New addition formulae for Weierstrass elliptic functions and for higher genus Abelian functions

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1. Introduction

Let $\wp(u)$ and $\sigma(u)$ be the Weierstrass functions satisfying

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3, \quad \sigma(u) = u\exp\left\{\int_0^u\int_0^u\left(\wp(u) - \frac{1}{u^2}\right)dudu\right\}.$$

Then we have ((Hermite and) Frobenius-Stickelberger, 1877)

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(v) - \wp(u) \left(= \begin{vmatrix} 1 & \wp(u) \\ 1 & \wp(v) \end{vmatrix}\right),$$

$$\frac{\sigma\left(u^{(1)} + u^{(2)} + \cdots + u^{(n)}\right)\prod_{i<j}\sigma(u^{(i)} - u^{(j)})}{\prod_{j=1}^{n}\sigma(u^{(j)})^n} = \frac{1}{\prod_j j!}\begin{vmatrix} 1 & \wp(u^{(1)}) & \wp'(u^{(1)}) & \cdots & \wp^{(n-2)}(u^{(1)}) \\ 1 & \wp(u^{(2)}) & \wp'(u^{(2)}) & \cdots & \wp^{(n-2)}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(u^{(n)}) & \wp'(u^{(n)}) & \cdots & \wp^{(n-2)}(u^{(n)}) \end{vmatrix}.$$

These formulae correspond to the canonical involution $v \mapsto -v$.

Today I will talk an extreme and elaborate generalization of these addition formulae.

2. Reformulation (1)

To step up higher genus cases smoothly, we reformulate the equalities for genus 1 case.

We start at the most general elliptic curve $\mathscr{C}: f(x,y) = 0$, where

$$f(x,y) = y^2 + (\mu_1 x + \mu_3)y - (x^3 + \mu_2 x^2 + \mu_4 x + \mu_6),$$
$$\mathrm{wt}(x) = -2, \ \mathrm{wt}(y) = -3, \ \ \mathrm{wt}(\mu_j) = -j,$$

with the point ∞ at infinity.

We define

$$\omega = \frac{dx}{f_y(x,y)} = \frac{dx}{2y + (\mu_1 x + \mu_3)}$$: the canonical differential.

Let $x(u)$ and $y(u)$ be the inverse functions defined by

$$u = \int_{\infty}^{(x(u),y(u))} \omega.$$

Then

$$x(u) = u^{-2} + \cdots, \quad y(u) = -u^{-3} + \cdots.$$

3. Reformulation (2)

The sigma function $\sigma(u)$ is defined by using the natural symplectic base of

$$H^1(\mathscr{C}, \mathbf{C}) \cong \varinjlim_n H^0(\mathscr{C}, d\mathscr{O}(n\cdot\infty))/d \varinjlim_n H^0(\mathscr{C}, \mathscr{O}(n\cdot\infty)).$$

Then

$$\sigma(u) = u + ((\tfrac{\mu_1}{2})^2 + \mu_2)\tfrac{u^3}{3!} + \cdots .$$

We define

$$\wp(u) := -\tfrac{d^2}{du^2}\log\sigma(u).$$

Then,

$$\wp(u) = x(u), \quad \wp'(u) = 2y(u) + \mu_1 x(u) + \mu_3.$$

4. The Reformulated Formula

Then we have

$$
\begin{aligned}
&\frac{\sigma(u^{(1)}+u^{(2)}+\cdots+u^{(n)})\prod_{i<j}\sigma(u^{(i)}-u^{(j)})}{\prod_j \sigma(u^{(j)})^n} \\
&= \frac{1}{\prod_j j!}\begin{vmatrix} 1 & \wp(u^{(1)}) & \wp'(u^{(1)}) & \cdots & \wp^{(n-2)}(u^{(1)}) \\ 1 & \wp(u^{(2)}) & \wp'(u^{(2)}) & \cdots & \wp^{(n-2)}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(u^{(n)}) & \wp'(u^{(n)}) & \cdots & \wp^{(n-2)}(u^{(n)}) \end{vmatrix} \\
&= \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & yx(u^{(1)}) & x^3(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & yx(u^{(2)}) & x^3(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & yx(u^{(n)}) & x^3(u^{(n)}) & \cdots \end{vmatrix}.
\end{aligned}
$$

5. Guide Function

We may extend this class of addition formulae by considering more general map

$$\varphi : \mathscr{C} \longrightarrow \mathbb{P}^1$$

which belongs to $\mathbb{Z}[\mu_1, \mu_2, \cdots, \mu_6][x(u), y(u)]$, and of homogeneous weight.
We suppose the coefficient of the lowest weight term w. r. t. $x(u)$ and $y(u)$ is 1.
For example $\varphi(u) = x(u)y(u) + \mu_2 y(u) + \mu_1\mu_4$.

Let $m \geq 2$ be the order of unique pole of φ, and u be the analytic variable of φ regarding $\mathscr{C}$ as a complex torus.

Then there exist

$$u,\ u^{\star},\ u^{\star^2}, u^{\star^3},\ \cdots,\ u^{\star^{m-1}} \quad \in \mathbb{C}$$

such that these m variables are generically different, vary continuously, and satisfy

$$\varphi(u) = \varphi(u^{\star}) = \cdots = \varphi(u^{\star^{m-1}}).$$

Moreover, we may choose them as

$$u + u^{\star} + \cdots + u^{\star^{m-1}} = 0.$$

Indeed $d(u + u^{\star} + \cdots + u^{\star^{m-1}})$ can be regarded as a holomorphic 1-form on $\mathbb{P}^1$.

6. An Example of New Addition Formula

Take the guide function $y(u)$.

Let $u = u^{(1)}$ and $v = u^{(2)}$ (two variable case). Then

Example. ([Eilbeck-England-Ô, 2014]) We have the addition formula

$$
\begin{aligned}
-\frac{\sigma(u+v)\sigma(u+v^{\star})\sigma(u+v^{\star\star})}{\sigma(u)^3\sigma(v)\sigma(v^{\star})\sigma(v^{\star\star})} &= y(v) - y(-u) \\
&= y(u) + y(v) + \mu_1 x(u) + \mu_3 \\
&= \left.\frac{f(x(u),Y) - f(x(u),W)}{Y - W}\right|_{Y=y(u),\,W=y(v)}.
\end{aligned}
$$

Remark. The RHS is given by an operation in “umbral calculus”, and obviously defined over $\mathbb{Z}[\mu] = \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$.

Remark. There is [Eilbeck-S.Matsutani-Ô, 2011] for $y^2 + \mu_3 y = x^3 + \mu_6$.

7. Second Example of New Addition Formula

Take the guide function $x^2(u)$.

Let $u = u^{(1)}$ and $v = u^{(2)}$ (two variable case). Then

Example. We have the addition formula

$$\frac{\sigma(u+v)\sigma(u+v^{\star})\sigma(u+v^{\star\star})\sigma(u+v^{\star\star\star})}{\sigma(u)^4\sigma(v)\sigma(v^{\star})\sigma(v^{\star\star})\sigma(v^{\star\star\star})} = x^2(u) - x^2(v).$$

Remark. The RHS is defined over $\mathbb{Z}[\mu] = \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$.

8. Higher Genus Curves

For coprime positive integers $q > d$, let $\mathscr{C}$ be the curve defined by $f(x,y) = 0$ with

$$f(x,y) = y^d - x^q + \sum_{i,j\,:\,dq>iq+jd} \text{(some coeff.)}\, x^i y^j, \quad (\mathrm{wt}(x) = -d, \ \ \mathrm{wt}(y) = -q)$$

adjoining unique point ∞ at infinity.

Call this (d,q)-curve. If $\mathscr{C}$ is non-singular, then its genus is given by $g = \frac{(d-1)(q-1)}{2}$.

For example, $$\begin{cases} f(x,y) = y^2 + (\mu_1 x + \mu_3)y - (x^3 + \mu_2 x^2 + \mu_4 x + \mu_6), \\ \mathrm{wt}(x) = -2, \ \mathrm{wt}(y) = -3, \ \ \mathrm{wt}(\mu_j) = -j. \end{cases}$$

$$\begin{cases} f(x,y) = y^3 + (\mu_1 x + \mu_4)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}) \\ \mathrm{wt}(x) = -3, \ \ \mathrm{wt}(y) = -4, \ \ \mathrm{wt}(\mu_j) = -j. \end{cases}$$

....

9. Weierstrass Gaps at ∞ and a Base of $\Gamma(\mathscr{C}, \Omega^1)$

Let $w_1, \cdots, w_g$ be the Weierstrass gap sequence at ∞.

For example, $(2,3)$-curve ... $w_1 = 1$.

$(3,4)$-curve ... $w_1 = 1$, $w_2 = 2$, $w_3 = 5$.

Let us fix the "natural" base $\vec{\omega} = (\omega_{w_g}, \omega_{w_{g-1}}, \cdots, \omega_{w_1})$ of $\Gamma(\mathscr{C}, \Omega^1)$.

Example. For $(3,4)$-curve

$$f(x,y) = y^3 + (\mu_1 x + \mu_4)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}) = 0,$$

we take the base $\vec{\omega}$ consists of $\omega_5 = \dfrac{dx}{f_y(x,y)}$, $\omega_2 = \dfrac{x dx}{f_y(x,y)}$, $\omega_1 = \dfrac{y dx}{f_y(x,y)}$.

Then we get the period lattice $\Lambda = \left\{ \oint \vec{\omega} \right\} \subset \mathbb{C}^g$.

10. The sigma function

We define the sigma function $\sigma(u)$ for $\mathscr{C}$ by using natural symplectic base of

$$H^1(\mathscr{C},\mathbb{C}) \cong \varinjlim_n H^0(\mathscr{C}, d\mathscr{O}(n\cdot\infty))/d\varinjlim_n H^0(\mathscr{C},\mathscr{O}(n\cdot\infty))$$

extending $\{\omega_{w_g}, \omega_{w_{g-1}}, \cdots, \omega_{w_1}\}$.

The sigma function $\sigma(u)$ is an entire function on $\mathbb{C}^g$ with g variables $u=(u_{w_g},\cdots,u_{w_1})$ which is a quite natural extension of certain Schur function.

Example. If $\mathscr{C}$ is $(3,4)$-curve,then

$$\sigma(u_5,u_2,u_1) = \Big(u_5 - u_1{u_2}^2 + \frac{1}{20}{u_1}^5\Big) + \Big(\frac{1}{12}\mu_1{u_1}^4u_2 - \frac{1}{3}\mu_1{u_2}^3\Big) + \cdots$$

There is an $\mathbb{R}$-bilinear form $L(\ ,\ ) : \mathbb{C}^g\times\mathbb{C}^g \to \mathbb{C}$, which is $\mathbb{C}$-linear on the 1st space, having the following properties:

(i) The map $(\ell,k)\mapsto L(\ell,k)-L(k,\ell)$ on $\Lambda\times\Lambda$ is $2\pi i\,\mathbb{Z}$-valued,

(ii) $\sigma(u+\ell)=\chi(\ell)\sigma(u)L(u+\frac{1}{2}\ell,\ell)$, $u\in\mathbb{C}^g$, $\ell\in\Lambda$,
with $\chi(\ell)\in\{\pm 1\}$ satisfying $\chi(\ell+k)=\chi(\ell)\chi(k)\exp\frac{1}{2}[L(\ell,k)-L(k,\ell)]$;

(iii) The set of zeroes of $u\mapsto\sigma(u)$ is exactly (pull-back of mod Λ of) the canonical image $\Theta^{[g-1]}$ of $\mathrm{Sym}^{g-1}\mathscr{C}$ w. r. t. Abel-Jacobi map, which zero set is of order 1.

11. On the Largest Stratum $\boxed{(3,4)\text{-curve}, \ g = 3}$

We define $\wp$-functions by

$$\wp_{ij}(u) := -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u), \quad \wp_{ijk}(u) := \frac{\partial}{\partial u_k} \wp_{ij}(u), \text{ etc.}$$

Then $\wp_{ij}(u) \in \Gamma\Big(\mathrm{Jac}(\mathscr{C}), \mathscr{O}(2\Theta^{[g-1]}) \Big)$, $\quad \wp_{ijk}(u) \in \Gamma\Big(\mathrm{Jac}(\mathscr{C}), \mathscr{O}(3\Theta^{[g-1]}) \Big)$, etc.

The case of the $(3,4)$-curve on the largest stratum in 2 variables.

Theorem. [J.C. Eilbeck, V.Enolskii, S.Matsutani, Y.Ô, E.Previato,2008]
For u, $v \in \mathbb{C}^3 = \kappa^{-1}(W^{[3]})$ (this notation is explained later), we have

$$\begin{aligned}
\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} &= -\wp_{55}(u) + \wp_{55}(v) - \wp_{52}(u)\wp_{21}(v) + \wp_{52}(v)\wp_{21}(u) \\
&- \wp_{51}(u)\wp_{22}(v) + \wp_{51}(v)\wp_{22}(u) - \tfrac{1}{3}\left(\wp_{11}(u)Q_{5111}(v) - \wp_{11}(v)Q_{5111}(u)\right) \\
&+ \tfrac{1}{3}\mu_1\left(\wp_{52}(u)\wp_{11}(v) - \wp_{52}(v)\wp_{11}(u)\right) + \mu_1\left(\wp_{51}(u)\wp_{21}(v) - \wp_{51}(v)\wp_{21}(u)\right) \\
&- \tfrac{1}{3}\left({\mu_1}^2 - \mu_2\right)\left(\wp_{51}(u)\wp_{11}(v) - \wp_{51}(v)\wp_{11}(u)\right) - \tfrac{1}{3}\mu_8\left(\wp_{11}(u) - \wp_{11}(v)\right),
\end{aligned}$$

where

$$Q_{5111} = \wp_{5111} - 6\wp_{51}\wp_{11}.$$

12. Differentials of the 1st kind and the Abel-Jacobi Maps

We fixed a base $\vec{\omega} = (\omega_{w_g}, \omega_{w_{g-1}}, \cdots, \omega_{w_1})$ of $\Gamma(\mathscr{C}, \Omega^1)$.

Example (revisited). For $(3,4)$-curve

$$f(x,y) = y^3 + (\mu_1 x + \mu_4)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}) = 0,$$

we took the base $\vec{\omega}$ consists of $\omega_5 = \dfrac{dx}{f_y(x,y)}$, $\omega_2 = \dfrac{xdx}{f_y(x,y)}$, $\omega_1 = \dfrac{ydx}{f_y(x,y)}$.

$\Lambda = \left\{ \oint \vec{\omega} \right\} \subset \mathbb{C}^g$ be the period lattice. We define, for each integer $k \geq 0$,

$$\iota : \mathrm{Sym}^k(\mathscr{C}) \to \mathbb{C}^g/\Lambda = \mathrm{Jac}(\mathscr{C})$$

$$(\mathrm{P}_1, \cdots, \mathrm{P}_k) \mapsto \sum_{j=1}^{k} \int_{\infty}^{\mathrm{P}_j} \vec{\omega} \bmod \Lambda.$$

We denote the mod Λ map by $\kappa : \mathbb{C}^g \to \mathbb{C}^g/\Lambda$.

We denote $W^{[k]} = \iota\big(\mathrm{Sym}^k(\mathscr{C})\big)$. Then $W^{[1]} \cong \mathscr{C}$. Let

$$\Theta^{[k]} = [-1]W^{[k]} \cup W^{[k]}.$$

13. The Stratification

Summing up, we have the following stratification:

$$
\begin{array}{ccccccccccc}
\infty & \in & \mathscr{C} = \mathrm{Sym}^1\mathscr{C} & \subset & \mathrm{Sym}^2\mathscr{C} & \subset & \cdots & \subset & \mathrm{Sym}^{g-1}\mathscr{C} & \subset & \mathrm{Sym}^g\mathscr{C} \\
\downarrow\iota & & \downarrow\iota & & \downarrow\iota & & & & \downarrow\iota & & \downarrow\iota \\
\mathbf{0} & \in & \iota(\mathscr{C}) = W^{[1]} & \subset & W^{[2]} & \subset & \cdots & \subset & W^{[g-1]} & \subset & W^{[g]} \\
\| & & \cap & & \cap & & & & \| & & \| \\
\mathbf{0} & \in & \Theta^{[1]} & \subset & \Theta^{[2]} & \subset & \cdots & \subset & \Theta^{[g-1]} & \subset & \Theta^{[g]} = \mathbf{C}^g/\Lambda \\
\uparrow\kappa & & \uparrow\kappa & & \uparrow\kappa & & & & \uparrow\kappa & & \uparrow\kappa \\
\Lambda & \subset & \kappa^{-1}(\Theta^{[1]}) & \subset & \kappa^{-1}(\Theta^{[2]}) & \subset & \cdots & \subset & \kappa^{-1}(\Theta^{[g-1]}) & \subset & \kappa^{-1}(\Theta^{[g]}) = \mathbf{C}^g.
\end{array}
$$

We note that Jacobi's theorem implies

$$\Theta^{[g-1]} = W^{[g-1]}.$$

We shall define certain function $\sigma_{\natural^k}(u)$ (higher derivative of $\sigma(u)$) on the k-th stratum in the next two slides.

14. Table of $\natural^n$

(d,p)	g	$\sharp = \natural^1$	$\flat = \natural^2$	$\natural^3$	$\natural^4$	$\natural^5$	$\natural^6$	$\natural^7$	$\natural^8$	$\cdots$
$(2,3)$	1	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(2,5)$	2	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(2,7)$	3	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(2,9)$	4	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(2,11)$	5	$\langle 3,7\rangle$	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(2,13)$	6	$\langle 1,5,9\rangle$	$\langle 3,7\rangle$	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(2,15)$	7	$\langle 3,7,11\rangle$	$\langle 1,5,9\rangle$	$\langle 3,7\rangle$	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$(3,4)$	3	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(3,5)$	4	$\langle 4\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(3,7)$	6	$\langle 1,6\rangle$	$\langle 1,5\rangle$	$\langle 4\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(3,9)$	7	$\langle 4,10\rangle$	$\langle 2,7\rangle$	$\langle 1,5\rangle$	$\langle 4\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

15. Higher Derivatives of the Sigma Function

We define, for the multi-index $\natural^n$ or for arbitrary multi-index I,

$$\sigma_I(u) = \left(\prod_{j\in I} \frac{\partial}{\partial u_j}\right)\sigma(u).$$

Examples. If $(d, q) = (3, 4)$ then $\flat = \natural^2 = \langle 1\rangle$ and $\sharp = \natural^1 = \langle 2\rangle$, and

$$\sigma_\flat(u) = \sigma_1(u) = \frac{\partial}{\partial u_1}\sigma(u_5, u_2, u_1),$$
$$\sigma_\sharp(u) = \sigma_2(u) = \frac{\partial}{\partial u_2}\sigma(u_5, u_2, u_1).$$

We define $\sigma_{\natural^0}(u) = 1$, a constant function.

16. Properties of the Satellite Sigma Functions (The most important slide!)

We call $\{\kappa^{-1}(\Theta^{[n]}) \ni u \mapsto \sigma_{\natural^n}(u) \mid 0 \leqq n \leqq g-1\}$ the <u>*satellite sigma functions*</u> for $\mathscr{C}$. They have the following very nice properties:

(i) $\sigma_{\natural^n}(u+\ell) = \chi(\ell)\sigma_{\natural^n}(u)L(u+\frac{1}{2}\ell,\ell)$, $u \in \kappa^{-1}(\Theta^{[n]})$, $\ell \in \Lambda$.

(ii) If $u \in \kappa^{-1}(W^{[n]} - W^{[n-1]})$, then the function $\kappa^{-1}(W^{[1]}) \ni v \mapsto \sigma_{\natural^{n+1}}(u+v)$ has a zero at Λ of order $w_{g-n} - g + n + 1$,

and other $g - (w_{g-n} - g + n + 1)$ zeroes elsewhere mod Λ.

Moreover, $\sigma_{\natural^{n+1}}(u+v) = \pm\sigma_{\natural^n}(u){v_1}^{w_{g-n}-g+n+1} +$ "higher terms in v_1".

The exact place of all zeroes of $v \mapsto \sigma_{\flat}(u+v) := \sigma_{\natural^2}(u+v)$ is known.

$\sigma_{\sharp}(u) := \sigma_{\natural^1}(u) = \pm {v_1}^g + \cdots$ (has only zero at Λ).

(iii) The set of zeroes of the function $\kappa^{-1}(W^{[n+1]}) \ni u \mapsto \sigma_{\natural^{n+1}}(u)$ is $\kappa^{-1}(\Theta^{[n]})$, which is of order 1.

(iv) For an index I, if $\mathbf{wt}(I) < \mathbf{wt}(\natural^n)$, then $\sigma_I(u) = 0$ on $\kappa^{-1}(\Theta^{[n]})$.

(v) If $\mathbf{wt}(I) = \mathbf{wt}(\natural^n)$, then the function $\sigma_I(u) =$ "an integer"$\cdot\,\sigma_{\natural^n}(u)$ on $\kappa^{-1}(\Theta^{[n]})$.

Proof : By certain expression of $\sigma(u)$ as the determinant of an infinite matrix (or by precise observation of power series expansions).

17. On the First Stratum, Two Variables $\boxed{(3,4)\text{-curve}, \ g=3}$

We define the functions $\kappa^{-1}(W^{[1]}) \ni u \mapsto x(u)$, $\kappa^{-1}(W^{[1]}) \ni u \mapsto y(u)$ by

$$u = (u_5, u_2, u_1) = \int_{\infty}^{(x(u),y(u))} \vec{\omega}.$$

Let us take $x(u)$ be the guide function. For a variable $v \in \kappa^{-1}(W^{[1]})$, let $\{v, v', v''\}$ be a complete representative modulo Λ of the inverse image of the map $v \mapsto x(v)$ such that v' and v'' vary continuously with respect to v and $v' = v'' = 0$ when $v = 0$.

Of course, $y(u)$, $y(u')$, $y(u'')$ are the three roots of the equation $f(x(v), Y) = 0$.

Lemma. [Ô, 2011] Then, for u, $v \in \kappa^{-1}(W^{[1]})$, we have

$$\frac{\sigma_\flat(u+v)\,\sigma_\flat(u+v')\,\sigma_\flat(u+v'')}{\sigma_\sharp(u)^3\,\sigma_\sharp(v)\,\sigma_\sharp(v')\,\sigma_\sharp(v'')} = \begin{vmatrix} 1 & x(u) \\ 1 & x(v) \end{vmatrix}^2.$$

Here we recall that

$$\sigma_\flat(u) = \sigma_{\natural^2}(u) = \sigma_2(u) = \tfrac{\partial}{\partial u_2}\sigma(u), \quad \sigma_\sharp(u) = \sigma_{\natural^1}(u) = \sigma_1(u) = \tfrac{\partial}{\partial u_1}\sigma(u).$$

18. On the First Stratum, n Variables $\boxed{(3,4)\text{-curve}, g=3}$

Theorem. [Ô, 2011] In n-variable case (Here $n \geq g$ for simplicity):

$$\frac{\sigma(u^{(1)}+\cdots+u^{(n)})\prod_{i<j}\sigma_1(u^{(i)}+u^{(j)\prime})\sigma_1(u^{(i)}+u^{(j)\prime\prime})}{\sigma_2(u)^{2n-2j+1}\sigma_2(u^{(j)\prime})^{j-1}\sigma_2(u^{(j)\prime\prime})^{j-1}}$$

$$= \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & yx(u^{(1)}) & y^2(u^{(1)}) & x^3(u^{(1)}) & yx^2(u^{(1)}) & y^2x(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & yx(u^{(2)}) & y^2(u^{(2)}) & x^3(u^{(2)}) & yx^2(u^{(2)}) & y^2x(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & yx(u^{(n)}) & y^2(u^{(n)}) & x^3(u^{(n)}) & yx^2(u^{(n)}) & y^2x(u^{(n)}) & \cdots \end{vmatrix}$$

$$\cdot \begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & \cdots & x^{n-1}(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & \cdots & x^{n-1}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^{n-1}(u^{(n)}) \end{vmatrix}.$$

19. Using Guide Function y, on the 1st Stratum $\boxed{(3,4)\text{-curve}, \ g = 3}$

Theorem. [EEÔ, 2014]

On the $\mathbf{1}$st stratum in $\mathbf{2}$-variables with guide function y (order 4), we have

$$\frac{\sigma_{1}(u+v)\sigma_{1}(u+v^{\star})\sigma_{1}(u+v^{\star\star})\sigma_{1}(u+v^{\star\star\star})}{\sigma_{2}(u)^4\sigma_{2}(v)\sigma_{2}(v^{\star})\sigma_{2}(v^{\star\star})\sigma_{2}(v^{\star\star\star})}$$

$$= y(u)^2 + y(u)y(v) + y(v)^2 + \big(\mu_1 x(u) + \mu_4\big)\big(y(u) + y(v)\big) + \mu_2 x(u)^2 + \mu_5\, x(u) + \mu_8$$

$$= \frac{f(x(u),Y) - f(x(u),W)}{Y - W}\Big|_{Y=y(u),W=y(v)} = \big(y(v) - y(u')\big)\big(y(v) - y(u'')\big).$$

Remark. Of course, $y(u) = y(u^{\star}) = y(u^{\star\star}) = y(u^{\star\star\star})$,

$y(u') = y(u^{\star\prime}) = y(u^{\star\star\prime}) = y(u^{\star\star\star\prime})$,

$y(u'') = y(u^{\star\prime\prime}) = y(u^{\star\star\prime\prime}) = y(u^{\star\star\star\prime\prime})$.

20. Using Guide Function y, on the 1st Stratum for (d,q)-curve

For the most general (d,q)-curve $(g{=}\frac{(d-1)(q-1)}{2})$

$$f(x,y) = (y^d + \cdots) - (x^q + \mu_d x^{q-1} + \cdots + \mu_{dq}) = 0,$$

we have

Theorem. [EEÔ, 2014]

On the 1st stratum in 2-variables with guide function y (order 4), we have

$$\frac{\sigma_\flat(u+v)\sigma_\flat(u+v^\star)\cdots\sigma_\flat(u+v^{\star^{q-1}})}{\sigma_\sharp(u)^q\,\sigma_\sharp(v)\,\sigma_\sharp(v^\star)\cdots\sigma_\sharp(v^{\star^{q-1}})} = \frac{f(x(u),Y)-f(x(u),W)}{Y-W}\Big|_{Y=y(u),W=y(v)}$$

$$= \big(y(v)-y(u')\big)\big(y(v)-y(u'')\big)\cdots\big(y(v)-y(u'^{d-1})\big).$$

Remark. As in the previous slide,

$$y(u) = y(u^\star) = \cdots = y(u^{\star^{q-1}}),$$
$$y(u') = y(u^{\star\prime}) = \cdots = y(u^{\star^{q-1}\prime}),$$
$$y(u'') = y(u^{\star\prime\prime}) = \cdots = y(u^{\star^{q-1}\prime\prime}),$$
$$\cdots\cdots\cdots.$$

Keys of the proof. For a fixed $u \in \kappa^{-1}(\Theta^{[1]})$, the map $v \mapsto \sigma_\flat(u+v)$ has a zero at $v=0$, u', $\cdots$, u'^{d-1} modulo Λ of order 1, and the map $u \mapsto \sigma_\sharp(u)$ has only zero at $u=0$ modulo Λ of order g, and no zeroes elsewhere.

21. Summary and Some Questions

For each curve $\mathscr{C}$, and for each setting of

(1) k $\cdots$ the stratum : the 1st stratum, by using $x(u)$ and $y(u)$;
the largest stratum, by using $\wp$-functions,

(2) n $\cdots$ the number of variables,

(3) φ $\cdots$ the guide function,

we have an addition formula of F-S type.

Some Questions:

Q1 Is there further natural generalization?

Q2 Why the coefficients of RHS belong to $\mathbb{Z}[\mu]$?
(It is obvious they belong to $\mathbb{Q}[\mu]$.)
(If the order of the guide function is small, Q2 is OK because the RHS is a determinant, etc.)

Q3 How do these formulae link with other existing mathematical world?
Or some applications?

Q4 Can the general RHS be regarded as a sort of higher generalization of the concept of "determinant"?

Bibliography

Please check

http://www2.meijo-u.ac.jp/~yonishi/

Thank you very much.

23a. Definition of $\natural^n$ (in order to define $\sigma_{\natural^n}(u)$)

For the given pair (d, q) of positive integers with $d < q$ and $\gcd(d, q) = 1$, we define multi-indices $\natural^n$ consist of numbers in $\{1, 2, \cdots, g\}$, where $g = \frac{(d-1)(q-1)}{2}$, as follows:

We explain by an example : $(d, q) = (3, 7)$, $g = 6$.

Write a $g \times g = 6 \times 6$ table as follows.

We first write the Weierstrass gap sequence with respect to (d, q) on the last column:

					11
					8
					5
					4
					2
					1

23a. Definition of $\natural^n$ (in order to define $\sigma_{\natural^n}(u)$)

For the given pair (d,q) of positive integers with $d < q$ and $\gcd(d,q) = 1$, we define multi-indices $\natural^n$ consist of numbers in $\{1,2,\cdots,g\}$, where $g = \frac{(d-1)(q-1)}{2}$, as follows:

We explain by an example : $(d,q) = (3,7)$, $g = 6$.

Write a $g \times g = 6 \times 6$ table as follows.

We first write the Weierstrass gap sequence with respect to (d,q) on the last column:

6	7	8	9	10	11
3	4	5	6	7	8
0	1	2	3	4	5
	0	1	2	3	4
			0	1	2
				0	1

Then, put into other boxes naturally increasing non-negative integers as above.

23b. Definition of $\natural^2$ for $(d,q) = (3,7)$, $g = 6$. (continuation)

If we wish to get $\natural^n = \natural^2$, extract $(g-n) \times (g-n) = 4 \times 4$ minor on the lower right corner. and Remove all rows and columns including $\mathbf{0}$.

6	7	8	9	10	11
3	4	5	6	7	8
0	1	2	3	4	5
	0	1	2	3	4
			0	1	2
				0	1

$\rightarrow$

2	3	4	5
1	2	3	4
	0	1	2
		0	1

$\rightarrow$

2	5
1	4

23b. Definition of $\natural^2$ for $(d,q)=(3,7)$, $g=6$. (continuation)

If we wish to get $\natural^n=\natural^2$, extract $(g-n)\times(g-n)=4\times 4$ minor on the lower right corner. and Remove all rows and columns including $\mathbf{0}$.

6	7	8	9	10	11
3	4	5	6	7	8
0	1	2	3	4	5
	0	1	2	3	4
			0	1	2
				0	1

$\rightarrow$

2	3	4	5
1	2	3	4
	0	1	2
		0	1

$\rightarrow$

2	$\boxed{5}$
$\boxed{1}$	4

Finally, by reading the numbers on the off-diagonal, we have

$$\natural^2=\langle 1,5\rangle$$