[g-1]})$.

modulus of $\mathscr{C}$

Note that $\sigma(u)$ is a "chunk of periods".

Examples of the power series expansion of $\sigma(u)$ at the origin.

(2,5)-curve

$$\sigma(u_3, u_1) = \underbrace{u_3 - \frac{u_1^3}{3!}}_{\text{Schur polynomial part}} - 4\mu_4 \frac{u^7}{7!} - 2\mu_4 \frac{u_3 u_1^4}{4!} + 64\mu_4 \frac{u_1^9}{9!} - 8\mu_6 \frac{u_3 u_1^6}{6!} - 2\mu_6 \frac{u_3^2 u_1^3}{2!\,3!} + \cdots \in \mathbb{Z}[\mu]\langle\langle u_3, u_1 \rangle\rangle^{wt=3}$$

(3,4)-curve

$$\sigma(u_5, u_2, u_1) = \underbrace{u_5 - u_1 u_2^2 + 6 \cdot \frac{u_1^5}{5!}}_{\text{Schur polynomial part}} - 30\mu_2^2 \cdot \frac{u_1^7}{7!} + (4\mu_4 - \mu_2^2)\frac{u_1^5 u_2^2}{5!\,2!} + \cdots$$

$$\in \mathbb{Z}[\mu]\langle\langle u_5, u_2, u_1 \rangle\rangle^{wt=5}$$

They belong to $\mathbb{Z}[\mu]\langle\langle u \rangle\rangle$. (Hurwitz integral power series)
and are of homogeneous weight $\frac{(e^2-1)(q^2-1)}{24}$.

They are natural generalization of the Schur polynomials.

Any Jacobian variety is an Abelian variety.
But, any J.V. has extremely exquisite structure as compared with $\overbrace{\text{A.V.}}^{\text{general}}$.

One of the tremendous structure (the stratification). 階層構造

(A refinement of Abel-Jacobi thm.)

$$\overrightarrow{\omega} \underset{(e=2)}{=} \left(\frac{dx}{f_y}, \frac{x\,dx}{f_y}, \cdots, \frac{x^{g-1}dx}{f_y}\right)$$

$P \longmapsto (P, \infty),\ (P_1, P_2) \longmapsto (P_1, P_2, \infty) \longmapsto \cdots\cdots$

$$\begin{tikzcd}
\infty \arrow[r, phantom, "\in"] \arrow[d, mapsto] & \mathcal{C}=\mathrm{Sym}^1\mathcal{C} \arrow[r, hook] \arrow[d, "\iota"'] & \mathrm{Sym}^2\mathcal{C} \arrow[r, hook] \arrow[d, "\iota"'] & \mathrm{Sym}^3\mathcal{C} \arrow[r, hook] \arrow[d, "\iota"'] & \cdots \arrow[r, hook] & \mathrm{Sym}^{g-1}\mathcal{C} \arrow[r, hook] \arrow[d, "\iota"'] & \mathrm{Sym}^{g}\mathcal{C} \arrow[r, phantom, "\ni"] \arrow[d, "\boxed{\iota}"'] & (P_1,\cdots,P_g) \arrow[d, mapsto] \\
0 \arrow[r, phantom, "\in"] \arrow[d, equal] & \iota(\mathcal{C})=W^{[1]} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & W^{[2]} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & W^{[3]} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & \cdots \arrow[r, phantom, "\subset"] & W^{[g-1]} \arrow[r, phantom, "\subset"] \arrow[d, equal] & W^{[g]} \arrow[r, phantom, "\ni"] \arrow[d, equal] & \sum_{j=1}^{g}\int_{\infty}^{P_j}\overrightarrow{\omega} \bmod \Lambda \\
0 \arrow[r, phantom, "\in"] & \Theta^{[1]}=W^{[1]}\cup[-1]W^{[1]} \arrow[r, phantom, "\subset"] & \Theta^{[2]} \arrow[r, phantom, "\subset"] & \Theta^{[3]} \arrow[r, phantom, "\subset"] & \cdots \arrow[r, phantom, "\subset"] & \Theta^{[g-1]} \arrow[r, phantom, "\subset"] & \Theta^{[g]} \arrow[r, phantom, "="] & \mathbb{C}^g/\Lambda\ (=\mathrm{Jac}(\mathcal{C})) \\
 & \kappa^{-1}\iota(\mathcal{C})=\boxed{\widetilde{\mathcal{C}}} \arrow[uu] \arrow[d, phantom, "\cap"] & & & & & & \\
\Lambda \arrow[uu] \arrow[r, phantom, "\subset"] & \kappa^{-1}(\Theta^{[1]}) \arrow[r, phantom, "\subset"] & \kappa^{-1}(\Theta^{[2]}) \arrow[uu, "\kappa"] \arrow[r, phantom, "\subset"] & \cdots & & \cdots \arrow[r, phantom, "\subset"] & \kappa^{-1}(\Theta^{[g]}) \arrow[uu, "\kappa"] \arrow[r, phantom, "="] & \mathbb{C}^g \arrow[uu, "\boxed{\kappa}"']
\end{tikzcd}$$

$(\sigma_{\sharp} = \sigma_{\natural^1})$ $(\sigma_{\flat} = \sigma_{\natural^2})$

$\natural^n$: the def. is simple but omitted here.

$\boxed{\sigma_{\natural^n}(u)}$: a higher derivative of $\sigma(u) = \sigma(u_{w_g}, \cdots, u_1)$

(be regarded as the "σ-fct." attached to the n-th stratum!

Only for the telescopic curves! →

$$\boxed{\sigma_{\natural^{n+1}}(u^{[n]} + v) = \sigma_{\natural^n}(u^{[n]}) \cdot v_1^{w_{g-n}-g+k} + (\text{higher terms in } v_1)}$$

"peeling expansion" ↗

$u^{[n]} \in \kappa^{-1}(\Theta^{[n]} \setminus \Theta^{[n-1]}),\quad v \in \kappa^{-1}(W^{[1]})$

Table of $\natural^n$

(e, q)	g	$\sharp = \natural^1$	$\flat = \natural^2$	$\natural^3$	$\natural^4$	$\natural^5$	$\natural^6$	$\cdots$
$(2, 3)$	1							$\cdots$
$(2, 5)$	2	$\{1\}$						$\cdots$
$(2, 7)$	3	$\{3\}$	$\{1\}$					$\cdots$
$(2, 9)$	4	$\{1, 5\}$	$\{3\}$	$\{1\}$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$(3, 4)$	3	$\{2\}$	$\{1\}$					$\cdots$
$(3, 5)$	4	$\{4\}$	$\{2\}$	$\{1\}$				$\cdots$
$(3, 7)$	6	$\{1, 6\}$	$\{1, 5\}$	$\{4\}$	$\{2\}$	$\{1\}$		$\cdots$
$(3, 8)$	7	$\{4, 10\}$	$\{2, 7\}$	$\{1, 5\}$	$\{4\}$	$\{2\}$	$\{1\}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

☆ The core of investigation on Abelian fcts. is just to research on the σ-fct..

☆ <u>Characterization of the σ-fct.</u>

$$\begin{aligned} \sigma : \quad \mathbb{C}^g &\longrightarrow \mathbb{C} \\ u=(u_{w_g},\cdots,u_1) &\longmapsto \sigma(u) \end{aligned}$$

1. $\sigma(u)$ is an entire fct. on $\mathbb{C}^g$;

$\sigma_{\natural^n}(u)$ has similar properties as well

2. $\sigma(u)$ vanishes only on $\kappa^{-1}(\Theta^{[g-1]})$ with order 1;
3. $\sigma(u)|_{\mu=0} =$ "the Schur polynom. attached to $W^{[g-1]}|_{\mu=0}$";
4. $\sigma(u+\ell)=\sigma(u)\cdot\chi(\ell)\cdot\exp L(u+\frac{1}{2}\ell,\ell)$ $(\forall u\in\mathbb{C}^g, \forall \ell\in\Lambda)$;
5. Modular invariance : $\sigma(u|\Omega^\gamma)=\sigma(u|\Omega)$ for $\forall\,\gamma\in Sp(2g,\mathbb{Z})$. $\quad \Omega=\left[\int_{\{\alpha_i,\beta_i\}}\begin{bmatrix}\vec{\omega}\\ \vec{\eta}\end{bmatrix}\right]$
6. $\wp_{ij}(u)=-\dfrac{\partial^2}{\partial u_i\partial u_j}\log\sigma(u)$: periodic w.r.t. Λ, and $\in\Gamma(\mathrm{Jac}(\mathcal{C}),\mathcal{O}(2\Theta^{[g-1]}))$.

The translational relation on the n-th stratum:

$$\sigma_{\natural^n}(u+\ell)=\sigma_{\natural^n}(u)\cdot\chi(\ell)\cdot\exp L(u+\tfrac{1}{2}\ell,\ell) \quad \text{for } {}^\forall u\in\kappa^{-1}(\Theta^{[n]}),\ {}^\forall\ell\in\Lambda.$$

The zeroes of $\sigma_{\natural^n}$

$$\sigma_{\natural^{n+1}}(u)=0 \iff u\in\kappa^{-1}(\Theta^{[n]})$$

Application (Taking (3,4)-curve as an example)

$\sigma_{\natural^n}=\sigma$ if $n \geqq g$.

$=\sigma_1$ for (3,4)-curve

in the cases of $(e,e+1)$-curves $\cdots$

$$\frac{\sigma_{\natural^n}(u^{(1)}+\cdots+u^{(n)})\prod_{i<j}\sigma_{\flat}(u^{(i)}-u^{(j)})}{\prod_{i}\sigma_{\sharp}(u^{(i)})^{n}} = \pm\begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & xy(u^{(1)}) & \cdots & x^a y^b(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & xy(u^{(2)}) & \cdots & x^a y^b(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & xy(u^{(n)}) & \cdots & x^a y^b(u^{(n)}) & \cdots \end{vmatrix}$$

$=\sigma_2$ for (3,4)-curve

(ends at the n-th column)

$$\begin{tikzcd} & \wp_{ij}(u)\ (u\in\mathbb{C}^g) \\ \wp(u) \arrow[ur, squiggly] \arrow[dr, squiggly] & \\ & x(u)\ (u\in\widetilde{\mathcal{C}}) \end{tikzcd}$$

} two kinds of generalizations

(x-coordinate)

$x(u)$ is a fct. on $\widetilde{\mathcal{C}}$

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{C}^g \arrow[d, "\bmod \Lambda"] \\ \mathcal{C} \arrow[r, hook] & \mathrm{Jac}(\mathcal{C}) \end{tikzcd}$$

$(=\mathbb{C}^g/\Lambda)$

- $u, v \in \widetilde{\mathcal{C}} \Rightarrow [\ \sigma_{\flat}(u-v)=0 \iff u-v\in\Lambda\ ]$
- $\sigma_{\sharp}(u)|_{\widetilde{\mathcal{C}}} = \sigma_{\sharp}(u_5, u_2, u_1)|_{\widetilde{\mathcal{C}}} = u_1^3 + (\text{highers in } u_1)$
- $\psi_n(u) = \dfrac{\sigma(nu)}{\sigma_{\sharp}(u)^{n^2}}$ $(u\in\widetilde{\mathcal{C}})$ (analytic expression of D.G. Cantor's <u>n-div. polynom.</u>)
- $x(u) = \dfrac{1}{(u_1-\ell_1)^e} + O(\underbrace{u_1-\ell_1}_{\text{CM-case}})$ (c.f. Weierstrass' $\wp(u)$)

A question: For example, let $\mathcal{C}$ be the curve

$$y^2 = x^5 - 1.$$

Around any lattice point $\ell = (\ell_3, \ell_1) \in \Lambda$, we have

$$\begin{cases} x(u) = \dfrac{1}{(u_1 - \ell_1)^2} + O(u_1 - \ell_1) \text{ for } u \in \widetilde{\mathcal{C}}. & \text{(c.f. } \cot(u) \text{ or Weierstrass' } \wp(u)\text{)} \\[2ex] x(u) = \dfrac{1}{u^2} + \displaystyle\sum_{n=1}^{\infty} \frac{C_{2n}}{2n} \cdot \frac{u^{2n-2}}{(2n-2)!}. & \end{cases}$$

Can you find a theory which gives the following?

$$\underbrace{\sideset{^*}{'}\sum_{\lambda \in \mathbb{Z}[\zeta_5]} \frac{1}{\lambda^{10m}}}_{\text{Hope to give some nice justification for this.}} = \frac{\varpi^{10m}}{(10m)!} \cdot C_{10m}, \leftarrow \text{“}10m\text{-th BHÔ number for } \mathcal{C}\text{”}$$

where $\varpi = \displaystyle\int_0^\infty \frac{x\,dx}{2y}.$

Another addition formula (F. Klein, H.F. Baker)

For instance, on the (2, 5)-curve (genus = 2)

$$\boxed{\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp_{33}(u) - \wp_{33}(v) + \wp_{13}(u)\,\wp_{11}(v) - \wp_{13}(v)\,\wp_{11}(u)}$$

where $\wp_{ij}(u) := -\dfrac{\partial^2}{\partial u_i\,\partial u_j}\log\sigma(u), \quad u \in \mathbb{C}^2,\ v \in \mathbb{C}^2.$

This is proved by Complex Analysis of several variables.

However, can be proved by using the previous add. formula.

The biggest calculation so far, in this direction of research, is written in [EEMÔP] on the (3, 4)-curve.

Eilbeck

Enol'ski

Matsutani

Previato

$\sigma(u)$ in the Theory of Gauss-Manin connections (Buchstaber-Leykin)

(2, 3)-curve : $f(x,y)=0$, 元々は…

where $f(X,Y)=Y^2-(X^3+\mu_4X+\mu_6)$.

→ slimmest definition.

This is equipped with symplectic structure: $\omega\cdot\eta := \oint_{\infty}\left(\int_{\infty}^{P}\omega\right)\eta(P)$

Horizontal Derivation Formula

$$H^1_{dR}(\mathcal{C}/\mathbb{Q}[\mu_4,\mu_6]) :\doteq \frac{\mathbb{Q}[\mu_4,\mu_6,x,y]\frac{dx}{2y}+d\left(\mathbb{Q}[\mu_4,\mu_6,x,y]\frac{1}{2y}\right)}{d\left(\mathbb{Q}[\mu_4,\mu_6,x,y]\frac{1}{2y}\right)}$$

(1st de Rham cohomology)

$$-6f\cdot : \underset{\text{unfolding}}{\mathbb{Q}[\mu_4,\mu_6,X,Y]/(f_X,f_Y)} \to \mathbb{Q}[\mu_4,\mu_6,X,Y]/(f_X,f_Y)$$ (multiplication by $-6f(X,Y)$)

The representation matix of this map w.r.t. the basis $(1, x)$ is $\boxed{T}=\begin{bmatrix}4\mu_4 & 6\mu_6\\ 6\mu_6 & -\frac{4}{3}\mu_4^2\end{bmatrix}$. Then $\Delta=2^2\cdot3\cdot\det(T)$.

$\det(T)=0 \iff \mathrm{coker}\neq0 \iff \mathbb{Q}[\mu_4,\mu_6,X,Y]/(f,f_X,f_Y)\neq0$
$\iff 1\in(f,f_X,f_Y) \iff \exists(x_0,y_0)$ s.t. $f(x_0,y_0)=f_X(x_0,y_0)=f_Y(x_0,y_0)=0$

$$\boxed{\mathbb{L}} := \mathbb{Q}[\mu_4,\mu_6]\underbrace{\left(4\mu_4\frac{\partial}{\partial\mu_4}+6\mu_6\frac{\partial}{\partial\mu_6}\right)}_{L_0:=} + \mathbb{Q}[\mu_4,\mu_6]\underbrace{\left(6\mu_6\frac{\partial}{\partial\mu_4}-\frac{4}{3}\mu_4^2\frac{\partial}{\partial\mu_6}\right)}_{=:L_2}$$

$\frac{\partial}{\partial\mu_j}$ だけでは うまくいかない

a symplectic basis of H^1_{dR}

$\mathbb{L}\ni L$ acts H^1_{dR} naturally! $$\left(L\left(\overset{\omega_1:=}{\boxed{\frac{dx}{2y}}}\right), L\left(\overset{=:\eta_{-1}}{\boxed{\frac{x\,dx}{2y}}}\right)\right) = \left(\frac{dx}{2y},\frac{x\,dx}{2y}\right)\boxed{\Gamma^L}.$$

Then $\Gamma^L\in \mathrm{Lie}(Sp(2,\mathbb{Q}[\mu_4,\mu_6]))$ $\{=\mathrm{Lie}(SL(2,\mathbb{Q}[\mu_4,\mu_6]))\}$

Sample calculation

$-\frac{3}{2}g_3$

$\|$

$$-6f(X,Y)\cdot 1 = -6Y^2 + 6X^3 + \underline{6\mu_4 X} + 6\mu_4$$

$$= \boxed{-\frac{3}{2}g_3} + \boxed{-g_2}\,X + \underline{(-2X)}\cdot f_X \underline{-3Y}\cdot f_Y$$

$6\mu_6$ $\quad$ $4\mu_4$

$-4\mu_4$

$4\mu_4 = -g_2$

$-3X^2 = f_X + \mu_4$

$f_X = -3X^2 - \mu_4$

$$6X^3 = -2X\cdot(-3X^2)$$

$$= -2X(f_X + \mu_4)$$

$$= (-2X)f_X - 2\mu_4 X$$

$$= (-2X)f_X + \frac{g_2}{2}X$$

$f_Y = 2Y$

$$-6f(X,Y)\cdot X = -6XY^2 + 6X^4 + 6\mu_4 X^2 + 6\mu_4 X$$

$$= (-3XY)2Y + (-2X^2)(f_X + \mu_4) - 2\mu_4(f_X + \mu_4) + 6\mu_4 X$$

$$= (-3XY)f_Y + (-2X^2)\cdot f_X + \left(\frac{2}{3}\mu_4\right)(f_X + \mu_4) - 2\mu_4(\quad) + 6\mu_4 X$$

$$= \boxed{-\frac{4}{3}\mu_4^2} + \boxed{6\mu_4}\,X + \underline{\left(-2X^2 - \frac{4}{3}\mu_4\right)} f_X + \underline{(-3XY)}f_Y$$

$$\Delta = 2^2\cdot 3\cdot \det(T) = g_2^3 - 27g_3^2$$

$\mathbb{L} \underset{\text{probably “$=$”}}{\subset}$ {the vector fields on $\{\Delta=0\}$; liftable w.r.t. $\mathcal{C} \to \operatorname{Spec}\mathbb{Q}[\mu]$}

$$L \in \mathbb{L} \curvearrowright H^1_{dR}(\mathcal{C}/\mathbb{Q}[\mu])$$ ← This is capsulated every things as expected.

form of the 1st kind, of the 2nd kind

$$(L(\vec{\omega}), L(\vec{\eta})) = (\vec{\omega}, \vec{\eta})\Gamma^L$$

symplectic basis

If $\Gamma = \begin{bmatrix} -{}^t\beta & \alpha \\ -\gamma & \beta \end{bmatrix} \rightsquigarrow {}^*\Gamma := \begin{bmatrix} \alpha & {}^t\beta \\ \beta & \gamma \end{bmatrix}$

$\Gamma^{L_0} = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}$, $\Gamma^{L_2} = \begin{bmatrix} & 1 \\ \frac{\mu_4}{3} & \end{bmatrix}$

$$H^L = \frac{1}{2}\left[\frac{\partial}{\partial u}\ \ u\right] {}^*\Gamma^L \begin{bmatrix} \frac{\partial}{\partial u} \\ u \end{bmatrix} - \frac{1}{8}\underbrace{L(\log\Delta)}_{\frac{L(\Delta)}{\Delta}} = \begin{cases} \frac{1}{2}\left[\frac{\partial}{\partial u}\ \ u\right]\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} \frac{\partial}{\partial u} \\ u \end{bmatrix} - \frac{3}{2} \\ \frac{1}{2}\left[\frac{\partial}{\partial u}\ \ u\right]\begin{bmatrix} 1 & \\ & -\frac{\mu_4}{3} \end{bmatrix}\begin{bmatrix} \frac{\partial}{\partial u} \\ u \end{bmatrix} - 0 \end{cases}$$

I don't know why this operator works well.

$$\boxed{(\underset{\text{lower}}{L} - \underset{\text{higher}}{H^L})\sigma(u) = 0}$$

$\frac{1}{2}\operatorname{tr}\beta$

$$\left(4\mu_4\frac{\partial}{\partial\mu_4} + 6\mu_6\frac{\partial}{\partial\mu_6}\right) - \left(u\frac{\partial}{\partial u} + \frac{1}{2} - \frac{3}{2}\right) = \boxed{4\mu_4\frac{\partial}{\partial\mu_4} + 6\mu_4\frac{\partial}{\partial\mu_6} - u\frac{\partial}{\partial u} + 1}$$

$$\left(6\mu_6\frac{\partial}{\partial\mu_4} - \frac{4}{3}\mu_4^2\frac{\partial}{\partial\mu_6}\right) - \left(\frac{1}{2}\frac{\partial^2}{\partial u^2} - \frac{1}{6}\mu_4 u^2 + 0 - 0\right) = \boxed{6\mu_4\frac{\partial}{\partial\mu_4} - \frac{4}{3}\mu_4^2\frac{\partial}{\partial\mu_6} - \frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{6}\mu_4 u^2}$$

→ a recursion relation on the coeffs. of $\sigma(u)$

$$\left(\overbrace{4\pi i\frac{\partial}{\partial\tau}}^{L} - \overbrace{\frac{\partial^2}{\partial z^2}}^{H}\right)\vartheta\begin{bmatrix} b \\ a \end{bmatrix}(z|\tau) = 0$$

$(+4\mu_4,\ +4\mu_6)$

These results coincide with those of Weierstrass

Technically integrating $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, he got a pair of lin. PDE's of $\sigma(u)$.

The theory of Gauss-Manin connections started at Manin's work.
It is aimed to get diff. eqn's satisfied by periods (integrals of diff. forms).
However, the σ-fct. is a "chunk" of periods.

⓪ Buchstaber-Leykin (BL) gave a system of linear PDE's (heat eqn's) satisfied by only absolute constant times the σ-fct.
It also is a theory of G.-M. connections. (The theory is quite algebraic!)
(H. Saito made me direct my attention to this.)

This knows (is capsulized with) every things! (This is equipped with symplectic structure. $\omega\cdot\eta := \oint_{\infty}\left(\int_{\infty}^{P}\omega\right)\eta(P)$)

$$\boxed{H^1_{dR}(\mathcal{C}/\mathbb{Q}[\mu]) \overset{\text{Horizontal Derivation Formula}}{\underset{\bigstar}{=}} \left(\mathbb{Q}[\mu,x,y]\frac{dx}{f_y} + d\left(\mathbb{Q}[\mu,x,y]\frac{1}{f_y}\right)\right)\Big/ d\left(\mathbb{Q}[\mu,x,y]\frac{1}{f_y}\right)}$$

$$L \in \bigoplus_i \mathbb{Q}[\mu] L_i = \mathbb{L} \curvearrowright H^1_{dR}(\mathcal{C}/\mathbb{Q}[\mu]) \simeq \bigoplus_i \mathbb{Q}[\mu]\,\omega_i \oplus \bigoplus_j \mathbb{Q}[\mu]\,\eta_{-j}$$

$\left(L_i = \sum_j T_{ij}\frac{\partial}{\partial\mu_j},\ T = [T_{ij}]\right)$ Thanks to the H.D.F., this is easily computed!

the rep. matrix of $-\mathrm{eq}\cdot f(x,Y)$-plication on $\mathbb{Q}[\mu,X,Y]/(f_X, f_Y)$.

$$L(\omega_{w_g},\cdots,\omega_1;\eta_{-w_g},\cdots,\eta_{-1}) = (\omega_{w_g},\cdots,\omega_1;\eta_{-w_g},\cdots,\eta_{-1})\boxed{\Gamma^L}$$

(Lightness)
Rightness of the def. $\bigstar$ is quite profound! ← (the "Horizontal Derivation Formula", due to Kouki Sato at Meijo Univ.)

Examples of (e, q)-curves

$g = \frac{(e-1)(q-1)}{2} =$ "the genus if smooth"

e	q	$f(x, y)$
2	3	$y^2 - (x^3 - \mu_4 x - \mu_6)$
2	5	$y^2 - (x^5 - \mu_4 x^3 - \mu_6 x^2 - \mu_8 x - \mu_{10})$
3	4	$y^3 - (\mu_2 x + \mu_5)y - (x^4 - \mu_3 x^2 - \mu_6 x - \mu_{12})$

Note that $\#\{\mu_j\} = 2g$ for these. (modality $= 0$)

We denote $\boxed{\mathcal{C}}$ the curve over $\operatorname{Spec} \mathbb{Q}[\mu]$.

3	7	$\mathcal{C}_{3,7} \left\{ \begin{aligned} & y^3 - (\overbrace{\mu_2 x^4 + \mu_5 x^3 + \mu_8 x^2 + \mu_{11} x + \mu_{14}}^{5\ \mu\text{'s}})y \\ & \qquad - (x^7 - \underbrace{\mu_6 x^6 - \mu_9 x^5 - \cdots - \mu_{21}}_{6\ \mu\text{'s}}) \end{aligned} \right\}$

$\mathcal{C}_{3,7} \rightarrow \operatorname{modal}(\mathcal{C}_{3,7}) = 1$

6 μ's → total $\boxed{11\ \mu\text{'s}}$

Recent Result (Theory of "heat equations" for $\sigma(u)$)

$$\mathbb{L} \underset{\nwarrow\ \text{probably "="}}{\subset} \{\text{the vector field on } \{\Delta=0\};\ \text{liftable w.r.t. } \mathcal{C} \to \mathrm{Spec}\,\mathbb{Q}[\mu]\}$$

$$\mathbb{L} \curvearrowright H^1_{dR}(\mathcal{C}/\mathbb{Q}[\mu])$$

$$T=(T_{ij})=\begin{pmatrix}\text{the rep. matrix for}\\ \text{the } -\text{eq}\cdot f(X,Y)-\text{plication}\\ \text{on } \mathbb{Q}[\mu,X,Y]/(f_X,f_Y)\\ \text{w.r.t. certain basis}\end{pmatrix}$$

forms of 1st kind ↙ of the 2nd kind ↘

$$(L(\vec{\omega}), L(\vec{\eta})) = (\vec{\omega}, \vec{\eta})\,\Gamma^L$$

simplectic basis

$$\Delta = e^{g-1}q^{e-1}\det(T)$$

$$H^L = \frac{1}{2}\left[\frac{\partial}{\partial u_{w_g}} \cdots \frac{\partial}{\partial u_1}\ \ u_{w_g} \cdots u_1\right]{}^{*}\Gamma^L \begin{bmatrix}\frac{\partial}{\partial u_{w_g}}\\ \vdots\\ \frac{\partial}{\partial u_1}\\ u_{w_g}\\ \vdots\\ u_1\end{bmatrix} - \frac{1}{8}L(\log\Delta)$$

旧G.M. theoryと比較するとき。"$\supset$"を示すところで $\frac{\partial}{\partial \mu_i}\omega', \omega''$ を知る場面がある。

$$L_i = \sum_j T_{ij}\frac{\partial}{\partial \mu_j}$$

Theorem (Eilbeck-Gibbons-Ô-Yasuda / Ô-K. Sato-M. Shibata)

$$\{\varphi(u) \in \mathbb{Q}[\mu][[u]] \mid (L_i - H^{L_i})\varphi(u) = 0 \text{ for } \forall i\} = \mathbb{Q}\cdot\sigma(u)$$

if modal$(\mathcal{C}) = 0$ or $\mathcal{C}$ is one of certain curves.

A fancy question

$\sigma(nu)/\sigma(u)^{n^2}$

$$\boxed{\sigma(u) = e^{-\mu u - \nu \frac{u^2}{2}} \cdot u \cdot \frac{\prod_{i=0}^{1} \Gamma_1(u \mid (-1)^i \omega_1) \prod_{j=0}^{1} \Gamma_1(u \mid (-1)^j \omega_2)}{\prod_{i,j=0}^{1} \Gamma_2(u \mid (-1)^i \omega_1, (-1)^j \omega_2)}}$$

a product divided in to the 4 quadrants (象限)

$$\left(\begin{aligned} &\mu = \pm \frac{\pi\sqrt{-1}}{\omega_1} \text{ or } \pm \frac{\pi\sqrt{-1}}{\omega_2} \text{ according to } \overbrace{\operatorname{Im}(\omega_1+\omega_2),\ \operatorname{Im}(\omega_1-\omega_2)}^{\text{the signs of}} \\ &\nu = -\underbrace{\sum_{m,n \in \mathbb{Z}\times\mathbb{Z}}{}' \frac{1}{(m\omega_1+n\omega_2)^2}}_{\text{(does not diverge)}} + \frac{u}{\omega_1\omega_2} \log \frac{\omega_1+\omega_2}{\omega_1-\omega_2} \end{aligned} \right)$$

★ σ-fct. is so sturdy for any degeneration of the curve.

So, I shall propose a question: Can you extract some meaningful relations on doube sine fcts. from, for example, the addition formulae of the σ-fct.?

Conclusion

While, we have nicely generalized the basic parts of the theory of elliptic fcts. to that of Abelian fcts., there are only a few applications for Number Theory.

(Why ?)

Thank you for
your attention!

Construct a theory over $\mathbb{Z}[\mu]$ (→ application to Numb. Th.)

A1 Constructions of $\sigma(u)$. (Using a ϑ-fct. / Find a system of PDE's)
(remarkable example of Gauss-Manin connections)

A2 Addition formulae. (There are two types)

A3 Generalization of the n-division polynomials. (D.G. Cantor's)
(Analytic expression by using σ-fct.)

A4 Generalization of Bernoulli-Hurwitz numbers. (v. S.-C., Kummer)

B1 A realization of Coble's theorem on vector bundles.

B2 Explicit expression of global height on J. V. by σ-fct.

B3 Express $\sigma(u)$ as a determinant of a matrix of size $\mathbb{N} \times \mathbb{N}$.

I won't speak about these today.

* * *

Fancy Dreams:

C1 Weierstrass' infinite product expression of $\sigma_{\#}(u)$

C2 Partial fractional expansion (→ relation with the periods)

C3 Generalization of the Dedekind η-fct.

C4 Generalization of the relation between fcts. $\Gamma_2(u)$ and $\sigma(u)$.