On a Generalization of the Theory of Heat Equations for the Weierstrass Sigma Function to Higher Genus Case

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Theory of Heat Equations for a Sigma Function

Main References

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Classical theory of elliptic functions (of Weierstraß)

Let Λ be a lattice in the complex plane $\mathbb{C}$.

$$\wp(u) = \frac{1}{u^2} + \sum_{u \in \Lambda - \{0\}} \left(\frac{1}{(u-\ell)^2} - \frac{1}{\ell^2} \right),$$

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\,\wp(u) - g_3,$$

where

$$g_2 = 60 \sum_{\ell \in \Lambda - \{0\}} \frac{1}{\ell^4}, \quad g_3 = 140 \sum_{\ell \in \Lambda - \{0\}} \frac{1}{\ell^6},$$

$$\zeta(u) = \frac{1}{u} - \int_0^u \left(\wp(u) - \frac{1}{u^2} \right) du,$$

$$\sigma(u) = u \exp\left(\int_0^u \int_0^u \left(\frac{1}{u^2} - \wp(u) \right) du\, du \right) = u \prod_{\ell \in \Lambda - \{0\}} \left(1 - \frac{u}{\ell} \right) e^{\frac{u}{\ell} + \frac{u^2}{2\ell^2}}.$$

Modern approach to elliptic functions (1)

In this talk we do not follow the classical theory in the previous slide.
We shall start from the elliptic curve :

$$\mathscr{C} \; : \; y^2 = x^3 + \mu_4 x + \mu_6.$$

The sigma function $\sigma(u)$ associate to this curve is

$$\sigma(u) = \Big(\frac{2\pi}{\omega'}\Big)^{1/2} \varDelta^{-\frac{1}{8}} \, \exp\Big(-\tfrac{1}{2}{\omega'}^{-1}\eta' u^2\Big)\cdot \vartheta\Big[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\Big]({\omega'}^{-1}u, \omega''/\omega'),$$

where

$$\varDelta = -16(4{\mu_4}^3 + 27{\mu_6}^2) = \text{the discriminant},$$

$$\begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix} = \begin{bmatrix} \int_{\alpha_1}\omega & \int_{\beta_1}\omega \\ \int_{\alpha_1}\eta & \int_{\beta_1}\eta \end{bmatrix} \quad \text{with} \quad \omega = \frac{dx}{2y}\,, \quad \eta = \frac{xdx}{2y}$$

and (α_1, β_1) is a symplectic basis of $H_1(\mathscr{C}, \mathbb{Z})$, and

$$\vartheta\Big[\begin{smallmatrix} b \\ a \end{smallmatrix}\Big](z,\tau) = \sum_{n\in\mathbb{Z}} \exp 2\pi \boldsymbol{i}\Big(\tfrac{1}{2}\tau(n+b)^2 + (n+b)(z+a)\Big) \quad (a, \ b \in \mathbb{R})$$

is Jacobi's theta series.

Modern approach to elliptic functions (2)

The sigma function $\sigma(u)$ is given by

$$\sigma(u) = \Big(\frac{2\pi}{\omega'}\Big)^{1/2} \Delta^{-\frac{1}{8}} \exp\Big(-\tfrac{1}{2}\omega'^{-1}\eta' u^2\Big) \cdot \vartheta\Big[\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\Big](\omega^{-1}u, \omega''/\omega').$$

Then

$$\zeta(u) = -\frac{\partial}{\partial u}\log\sigma(u), \quad \wp(u) = -\frac{\partial^2}{\partial^2 u}\log\sigma(u),$$

$$\wp'(u)^2 = 4\,\wp(u)^3 + \mu_4\,\wp(u) + \mu_6, \quad -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma(u)^2\,\sigma(v)^2} = \wp(u) - \wp(v).$$

The formula of Frobenius-Stickelberger : for n variables $u^{(1)}, \cdots, u^{(n)}$, we have

$$(-1)^{(n-1)(n-2)/2}(1!2!\cdots(n-1)!)\,\frac{\sigma(u^{(1)}+\cdots+u^{(n)})\,\prod_{i<j}\sigma(u^{(i)}-u^{(j)})}{\sigma(u^{(1)})^n\cdots\sigma(u^{(n)})^n}$$

$$= \begin{vmatrix} 1 & \wp(u^{(1)}) & \wp'(u^{(1)}) & \cdots & \wp^{(n-2)}(u^{(1)}) \\ 1 & \wp(u^{(2)}) & \wp'(u^{(2)}) & \cdots & \wp^{(n-2)}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(u^{(n)}) & \wp'(u^{(n)}) & \cdots & \wp^{(n-2)}(u^{(n)}) \end{vmatrix}.$$

On the first factors of definition of $\sigma(u)$

—. It is not clear that $\sigma(u)$ is independent of the choice of α_1 and β_1.
(This is obvious if we adapt the classical definition.)

—. However, eventually the changes of the Two factors cancel!

—. Using the Dedekind eta function

$$\boldsymbol{\eta}(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

the first part in

$$\sigma(u) = \left(\frac{2\pi}{\omega'}\right)^{1/2} \Delta^{-\frac{1}{8}} \exp\left(-\frac{1}{2}\omega'^{-1}\eta' u^2\right) \cdot \vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\omega^{-1}u, \omega''/\omega')$$

is explicitly written as

$$\left(\frac{2\pi}{\omega'}\right)^{1/2} \Delta^{-\frac{1}{8}} = -\frac{\omega'}{2\pi} \boldsymbol{\eta}(\omega''/\omega')^{-3}.$$

Why do we particularly investigate sigma functions?

Because :

(1) it is a fundamental functions as $\wp_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u_1, \cdots, u_g)$;

(2) it behaves nicer than theta functions in algebraic sence (which less depends on base ring);

(3) it is useful for analytic calculation more than tau functions;

(4) addition formula, (Hermite-)Frobenius-Stickelberger formulae

(5) n -plication formula

(6) it is a natural generalization of the Schur polynomials.

Why do we investigate power series expansions of the sigma functions?

Because:

—. Weierstrass investigated; (negative reason ...)

—. it has application to n-plication formula by using $\sigma(nu)/\sigma(u)^{n^2}$;

—. the coefficients of the power series expansions of sigmas are polynomial of the coefficient of the defining equation of the curve, and has Hurwitz integrality which permits to define p-adic sigma functions

The function $\sigma(u)$ has a power series expansion at the origin as follows:

$$\sigma(u) = u \sum_{n_4,\,n_6 \geqq 0} b(n_4, n_6) \frac{(\mu_4 u^4)^{n_4} (\mu_6 u^6)^{n_6}}{(1+4n_4+6n_6)!}$$

$$= u + 2\mu_4 \frac{u^5}{5!} + 24\mu_6 \frac{u^7}{7!} - 36\mu_4^2 \frac{u^9}{9!} - 288\mu_4\mu_6 \frac{u^{11}}{11!} + \cdots,$$

where $b(n_4, n_6) \in \mathbb{Z}$.

(*This expansion also shows the independence of* $\sigma(u)$ *with respect to the choice of* α_1 *and* β_1).

In general, for $\mathscr{C} : y^2 + (\mu_1 x + \mu_3)y = x^3 + \mu_2 x^2 + \mu_4 + \mu_6$, we have

$$\sigma(u) = u + ((\tfrac{\mu_1}{2})^2 + \mu_2)\frac{u^3}{3!} + ((\tfrac{\mu_1}{2})^4 + 2\mu_2(\tfrac{\mu_1}{2})^2 + \mu_3\mu_1 + {\mu_2}^2 + 2\mu_4)\frac{u^5}{5!}$$

$$+ ((\tfrac{\mu_1}{2})^6 + 3\mu_2(\tfrac{\mu_1}{2})^4 + 6\mu_3(\tfrac{\mu_1}{2})^3 + 3{\mu_2}^2(\tfrac{\mu_1}{2})^2 + 6\mu_4(\tfrac{\mu_1}{2})^2$$

$$+ 6\mu_3\mu_2\tfrac{\mu_1}{2} + {\mu_2}^3 + 6\mu_4\mu_2 + 6{\mu_3}^2 + 24\mu_6)\frac{u^7}{7!} + \cdots.$$

Characterization of σ in genus 1 case

$\Lambda := \left\{ \oint \omega \right\}$, 但し, $\omega = \dfrac{dx}{2y + \mu_1 x + \mu_3}$.

For $u \in \mathbb{C}$, we define $u', u'' \in \mathbb{R}$ by $u = u'\omega' + u''\omega''$.

For $\ell \in \Lambda$, we define $\ell', \ell'' \in \mathbb{Z}$ by $\ell = \ell'\omega' + \ell''\omega''$.

$L(u, v) := u(v'\eta' + v''\eta'')$, $\chi(\ell) := \exp \pi \boldsymbol{i}(\ell' + \ell'' + \ell'\ell'')$.

Proposition (Characterization of σ of genus 1)

The sigma function for $\mathscr{C}$ is characterize by the following 5 properties :

(S1) $\sigma(u)$ is an entire function on $\mathbb{C}$;

(S2) $\sigma(u + \ell) = \chi(\ell)\,\sigma(u) \exp L(u + \frac{1}{2}\ell, \ell)$ for any $u \in \mathbb{C}$ and $\ell \in \Lambda$;

(S3) $\sigma(u)$ is expanded as a power series around the origin with coefficients in $\mathbb{Q}[\boldsymbol{\mu}]$ of homogeneous weight $1 \left(= \frac{(2^2-1)(3^2-1)}{24} \right)$; ただし, $\mathrm{wt}(\mu_j) = -j$, $\mathrm{wt}(u) = 1$;

(S4) $\sigma(u)|_{\boldsymbol{\mu}=\mathbf{0}} = u$ (the Schur polynomial $s_{2,3}(u) = u$ for genus one) ;

(S5) $\sigma(u) = 0 \iff u \in \Lambda$.

These properties might be independent each other.

Heat equation finds $\Delta^{-\frac{1}{8}}$

Analytic construction of the sigma in genus 1 case:

$$\sigma(u) = -\boldsymbol{\eta}(\omega'^{-1}\omega'')^{-3} \cdot \frac{\omega'}{2\pi} \cdot \exp\left(-\tfrac{1}{2}u^2\eta'\omega'^{-1}\right)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\omega'^{-1}u|\omega'^{-1}\omega'').$$

It is expected that the sigma function $\sigma(u)$ corresponding to a plane telescoplic curve (which we mention later) $\mathscr{C}$ of genus g is a function of g variables is constructed by

$$\hat{\sigma}(u) = \hat{\sigma}(u,\Omega) = \Delta^{-\frac{1}{8}}\left(\frac{(2\pi)^g}{|\omega'|}\right)^{\frac{1}{2}}\exp\left(-\tfrac{1}{2}\,{}^t u\,\eta'\omega'^{-1}u\right)$$

$$\cdot\sum_{n\in\mathbb{Z}^g}\exp\left(\tfrac{1}{2}\,{}^t(n+\delta'')\,\omega'^{-1}\omega''(n+\delta'') + {}^t(n+\delta'')(\omega'^{-1}u+\delta')\right)\left(=\Delta^{-\frac{1}{8}}\,\widetilde{\sigma}(u),\ \text{say}\right).$$

Here, Δ is the discriminant of $\mathscr{C}$ i.e. $\Delta \in \mathbb{Z}[\boldsymbol{\mu}]$ which is irreducible and "$\Delta \neq 0 \iff \mathscr{C}$ is smooth".
Theta characteristic $\begin{bmatrix}\delta''\\ \delta'\end{bmatrix} \in \left(\tfrac{1}{2}\mathbb{Z}\right)^{2g}$ is determined by the Riemann constant of $\mathscr{C}$.

Fact : $\sigma(u)$ is $\widetilde{\sigma}(u)$ times a function of μ_j.

Let $\hat{\sigma}(u) = \Delta^{-\frac{1}{8}}\,\widetilde{\sigma}(u)$.

Conj. : $\sigma(u) = \hat{\sigma}(u)$.

There is a proof for genus 2 case (by D.Grant using Thomae's formula)).

Theorem (Buchstaber-Leykin + EGÔY)

If the (plane telescopic) curve is of genus 3 or smaller, we have up to non-zero multiplicative absolute constant that

$$\sigma(u) = \hat{\sigma}(u).$$

One glance on Abelian functions of higher genus (H.F. Baker)

For simlicity we avoid $y^2+(\mu_1x^2+\mu_3x+\mu_5)y=x^5+\mu_2x^4+\mu_4x^3+\mu_6x^2+\mu_8x+\mu_{10}$ and consider

$$\mathscr{C}\ :\ y^2=x^5+\mu_2x^4+\mu_4x^3+\mu_6x^2+\mu_8x+\mu_{10}.$$

Then

$$\omega_1=\frac{dx}{2y},\ \ \omega_2=\frac{xdx}{2y},\ \ \eta_1=-\frac{(3x^3+2\mu_2x^2+\mu_4x)dx}{2y},\ \ \eta_2=-\frac{x^2dx}{2y}.$$

We denote $\boldsymbol{\omega}=(\,\omega_1\ \ \omega_2\ \ \eta_1\ \ \eta_2\,)$.

$$\omega'=\Big[\int_{\alpha_i}\omega_j\Big],\quad \omega''=\Big[\int_{\beta_i}\omega_j\Big],\quad \eta'=\Big[\int_{\alpha_i}\eta_j\Big],\quad \eta''=\Big[\int_{\beta_i}\eta_j\Big],$$

$$\sigma(u)=\sigma(u_3,u_1)=\Big(\frac{2\pi}{\omega'}\Big)^{1/2}\Delta^{-\frac{1}{8}}\ \exp\Big(-\tfrac{1}{2}\omega'^{-1}\eta' u^2\Big)\cdot\vartheta\begin{bmatrix}1/2\\1/2\\1\\1/2\end{bmatrix}(\omega^{-1}u,\omega''\omega'^{-1}).$$

$$\wp_{ij}(u)=-\frac{\partial^2}{\partial u_i\partial u_j}\log\sigma(u),\qquad \wp_{33}(u),\quad \wp_{31}(u),\quad \wp_{11}(u),$$

$$-\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma(u)^2\,\sigma(v)^2}=\wp_{33}(u)-\wp_{33}(v)+\wp_{31}(u)\wp_{11}(v)-\wp_{31}(v)\wp_{11}(u).$$

The classical heat equation

Let z and τ $(\operatorname{Im}\tau > 0)$ are complex variables. For the theta series

$$\vartheta\begin{bmatrix} b \\ a \end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}} \exp 2\pi \boldsymbol{i}\left(\tfrac{1}{2}\tau(n+b)^2 + (n+b)(z+a)\right) \quad (\, a,\ b \in \mathbb{R}\,)$$

and operators $L = 4\pi\boldsymbol{i}\dfrac{\partial}{\partial\tau}$, $H = \dfrac{\partial^2}{\partial z^2}$, we have a Heat Equation

$$(L - H)\,\vartheta\begin{bmatrix} b \\ a \end{bmatrix}(z,\tau) = 0,$$

which is already holds for individula terms.

Weierstrass work on σ

Theoretically it suffice to rewrite the heat equation above into a diff. eq. in terms of g_2, g_3 and u.
However, it is not easy task.
Starting from the diff. eq. of $\wp(u)$, he derived a recurion equation of the coefficients of the power series expansion of $\sigma(u)$ around $u = 0$.
(Considering the definition of $\wp(u)$, it looks that some information of $\sigma(u)$ might be dropped.
But, by regarding g_2 and g_3 as variables ...)

Buchstaber-Leykin's work on σ

Significantly reformulates and generalises Weierstrass work to multivariate σ.
For the purpose, they investigate the tangent space of the variety $\Delta = 0$.
Firstly, they gave a heat equation of the top exponetial function of the sigma.
Then proceeded to a heat equation for the sigma function.

The recursion in the $(2,3)$-case (Weierstrass' work)

[Weierstrass 1882] Regarding $\wp'^2 = 4\wp^3 + 4\mu_4\wp + 4\mu_6$ as a differential equation for $\sigma(u)$, by highly technical feat he got the heat eq. of $\sigma(u)$:

$$(L_0 - H_0)\sigma(u) = \left(4\mu_4\frac{\partial}{\partial\mu_4} + 6\mu_6\frac{\partial}{\partial\mu_6} - u\frac{\partial}{\partial u} + 1\right)\sigma(u) = 0,$$

$$(L_2 - H_2)\sigma(u) = \left(6\mu_6\frac{\partial}{\partial\mu_4} - \frac{4}{3}{\mu_4}^2\frac{\partial}{\partial\mu_6} - \frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{6}\mu_4u^2\right)\sigma(u) = 0.$$

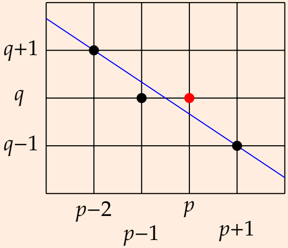
From the 1st formula, we see the sigma is of the form:

$$\sigma(u) = u\sum_{n_4,n_6\geqq 0} b(n_4,n_6)\,\frac{u\,(\mu_4u^4)^{n_4}(\mu_6u^6)^{n_6}}{(1+4n_4+6n_6)!}.$$

From the 2nd formula (by rewirting $p = n_4,\ q = n_6$),

$$\begin{aligned} b(p,q) = {} & \tfrac{2}{3}(4p+6q-1)(2p+3q-1)\,b(p-1,q) \\ & -\tfrac{8}{3}(q+1)\,b(p-2,q+1) \\ & +12(p+1)\,b(p+1,q-1), \end{aligned}$$

$b(p,q) = 0$ if $p < 0$ or $q < 0$.



From this, we get

$$\sigma(u) = u + 2\mu_4\frac{u^5}{5!} + 24\mu_6\frac{u^7}{7!} - 36\mu_4^2\frac{u^9}{9!} - 288\mu_4\mu_6\frac{u^{11}}{11!} + \cdots.$$

Work of Frobenius-Stickelberger

Work of Frobenius-Stickelberger (published in the same year as Weierstrass' paper).
This is anothor observation of Weierstrass' heat equations. In the paper, they investigated

$$-4\mu_4 = g_2 = 60 \sum_{n',n''}{}' \frac{1}{(n'\omega' + n''\omega'')^4}, \quad -4\mu_6 = g_3 = 140 \sum_{n',n''}{}' \frac{1}{(n'\omega' + n''\omega'')^6},$$

$$\wp(u) = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + \frac{{g_2}^2}{1200}u^6 + \cdots, \quad \zeta(u + n'\omega' + n''\omega'') = \zeta(u) + n'\eta' + n''\eta'',$$

and got the following :

$$\begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix} \begin{bmatrix} \frac{\partial g_2}{\partial \omega'} & \frac{\partial g_3}{\partial \omega'} \\ \frac{\partial g_2}{\partial \omega''} & \frac{\partial g_3}{\partial \omega''} \end{bmatrix} = \begin{bmatrix} -4g_2 & -6g_3 \\ -6g_3 & -\frac{1}{3}{g_2}^2 \end{bmatrix}.$$

Multiplying $\begin{bmatrix} \frac{\partial}{\partial g_2} \\ \frac{\partial}{\partial g_3} \end{bmatrix}$, we have

$$\omega' \frac{\partial}{\partial \omega'} + \omega'' \frac{\partial}{\partial \omega''} = -4g_2 \frac{\partial}{\partial g_2} - 6g_3 \frac{\partial}{\partial g_3} \quad \text{(former part of } L_0 - H_0\text{)},$$

$$\eta' \frac{\partial}{\partial \omega'} + \eta'' \frac{\partial}{\partial \omega''} = -6g_3 \frac{\partial}{\partial g_2} - \frac{1}{3}{g_2}^2 \frac{\partial}{\partial g_3} \quad \text{(former part of } L_2 - H_2\text{)}.$$

Analytic Side = Algebraic Side

Moreover, $\left(\omega' \frac{\partial}{\partial \omega'} + \omega'' \frac{\partial}{\partial \omega''}\right)(a\eta' + b\eta'') = a\eta' + b\eta''$, $\left(\eta' \frac{\partial}{\partial \omega'} + \eta'' \frac{\partial}{\partial \omega''}\right)(a\eta' + b\eta'') = -\frac{1}{12}g_2\,(a\omega' + b\omega'')$.

—. We want to get similar fourmulae for $g > 1$. But there are no longer such Eisenstein seres!

—. Here we note that the RHSs of the operators are tangent to ${g_2}^3 - 27{g_3}^2$.

Overview of the Work

(1) Get the primary heat equation.

(2) Hypothesise $\sigma(u) = \hat{\sigma}(u) := \Delta^{-\frac{1}{8}} \cdot \widetilde{\sigma}$ (clue is in genus one case).

(3) Check the tangent space of $\Delta = 0$.

(4) Find a basis $L_0 = L_{v_1}$, ... , $L_{v_{2g}}$ (if modality is 0) using a method known in the *singularity theory* (but number theorists may not know) :

(5) Using the primary heat equation, we get algebraic heat operators :

$$L_{v_j} - H^{L_{v_j}} - (L_{v_j} \log \Delta).$$

(6) Solve alg. heat equations for our higher genus cases $(2,3)$, $\cdots$, $(3,4)$,

and show that the solution space is of dimension 1. This implies the standard solution is no other than $\sigma(u)$:

$$\sigma(u) = \hat{\sigma}(u)$$

We express the whole theory in one breath : Weierstrass' heat equation

$$\left(4\mu_4 \frac{\partial}{\partial \mu_4} + 6\mu_6 \frac{\partial}{\partial \mu_6} - u\frac{\partial}{\partial u} - \frac{1}{2} + \frac{3}{2}\right)\sigma(u) = 0,$$

$$\left(6\mu_6 \frac{\partial}{\partial \mu_4} - \frac{4}{3}{\mu_4}^2 \frac{\partial}{\partial \mu_6} - \frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{6}\mu_4 u^2 + 0 + 0\right)\sigma(u) = 0$$

is generalized to $\left(L_j - H^{L_j} + \frac{1}{8}L_j(\log \Delta)\right)\sigma(u) = 0$, where ($j$ runs certain $2g$ integers in $1, \cdots,$ $4g-2$.

L_j are Δ tangent vectors of Δ, and H^{L_j} is determined by the action of L_js on $H^1(\mathscr{C}, \mathbb{Q}[\mu])$.

Plane Telescopic Curves

We will treat the curves of the following type: Let e and q be a fixed pair of positive integers such that $e < q$, $\gcd(e, q) = 1$. For indeterminates x and y, we let

$$f(x, y) = y^e + p_1(x)y^{e-1} + \cdots + p_{e-1}(x)y - p_e(x).$$

Here $p_j(x)$ is a polynomial of x of degree not exceeding $\left\lceil \frac{jq}{e} \right\rceil$. We write this as

$$p_j(x) = \sum_{k\,:\,jq-ek>0} \mu_{jq-ek}\, x^k \quad (1 \le j \le e-1), \quad p_e(x) = x^q + \mu_{e(q-1)}x^{q-1} + \cdots + \mu_{eq}.$$

The base ring can be taken quite arbitrary, However, for simplicity, we assume μ_i are generic values in $\mathbb{C}$.

Definition

We denote by $\mathscr{C}$ the (non-singular) curve defined by $f(x, y) = 0$ which is added unique point ∞ at infinity, which is called (e, q)-curve, or plane telescopic curve.

Examples :

$(2, 3)$-curve : $f = y^2 + (\mu_1 x + \mu_3)y - (x^3 + \mu_2 x^2 + \mu_4 + \mu_6)$.

$(2, 5)$-curve : $f = y^2 + (\mu_1 x^2 + \mu_3 x + \mu_5)y - (x^5 + \mu_2 x^4 + \mu_4 x^3 + \mu_6 x^2 + \mu_8 x + \mu_{10})$.

$(2, 7)$-curve : $f = y^2 + (\mu_1 x^3 + \mu_3 x^2 + \mu_5 x + \mu_7)y - (x^7 + \mu_2 x^6 + \mu_4 x^5 + \mu_6 x^4 + \mu_8 x^3 + \mu_{10} x^2 + \mu_{12} x + \mu_{14})$.

$(3, 4)$-curve : $f = y^3 + (\mu_1 x + \mu_4)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12})$.

We introduce a weight defined by

$$\mathrm{wt}(\mu_j) = -j, \quad \mathrm{wt}(x) = e, \quad \mathrm{wt}(y) = q.$$

Then all the formulae and expressions are of homogeneous weight, e.g. f is of homog. wt. eq.

Discriminant of the Curve

A discriminant Δ of the curve $\mathscr{C}$ is defined by the property

$\Delta \in \mathbb{Z}[\boldsymbol{\mu}]$ is irreducible, and "$\Delta \neq 0 \iff \mathscr{C}$ is smooth".

Conjecture

Let the coefficients μ_j of the defining equation $f(x,y) = 0$ be indeterminates, and define

$$R_1 = \mathrm{rslt}_x\Big(\mathrm{rslt}_y\Big(f(x,y), \tfrac{\partial}{\partial x}f(x,y)\Big), \mathrm{rslt}_y\Big(f(x,y), \tfrac{\partial}{\partial y}f(x,y)\Big)\Big),$$

$$R_2 = \mathrm{rslt}_y\Big(\mathrm{rslt}_x\Big(f(x,y), \tfrac{\partial}{\partial x}f(x,y)\Big), \mathrm{rslt}_x\Big(f(x,y), \tfrac{\partial}{\partial y}f(x,y)\Big)\Big),$$

$$R = \gcd(R_1, R_2) \quad \text{in } \mathbb{Z}[\boldsymbol{\mu}].$$

Here rslt_z is the Sylvester resultant with respect to z. This R might be a square : $R = {\Delta'}^2$. Then we restore μ_j to their original values. It is quite plausible that $\Delta' = \pm\Delta$.

—. It is so hard to compute Δ naively following this definition.

—. If $(e,q) = (2, 2g+1)$, Δ is no other than a discriminant of a polynomial in one variable.

—. If $e \geq 3$, we require another method.

—. I will explain later a method which works if modality is 0.

—. It is checked at least for modality 0 case that

$$\mathrm{wt}(\Delta) = -eq(e-1)(q-1).$$

We write $g = \dfrac{(e-1)(q-1)}{2}$, which is the genus of $\mathscr{C}$ if this is a non-singular curve.

Characterization for the Sigma Functions

We shall characterize the sigma function as follows.

Proposition (F.Klein, H.F.Baker, ... , Nakayashiki)

Assume the coefficients $\{\mu_j\}$ are complex numbers, and $\Delta \neq 0$.
There is unique function $\sigma(u)$ satisfying the following :

(S1) $\sigma(u)$ is an entire function on $\mathbb{C}^g$;

(S2) $\sigma(u+\ell) = \chi(\ell)\,\sigma(u)\exp L(u+\frac{1}{2}\ell,\ell)$ for any $u \in \mathbb{C}^g$ and any $\ell \in \Lambda$;

(S3) $\sigma(u)$ is expanded around the origin as a power series with coefficients in $\mathbb{Q}[\boldsymbol{\mu}]$ and is homoge. weight of $(e^2-1)(q^2-1)/24$;

(S4) The top part $\sigma(u)\,|_{\boldsymbol{\mu}=\mathbf{0}}$ is the Schur plynomial $s_{e,q}(u)$;

(S5) $\sigma(u) = 0 \iff u \in \kappa^{-1}(\Theta)$, and the order of zeroes along Θ is 1.

Here $[\,\delta''\ \delta'\,] \in \left(\frac{1}{2}\mathbb{Z}\right)^{2g}$ gives the Riemann constant vector of $\mathscr{C}$,

$$\chi(\ell) := \exp\Big(2\pi\boldsymbol{i}\big({}^t\ell'\delta'' + {}^t\ell''\delta' + \tfrac{1}{2}{}^t\ell'\ell''\big)\Big), \quad L(u,v) := {}^tu(v'\eta' + v''\eta''),$$

where η' and η'' are period matices for η_js of a symplectic basis $\omega_1,\cdots,\omega_g,\eta_1,\cdots,\eta_g$ of $H^1(\mathscr{C},\mathbb{Q}[\boldsymbol{\mu}])$,

$$\boldsymbol{\omega} := (\omega_1,\ \cdots,\omega_g)\ \Big(\text{e.g. if } (e,q)=(2,5), \text{ then } \omega_1 = \frac{dx}{f_y},\ \omega_2 = \frac{xdx}{f_y}\Big),\ \Lambda := \left\{\oint\boldsymbol{\omega}\right\} \subset \mathbb{C}^g,$$

$\Theta :=$ Abel-Jacobi image of $\mathrm{Sym}^{g-1}(\mathscr{C})$ with the base ∞, $\kappa : \mathbb{C}^g \longrightarrow \mathrm{Jac}(\mathscr{C})(\mathbb{C}) = \mathbb{C}^g/\Lambda$,

$$s_{2,3} = u_1,\ s_{2,5} = u_3 - 2\frac{{u_1}^3}{3!},\quad s_{2,7} = u_1u_5 - 2\frac{{u_3}^2}{2!} - 2\frac{{u_1}^3u_3}{3!} + 16\frac{{u_1}^6}{6!},\quad s_{3,4} = u_5 - u_1{u_2}^2 + 6\frac{{u_1}^5}{5!},\ \cdots.$$

The zeroes of $\sigma(u)$ and a Schur polynomial

We explain for $(e, q) = (2, 5)$ $(g = 2)$ what is the Schur polynomial.
Let $t = x^2/y$ which is a local parameter around ∞.
Note that the zeroes of $\sigma(u)$ is the pul-back of $\Theta := \mathrm{Sym}^{g-1}(\mathscr{C})$ with respect to Abel-Jacobi map w. r. t. $\kappa := \mathrm{mod}\,\Lambda$.
So that in our case the coordinates $(u_3, u_1) \in \mathbb{C}^2$ is expressed by using the value t of the local parameter on $\mathscr{C}$

$$u_3 = \int_\infty^{(x,y)} \frac{1}{f_y} dx = \int_0^t (t^2 + \cdots) dt = \tfrac{1}{3} t^3 + \text{“higher terms”},$$

$$u_1 = \int_\infty^{(x,y)} \frac{x}{f_y} dx = \int_0^t (1 + \cdots) dt = t + \text{“higher terms”}.$$

Namely, the expansion of $\sigma(u)$ should be

$$u_3 - \frac{1}{3} {u_1}^3 + \text{“higher terms”}.$$

This top part is the Schur polynomial $s_{2,5}$,
which is the Schur polynom. attached the semi-group generated by 2 and $5 \in \mathbb{Z}$.

What was known without using heat equations?

Forgetting $\Delta^{-\frac{1}{8}}$, we define

$$\widetilde{\sigma}(u) = \widetilde{\sigma}(u,\Omega) = \Big(\frac{(2\pi)^g}{\det\omega'}\Big)^{\frac{1}{2}} \cancel{\Delta^{-\frac{1}{8}}} \exp\Big(-\tfrac{1}{2}\,{}^t u\,\eta'\omega'^{-1}u\Big)$$

$$\cdot \sum_{n\in\mathbb{Z}^g} \exp\Big(\tfrac{1}{2}\,{}^t(n+\delta'')\,\omega'^{-1}\omega''(n+\delta'') + {}^t(n+\delta'')(\omega'^{-1}u+\delta')\Big).$$

Then $\widetilde{\sigma}(u)$ satisfies (S1) $\sim$ (S5) except (S4) (the top part is certain Schur plynomial).

This is roughly known by H.F.Baker ,..., Leykin, and completely proved by Nakayashiki for general plain telescoplic curves curves.

Moreover (S4) is satisfied up to non-zero multiplicative constant which may contain μ_js.
Namely, we need only to fix such multiplicative constant which is a function of μ_js.

To fix it, the discriminant Δ of $\mathscr{C}$ takes very important rolet.

The Primary Heat Equation (1)

We proceed to the case of a general curve $\mathscr{C}$. We shall fix "correct"L later.
Here, for arbitrary chosen linear operator (vector field) $L \in \bigoplus_j \mathbb{Q}[\boldsymbol{\mu}]\frac{\partial}{\partial \mu_j}$, which is regraded to correspond Algebraic Side, we want to find the corresponding operator in Analytic Side.
However, it is sufficient to know only $L(\Omega)$.

By a lemma due to Chevalley, thhe operator L acts

$$H^1(\mathscr{C}, \mathbb{Q}[\boldsymbol{\mu}]) = \varinjlim_k \Gamma(\mathscr{C}, d\mathscr{O}(k\cdot\infty))/d\Gamma(\mathscr{C}, \mathscr{O}(k\cdot\infty)) \simeq \frac{\text{“The forms of the 2nd kind with only pole at } \infty\text{”}}{\text{“The exact forms in that space”}},$$

we have $\Gamma = \Gamma^L$ such that $L({}^t\boldsymbol{\omega}) = \Gamma\,{}^t\boldsymbol{\omega}$.
Integrating this along various roop paths, we have $L(\Omega) = \Gamma\Omega$, and operating L to the Legendre relation

$${}^t\,\Omega J \Omega = 2\pi i\, J, \quad \text{where} \quad J = \begin{bmatrix} & 1_g \\ -1_g & \end{bmatrix},$$

we get ${}^t\Gamma J + J\Gamma = 0$. This means that $K := \Gamma J$ is a symmetric matrix. Denoting

$$\Omega = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}, \quad K = \begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix} \quad (\ \alpha,\ \gamma \ \text{ is symmetric }),$$

we have

$$\Gamma = KJ = K\begin{bmatrix} & 1_g \\ -1_g & \end{bmatrix} = \begin{bmatrix} \beta & -\alpha \\ \gamma & -{}^t\beta \end{bmatrix},$$

$$\Gamma\Omega = \begin{bmatrix} \beta\omega' - \alpha\eta' & \beta\omega'' - \alpha\eta'' \\ \gamma\omega' - {}^t\beta\eta' & \gamma\omega'' - {}^t\beta\eta'' \end{bmatrix} = L(\Omega), \quad (\text{all the entries of } \alpha,\ \beta,\ \gamma \text{ are } \in \mathbb{Q}[\boldsymbol{\mu}]).$$

The Primary Heat Equation (2) (作用素 L and H^L)

従つて,（作用素 L は Ω の成分を変数に関する微分作用素として書き下せてはゐないが,）$L(\Omega)$ の求め方はわかつた. つまり, Analytic Side $\leftsquigarrow$ Algebraic Side である.

いま Weierstrass の得た微分作用素

$$L_0 - H^{L_0} = 4\mu_4 \frac{\partial}{\partial \mu_4} + 6\mu_6 \frac{\partial}{\partial \mu_6} - u\frac{\partial}{\partial u} + 1,$$

$$L_2 - H^{L_2} = 6\mu_6 \frac{\partial}{\partial \mu_4} - \frac{4}{3}{\mu_4}^2 \frac{\partial}{\partial \mu_6} - \frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{6}\mu_4 u^2$$

の前半 L_0 と L_2 （青色）を L として採れば,

$$\Gamma^{L_0} = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \quad \Gamma^{L_2} = \begin{bmatrix} & 1 \\ \frac{\mu_4}{3} & \end{bmatrix}.$$

後半の H^{L_0} と H^{L_2} の正しい一般化は以下の通り. 先に L から得られた対称行列

$$K = \begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix} = \Gamma^L J \quad (\alpha,\ \gamma \text{ は対称行列}), \quad \left(\Gamma^L = \begin{bmatrix} \beta & -\alpha \\ \gamma & -{}^t\beta \end{bmatrix} \right)$$

に対して, 作用素 H^L を

$$H^L = \tfrac{1}{2}\left[\ {}^t\tfrac{\partial}{\partial \boldsymbol{u}} \quad {}^t u\ \right] K \begin{bmatrix} \frac{\partial}{\partial \boldsymbol{u}} \\ u \end{bmatrix} + \tfrac{1}{2}\,\mathrm{Tr}(\beta) \;=\; \frac{1}{2}\sum_{i=1}^{g}\sum_{j=1}^{g}\left(\alpha_{ij}\,\frac{\partial^2}{\partial u_i \partial u_j} + 2\beta_{ij}\,u_i\frac{\partial}{\partial u_j} + \gamma_{ij}\,u_i u_j\right) + \tfrac{1}{2}\,\mathrm{Tr}(\beta)$$

で定義する（[BL]）.

The Primary Heat Equation (3)

Theta 級数と sigma 函数を調整する因子

$$G_0(u,\Omega) = \Big(\frac{(2\pi)^g}{\det\omega'}\Big)^{\frac{1}{2}} \exp\Big(-\tfrac{1}{2}\,{}^t u\,\eta'\omega'^{-1}u\Big)$$

について

$$(L - H^L)\,G_0(u,\Omega) = 0$$

が成り立つ.

Proof. もちろん, $L(\Omega) = \Gamma\Omega$ を使つて check する. □

The Primary Heat Equation (4)

これに characteristic $b = {}^t[b'\ b'']$ の theta の項を掛けた

$$G(b,u,\Omega) = \left(\frac{(2\pi)^g}{\det(\omega')}\right)^{\frac{1}{2}} \exp\Big(-\tfrac{1}{2}{}^t u\eta'\omega'^{-1}u\Big)\cdot\exp\Big(2\pi i\Big(\tfrac{1}{2}{}^t b''\omega'^{-1}\omega'' b'' + {}^t b''(\omega'^{-1}u + b')\Big)\Big)$$

を考へる. これは $\sigma(u)$ の定義式の第 n 項で ${}^t[n+\delta''\ \delta'] = b = {}^t[b''\ b']$ としたものに他ならない.
これについても次の定理が成り立つ.
これが以降の理論の基礎なので primary heat equation と呼ぶことにする.
([BL] の Thm.13 の修正版)

Theorem (Primary heat equation)

For the function $G(b,u,\Omega)$ above, one has

$$(L - H^L)\,G(b,u,\Omega) = 0.$$

これは自明ではなく, かなりの計算を要する.
ここでも, もちろん $L(\Omega) = \Gamma\Omega$ を使つて計算する.

The Primary Heat Equation (5)

(*Proof of the Primary Heat Equation*)

最初は $L\Big(G(b,u,\Omega)\Big)$ と $H^L\Big(G(b,u,\Omega)\Big)$ を straight forward に計算して証明してゐた.

J. Gibbons 氏がある程度, 見通しのよい証明をこしらへたが, 成り立つ理由を明快にできないまま.

$\dfrac{L\Big(G(b,u,\Omega)\Big)}{G(b,u,\Omega)}$ と $\dfrac{H^L\Big(G(b,u,\Omega)\Big)}{G(b,u,\Omega)}$ を計算すれば, どちらも

$$\frac{1}{2}{}^t u\, {}^t\omega'^{-1}\, {}^t\eta' \alpha \eta' \omega'^{-1} u - 2\pi \boldsymbol{i}\, {}^t b'' \omega'^{-1} \alpha \eta' \omega'^{-1} u - 2\pi^2\, {}^t b''\, {}^t\omega'^{-1} \alpha\, {}^t\omega'^{-1} b'' - \frac{1}{2} \sum_{i,j} \alpha_{ij} (\eta' \omega'^{-1})_{ij}$$
$$+ 2\pi \boldsymbol{i}\, {}^t b'' \omega'^{-1} \beta u - {}^t u \omega'^{-1} \eta' \beta u + \frac{1}{2} \mathrm{tr}\beta + \frac{1}{2}{}^t u \gamma u$$

となる. (QED)

The Primary Heat Equation (6)

これより, 直ちに

$$\widetilde{\sigma}(u) := \left(\frac{(2\pi)^g}{|\omega'|}\right)^{\frac{1}{2}} \cancel{\Delta^{-\frac{1}{8}}} \exp\left(-\tfrac{1}{2}\,{}^t u\,\eta'\omega'^{-1}u\right)\cdot\vartheta\begin{bmatrix}\delta''\\ \delta'\end{bmatrix}(\omega^{-1}u\,|\,\omega''/\omega')$$

についても

$$(L - H^L)\,\widetilde{\sigma}(u) = 0$$

が成り立つことがわかる.

我々の考察する場合については，結果的に $\widetilde{\sigma}(u) \notin \mathbb{Q}[\mu][[u]])$ であることがわかるから，どんな $L \in \bigoplus_j \mathbb{Q}[\mu]\dfrac{\partial}{\partial \mu_j}$ に対しても, 方程式

$$(L - H^L)\,\varphi(u) = 0, \quad \varphi(u) \in \mathbb{Q}[\mu][[u]]$$

は非自明な解を持たない.

$\sigma(u)$ に一致すると予想される $\hat{\sigma}(u)$ を消す作用素

$$L - H^L \ + \text{“ a constant ”} \in \mathbb{Q}[\boldsymbol{\mu}]\left[\tfrac{\partial}{\partial u_i}, \tfrac{\partial}{\partial \mu_j}\right]$$

を与える L は, Δ を積分多様体に持つものでなくてはならない. （これを後で実行する. ）

Outline of Proof of the Main Result

(1) $\hat{\sigma}(u) = \Delta^{-\frac{1}{8}}\,\widetilde{\sigma}(u)$ を消す operators を探す.
それには, 特異点論の結果を援用し
Δ を極大積分多様体に持つ様な space of vector fields
を構成する. (→ Algebraic Side および "a constant " のところの決定)

(2) [BL] の方法によつて, 一般に $\Delta^{-\frac{1}{8}}\,\widetilde{\sigma}(u)$ を解に含む熱方程式系を構成できるが,
$(e, q) = (2, 3), (2, 5), (2, 7), (3, 4)$
の場合には, それらから得られる展開係数の関係式が一意的に解けることを示すことにより,
解が, ある（重さ $(e^2-1)(q^2-1)/24$ で斉重な）級数の絶対定数倍に限られるこがわかる.

(3) しかるに $\sigma(u)$ と $\Delta^{-\frac{1}{8}}\,\widetilde{\sigma}(u)$ は定数倍（μ_j を含み得る）しか違はないのであるから,
解には所望の $\sigma(u)$ の定数（$\{\mu_j\}$ の函数）が含まれることはわかつてゐる.

(4) 従つて, 得られた級数が $\sigma(u)$ でなければならない.

Modality, Weierstrass Form

(e,q)-curve の方程式 $f(x,y)=0$ に対して $\mathbb{Q}[\boldsymbol{\mu}]$ 上の Tschirnhaus 変換で x^{q-1} と y^{e-1} の項を落した形のものを Weierstrass form と呼ぶ.
Weierstrass form に変換したのち μ_j の名前を元の付け方に戻したものを考へる.
例へば $(e,q)=(3,4)$ ならば

$$f = y^3 \xcancel{+(\mu_1 x+\mu_4)y^2} + (\mu_2 x^2+\mu_5 x+\mu_8)y - (x^4 \xcancel{+\mu_3 x^3} + \mu_6 x^2 + \mu_9 x + \mu_{12}).$$

その結果残る μ_j の個数は $2g$ 以下になるが, この差を modality と呼ぶ.
Modality は

$$(e-3)(q-3)+\left\lfloor \frac{q}{e} \right\rfloor - 1$$

で与へられる.　特に

$$\text{modality} = 0 \iff (e,q) = (2,2g+1),\ \ (3,4),\ \ (3,5).$$

The Operator Space which is tangent to the Discriminant Variety (1)

一般の代数曲線の判別式をどうやつて計算するか.

Modality が 0 の曲線のみ考察する.

$(2,5)$-case を例にし, discriminant と tangent になる operators L の構成を説明する.

$\mathbb{Q}[\mu]$ 上の加群 $\mathbb{Q}[\mu][x,y]/(f_x, f_y)$ （modality 0 ゆゑ階数は $2g$）から自身への $-eqf(x,y)$ 倍写像を考へて，基底 $\{\, x^3,\ x^2,\ x,\ 1\,\}$ に関する表現行列を $T = [\,T_{i,j}\,]$ （$2g$ 次正方行列）とおくと

$$\det(T) = \text{“non-zero rational”} \cdot \Delta.$$

しかし, この T より, 以下の様に簡単に計算される対称行列 V の方がはるかに有用である.

$$H := \frac{1}{2}\begin{vmatrix} \dfrac{f_1(x,y)-f_1(z,w)}{x-z} & \dfrac{f_2(x,y)-f_2(z,w)}{x-z} \\ \dfrac{f_1(z,y)-f_1(x,w)}{y-w} & \dfrac{f_2(z,y)-f_2(x,w)}{y-w} \end{vmatrix}, \quad f_1(x,y) = \frac{\partial}{\partial x} f(x,y), \quad f_2(x,y) = \frac{\partial}{\partial y} f(x,y),$$

$$M = [\ x^3\ \ x^2\ \ x\ \ 1\ ] \quad (4 = 2g)$$

とおき，$\mathbb{Q}\left[x,y,z,w\right]\big/\Big(f_1(x,y),\, f_2(x,y),\, f_1(z,w),\, f_2(z,w)\Big)$ の中で

$${}^tM\,V\,M = f(x,y)\,H$$

で $V = [V_{i,j}] \in \mathrm{Mat}\,(2g, \mathbb{Q}[\mu])$ を定義する.

このとき V は T に簡単な左上三角行列を掛けたものになることが容易にわかり,

$$\det(V) = \det(T) = \text{“non-zero rational”} \cdot \Delta$$

となる.

The Operator Space which is tangent to the Discriminant Variety (2)

さうして

$$L_{v_i} = \sum_{j=1}^{2g} V_{i,j} \frac{\partial}{\partial \mu_{eq-v_j}}$$

とおく. ここに

$$j \mapsto v_j$$

は $\mathrm{wt}(L_{v_j}) = v_j$ となる様に仕組んだ函数.
これらの L_{v_j} が丁度 $\Delta = 0$ を極大積分多様体とする operators の空間を張る.
（齋藤恭司氏の理論）

例へば $(e,q) = (2,3)$ の場合は

$$V = \begin{bmatrix} 4\mu_4 & 6\mu_6 \\ 6\mu_6 & -\frac{4}{3}{\mu_4}^2 \end{bmatrix}$$

であり, $\det(V) = \frac{4}{3}\Delta$ である. また $v_1 = 0$, $v_2 = 2$ であつて, L_0 と L_2 は

$$L_0 = 4\mu_4 \frac{\partial}{\partial \mu_4} + 6\mu_6 \frac{\partial}{\partial \mu_6},$$

$$L_2 = 6\mu_6 \frac{\partial}{\partial \mu_4} - \frac{4}{3}{\mu_4}^2 \frac{\partial}{\partial \mu_6}$$

と定義される.

Algebraic Heat Operator L and H^L

予備的考察

L は $\{\mu_j\}$ に関する導分で，H^L は $\{u_j\}$ に関する導分.
いま $\varXi$ を $\{\mu_j\}$ のみの（$\{u_j\}$ に依存しない）函数とすれば，

$$L\,(\varXi\,\widetilde{\sigma}(u)) = (L\varXi)\,\widetilde{\sigma}(u) + \varXi(L\,\widetilde{\sigma}(u)), \quad H^L\,\varXi\,\widetilde{\sigma}(u) = \varXi(H^L\,\widetilde{\sigma}(u)).$$

従つて，$\varXi\,\widetilde{\sigma}(u)$ について

$$(L - H^L)(\varXi\,\widetilde{\sigma}(u)) = \frac{L\varXi}{\varXi}\,\varXi\,\widetilde{\sigma}(u) = (L\log\varXi)\,\varXi\,\widetilde{\sigma}(u),$$

つまり $\big(L - H^L - L\,(\log\,\varXi)\big)\big(\varXi\,\widetilde{\sigma}(u)\big) = 0.$

もし $\varXi\,\widetilde{\sigma}(u)$ が求める $\sigma(u) \in \mathbb{Q}[\mu][[u]]$ であるならば，$L\,(\log\,\varXi)\,\varXi\,\widetilde{\sigma}(u) \in \mathbb{Q}[\mu][[u]]$ である.
ゆゑに，$\sigma(u) = \hat{\sigma}(u)$ であると思ふなら，$L(\log\varDelta) \in \mathbb{Q}[\mu]$ なる L を探すべきである.

帰 結

以上より，

$L(\log\varDelta) \in \mathbb{Q}[\mu]$ なる $L \in \bigoplus_j \dfrac{\partial}{\partial\mu_j}$ について

$$\Big(L - H^L - L(\log\varDelta^{-\frac{1}{8}})\Big)\,\hat{\sigma}(u) = 0.$$

従つて，L として 前 page の L_{v_j} 達を採ればよいであらう.

Values $L_{v_j}\log(\Delta) = L_{v_j}(\Delta)/\Delta$

Let $M(x,y)\,\omega_1$ be the canonical basis of $H^1(\mathscr{C},\mathbb{Q}[\boldsymbol{\mu}])$.
For example, if $\mathscr{C}$ is the $(2,2g+1)$-curve, we have

$$M(x,y) = [\,1 \;\; x \;\; \cdots \;\; x^{2g-1}\,], \qquad \omega_1 = \frac{1}{2y}\,dx.$$

Then we have (proved by S. Yasuda)

$$M(x,y)\,{}^t[\,L_{v_1} \;\; \cdots \;\; L_{v_{2g}}\,](\Delta) = \mathrm{Hess}\,f(x,y)\cdot\Delta.$$

(これはかなり有用な公式！ しかしあまり知られてゐない？ 齋藤恭司氏の理論にある ?)

Proof. If $e=2, f(x,y)$ is of the form $y^2-p(x)$.
Compute $\dfrac{\mathrm{Hess}\,f}{f} = -\dfrac{p''(x)}{p(x)}$ in the localized ring $\big(F[x]/(p'(x))\big)_{p(x)}$ of $\big(F[x]/(p'(x))\big)$ w.r.t. the multiplicative set $\{1,\,p(x),\,p(x)^2,\,\cdots\}$.
For other cases, this was checked directly. □

この式から, 明らかに

$$L_{v_j}(\log\Delta) = \frac{L(\Delta)}{\Delta} \in \mathbb{Q}[\mu].$$

Algebraic Heat Equations

Proposition (ここまでのまとめ)

(1) 行列 $V = [V_{i,j}]$ は対称行列で, $\det(V) = \varDelta$ となる.
(2) $L_{v_j}(\log \varDelta) \in \mathbb{Q}[\boldsymbol{\mu}]$ であり, $\{L_{v_j}\}$ は $\{\varDelta = 0\}$ の tangent space を張る.
(3) $\{\varDelta = 0\}$ は $\{L_{v_j}\}$ の唯 1 つの極大積分多様体である.

ある補題（Chevalley, 局所径数を交換しても完全微分の差しか生じない）から L_{v_j} は

$$H^1(\mathscr{C}, \mathbb{Q}[\boldsymbol{\mu}]) = \varinjlim_k \Gamma(\mathscr{C}, d\mathscr{O}(k\cdot\infty))/d\Gamma(\mathscr{C}, \mathscr{O}(k\cdot\infty)) \simeq \frac{\text{“The forms of the 2nd kind with only pole at } \infty\text{”}}{\text{“The exact forms in that space”}}$$

に作用する. そこで

$$L_{v_j}({}^t\boldsymbol{\omega}) = \Gamma_{v_j}{}^t\boldsymbol{\omega}$$

によつて Γ_{v_j} を定めると, これの積分をすることで周期の関係式が得られる :

$$L_{v_j}(\varOmega) = \Gamma_{v_j}\varOmega.$$

周期行列 $\varOmega$ についての Legendre の関係式から ΓJ は対称行列である. Γ_{v_j} から $H^{L_{v_j}}$ を作れば,

(4) $\left(L_{v_j} - H^{L_{v_j}} - L_{v_j}(\log \varDelta^{-\frac{1}{8}})\right) \varDelta^{-\frac{1}{8}}\ \widetilde{\sigma}(u) = 0.$

One Dimesionality of the Heat Equations (Main Result)

Theorem (E-G-Ô-Y)

級数 $\varphi(u) \in \mathbb{Q}[\mu][[u]]$ $(u = (u_{w_g}, \cdots, u_{w_1}))$ についての方程式系

$$\left(L_{v_j} - H^{L_{v_j}} - L_{v_j}(\log \Delta^{-\frac{1}{8}})\right) \varphi(u) = 0 \quad (j = 0, \cdots, 2g)$$

の解空間は, 種数 3 以下の場合

$$(e, q) = (2,3), \ (2,5), \ (2,7), \ (3,4)$$

のすべてにおいて 1 次元, つまり $\hat{\sigma}(u) = \Delta^{-\frac{1}{8}} \widetilde{\sigma}(u)$ の絶対定数倍の全体である.

Proof.

We can explicitly construct a recursion system for the coefficients and check uniqueness of the solution once the initial coefficient is given. It is easy to check the solution is independent of the choice of such a recursion system. □

—. It would be very nice if one has find a proof which reveals intrinsic structure of the heat equations for any plane telescopic curve.

Sample calculation in the $(2,3)$ case (1)

$(e,q)=(2,3)$ の場合：

$$M=[\ x\ \ 1\ ],\quad f_1(x,y)=\frac{\partial}{\partial x}f(x,y)=-3x^2-\mu_4,\quad f_2(x,y)=\frac{\partial}{\partial y}f(x,y)=2y.$$

$$H:=\frac{1}{2}\begin{vmatrix} \dfrac{f_1(x,y)-f_1(z,w)}{x-z} & \dfrac{f_2(x,y)-f_2(z,w)}{x-z} \\ \dfrac{f_1(z,y)-f_1(x,w)}{y-w} & \dfrac{f_2(z,y)-f_2(x,w)}{y-w} \end{vmatrix}=6(x+z)$$

となり，$\mathbb{Q}\left[x,y,z,w\right]/\left(f_1(x,y),\,f_2(x,y),\,f_1(z,w),\,f_2(z,w)\right)$ の中で

$${}^tM\,V\,M=f(x,y)\,H=4\mu_4xz+6\mu_6z+6\mu_6x-\frac{4}{3}\mu_4{}^2.$$

よって

$$V=\begin{bmatrix} 4\mu_4 & 6\mu_6 \\ 6\mu_6 & -\frac{4}{3}\mu_4{}^2 \end{bmatrix}$$

であり，確かに $\det(V)=\frac{4}{3}\Delta$ である.
以上により L_0 と L_2 は

$$L_0=4\mu_4\frac{\partial}{\partial\mu_4}+6\mu_6\frac{\partial}{\partial\mu_6},$$

$$L_2=6\mu_6\frac{\partial}{\partial\mu_4}-\frac{4}{3}\mu_4{}^2\frac{\partial}{\partial\mu_6}.$$

Sample calculation in the $(2,3)$ case (2)

Choose the differential forms and the local parameter by

$$\boldsymbol{\omega} = (\,\omega_1,\ \eta_1\,) = \left(\frac{dx}{2y},\ \ \frac{xdx}{2y}\right), \qquad t = x^{-\frac{1}{2}},$$

and suppose $\frac{\partial}{\partial \mu_j} t = 0$ for any j.

So, we have $\frac{\partial}{\partial \mu_j} x = 0$ for $j = 4,\, 6$, and we compute the matrix Γ as follows.

Using $f(x,y) = y^2 - (x^3 + \mu_4 x + \mu_6)$, we see $2y\frac{\partial}{\partial \mu_4} y = x$ and $2y\frac{\partial}{\partial \mu_6} y = 1$, so that

$$\frac{\partial}{\partial \mu_4} y = \frac{x}{2y}, \quad \frac{\partial}{\partial \mu_6} y = \frac{1}{2y}.$$

Therefore, we have

$$\frac{\partial}{\partial \mu_6}\omega_1 = -\frac{1}{4y^3}dx, \quad \frac{\partial}{\partial \mu_4}\omega_1 = \frac{\partial}{\partial \mu_6}\eta_1 = \frac{x}{4y^3}dx, \quad \frac{\partial}{\partial \mu_4}\eta_1 = \frac{x^2}{4y^3}dx.$$

By computing $d\left(\frac{1}{y}\right)$, $d\left(\frac{x}{y}\right)$, $d\left(\frac{x^2}{y}\right)$, we get

$$L_0(\omega_1) = -\omega_1 + d\left(\frac{1}{y}\right), \qquad L_0(\eta_1) = \eta_1 - d\left(\frac{x^2}{y}\right),$$

$$L_2(\omega_1) = \eta_1 - d\left(\frac{x^2}{y}\right) - \frac{2}{3}\, d\left(\frac{1}{y}\right), \quad L_2(\eta_1) = \frac{\mu_4}{3}\,\omega_1 + \mu_6\, d\left(\frac{1}{y}\right).$$

Sample calculation in the $(2,3)$ case (3)

We have got

$$L_0(\omega_1) = -\omega_1 + d\Big(\frac{1}{y}\Big), \qquad L_0(\eta_1) = \eta_1 - d\Big(\frac{x^2}{y}\Big),$$

$$L_2(\omega_1) = \eta_1 - d\Big(\frac{x^2}{y}\Big) - \frac{2}{3}\, d\Big(\frac{1}{y}\Big), \quad L_2(\eta_1) = \frac{\mu_4}{3}\,\omega_1 + \mu_6\, d\Big(\frac{1}{y}\Big).$$

Therefore, we have on $H^1(\mathscr{C}, \mathbb{Q}[\boldsymbol{\mu}])$ that

$$L_0\Big(\begin{bmatrix}\omega_1\\ \eta_1\end{bmatrix}\Big) = \Gamma_0 \begin{bmatrix}\omega_1\\ \eta_1\end{bmatrix}, \ L_2\Big(\begin{bmatrix}\omega_1\\ \eta_1\end{bmatrix}\Big) = \Gamma_2 \begin{bmatrix}\omega_1\\ \eta_1\end{bmatrix}, \ \text{where } \Gamma_0 = \begin{bmatrix}-1 & \\ & 1\end{bmatrix}, \ \Gamma_2 = \begin{bmatrix} & 1\\ \frac{\mu_4}{3} & \end{bmatrix}.$$

Recall $\Gamma = \begin{bmatrix}\beta & -\alpha\\ \gamma & -{}^t\beta\end{bmatrix}$ and the definition of H^L :

$$H^L = \tfrac{1}{2}\begin{bmatrix}{}^t\frac{\partial}{\partial \boldsymbol{u}} & {}^t u\end{bmatrix}\Gamma^L J \begin{bmatrix}\frac{\partial}{\partial \boldsymbol{u}}\\ u\end{bmatrix} + \tfrac{1}{2}\,\mathrm{Tr}(\beta) = \frac{1}{2}\sum_{i=1}^{g}\sum_{j=1}^{g}\Big(\alpha_{ij}\,\frac{\partial^2}{\partial u_i \partial u_j} + 2\beta_{ij}\, u_i \frac{\partial}{\partial u_j} + \gamma_{ij}\, u_i u_j\Big) + \tfrac{1}{2}\,\mathrm{Tr}(\beta).$$

Since

$$[\, L_0 \ \ L_2\,](\Delta) = [\,12 \ \ 0\,]\,\Delta,$$

we have arrived at

$$(L_0 - H_0)\,\hat{\sigma}(u) = \Big(4\mu_4\frac{\partial}{\partial \mu_4} + 6\mu_6\frac{\partial}{\partial \mu_6} - u\frac{\partial}{\partial u} + 1\Big)\hat{\sigma}(u) = 0,$$

$$(L_2 - H_2)\,\hat{\sigma}(u) = \Big(6\mu_6\frac{\partial}{\partial \mu_4} - \frac{4}{3}{\mu_4}^2\frac{\partial}{\partial \mu_6} - \frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{6}\mu_4 u^2\Big)\hat{\sigma}(u) = 0,$$

where $H_j = H^{L_j} + \frac{1}{8}L_j \log\Delta$ for $j = 0$ and 2.

Explicit Expression of the Heat Operators (1)

For $(2,5)$-curve $y^2 = x^5 + \mu_2 x^4 + \mu_4 x^3 + \mu_6 x^2 + \mu_8 x + \mu_{10}$, we have

$$\omega_1 = \frac{dx}{2y},\ \ \omega_2 = \frac{xdx}{2y},\ \ \eta_1 = \frac{(3x^3 + 2\mu_2 x^2 + \mu_4 x)dx}{2y},\ \ \eta_2 = \frac{x^2 dx}{2y},$$

$$\begin{aligned}
L_0 &= 4\mu_4 \tfrac{\partial}{\partial \mu_4} + 6\mu_6 \tfrac{\partial}{\partial \mu_6} + 8\mu_8 \tfrac{\partial}{\partial \mu_8} + 10\mu_{10} \tfrac{\partial}{\partial \mu_{10}},\\
L_2 &= 6\mu_6 \tfrac{\partial}{\partial \mu_4} + \tfrac{4}{5}(10\mu_8 - 3{\mu_4}^2) \tfrac{\partial}{\partial \mu_6} + \tfrac{2}{5}(25\mu_{10} - 4\mu_6\mu_4) \tfrac{\partial}{\partial \mu_8} - \tfrac{4}{5}\mu_8\mu_4 \tfrac{\partial}{\partial \mu_{10}},\\
L_4 &= 8\mu_8 \tfrac{\partial}{\partial \mu_4} + \tfrac{2}{5}(25\mu_{10} - 4\mu_4\mu_6) \tfrac{\partial}{\partial \mu_6} + \tfrac{4}{5}(5\mu_4\mu_8 - 3{\mu_6}^2) \tfrac{\partial}{\partial \mu_8} + \tfrac{6}{5}(5\mu_4\mu_{10} - \mu_6\mu_8) \tfrac{\partial}{\partial \mu_{10}},\\
L_6 &= 10\mu_{10} \tfrac{\partial}{\partial \mu_4} - \tfrac{4}{5}\mu_8\mu_4 \tfrac{\partial}{\partial \mu_6} + \tfrac{6}{5}(5\mu_{10}\mu_4 - \mu_8\mu_6) \tfrac{\partial}{\partial \mu_8} + \tfrac{4}{5}(5\mu_{10}\mu_6 - 2{\mu_8}^2) \tfrac{\partial}{\partial \mu_{10}}.
\end{aligned}$$

The action of these operators on the discriminant Δ is as follows:

$$[\, L_0 \ \ L_2 \ \ L_4 \ \ L_6 \,]\Delta = [\, 40 \quad 0 \quad 12\,\mu_4 \quad 4\,\mu_6 \,]\Delta.$$

The representation matrices Γ_j for L_j's acting on the space $H^1(\mathscr{C}, \mathbb{Q}[\mu])$ are

$$\Gamma_0 = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\quad \Gamma_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ \frac{4}{5}\,\mu_4 & 0 & 0 & 1 \\ \frac{4}{5}\,{\mu_4}^2 - 3\,\mu_8 & 0 & 0 & -\frac{4}{5}\,\mu_4 \\ 0 & \frac{3}{5}\,\mu_4 & 1 & 0 \end{bmatrix},$$

$$\Gamma_4 = \begin{bmatrix} -\mu_4 & 0 & 0 & 1 \\ \frac{6}{5}\,\mu_6 & 0 & 1 & 0 \\ \frac{6}{5}\,\mu_4\mu_6 - 6\,\mu_{10} & -\mu_8 & \mu_4 & -\frac{6}{5}\,\mu_6 \\ -\mu_8 & \frac{2}{5}\,\mu_6 & 0 & 0 \end{bmatrix},\quad \Gamma_6 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \frac{3}{5}\,\mu_8 & 0 & 0 & 0 \\ \frac{3}{5}\,\mu_4\mu_8 & -2\,\mu_{10} & 0 & -\frac{3}{5}\,\mu_8 \\ -2\,\mu_{10} & \frac{1}{5}\,\mu_8 & 0 & 0 \end{bmatrix}.$$

Explicit Expression of the Heat Operators (2)

Therefore, we have the operators $H_j = H^{L_j} + \frac{1}{8}L_j \log\Delta$ as follows:

$$H_0 = 3u_3\frac{\partial}{\partial u_3} + u_1\frac{\partial}{\partial u_1} + 3,$$

$$H_2 = \frac{1}{2}\frac{\partial^2}{\partial {u_1}^2} + u_1\frac{\partial}{\partial u_3} - \frac{4}{5}\mu_4 u_3\frac{\partial}{\partial u_1} - \frac{3}{10}\mu_4 {u_1}^2 - \Big(\frac{3}{2}\mu_8 - \frac{2}{5}{\mu_4}^2\Big){u_3}^2,$$

$$H_4 = \frac{\partial^2}{\partial u_1 \partial u_3} - \frac{6}{5}\mu_6 u_3\frac{\partial}{\partial u_1} + \mu_4 u_3\frac{\partial}{\partial u_3} - \frac{1}{5}\mu_6 {u_1}^2 + \mu_8 u_1 u_3 - \Big(\frac{3}{5}\mu_4\mu_6 + 2\mu_{10}\Big){u_3}^2 + \mu_4,$$

$$H_6 = \frac{1}{2}\frac{\partial^2}{\partial {u_3}^2} - \frac{3}{5}\mu_8 u_3\frac{\partial}{\partial u_1} - \frac{1}{10}\mu_8 {u_1}^2 + 2\mu_{10}u_3u_1 - \frac{3}{10}\mu_8\mu_4{u_3}^2 - \frac{1}{2}\mu_6.$$

By the first heat equation $(L_0 - H_0)\,\sigma(u) = 0$, where

$$L_0 - H_0 = \Big(4\mu_4\tfrac{\partial}{\partial\mu_4} + 6\mu_6\tfrac{\partial}{\partial\mu_6} + 8\mu_8\tfrac{\partial}{\partial\mu_8} + 10\mu_{10}\tfrac{\partial}{\partial\mu_{10}}\Big) - \Big(3u_3\tfrac{\partial}{\partial u_3} - u_1\tfrac{\partial}{\partial u_1} + 3\Big),$$

the sigma function should be of the form

$$\sigma(u_3,u_1) = \sum_{\substack{m,n_4,n_6,n_8,n_{10}\geq 0\\ 3-3m+4n_4+6n_6+8n_8+10n_{10}\geq 0}} b(m,n_1,n_4,n_6,n_8,n_{10})\,\frac{{u_1}^3\Big(\frac{u_3}{{u_1}^3}\Big)^m\big(\mu_4{u_1}^4\big)^{n_4}\big(\mu_6{u_1}^6\big)^{n_6}\big(\mu_8{u_1}^8\big)^{n_8}\big(\mu_{10}{u_1}^{10}\big)^{n_{10}}}{m!\,(3-3m+4n_4+6n_6+8n_8+10n_{10})!}.$$

Recursion in the $(2,5)$-case

Let $k = 3 - 3m + 4n_4 + 6n_6 + 8n_8 + 10n_{10}$.
Then the other heat equations $(L_j - H_j)\,\sigma(u) = 0$ ($j = 2, 4, 6$) imply the following recursion relations:

$$
b(m, n_4, n_6, n_8, n_{10})
= \begin{cases}
20(n_8+1)\,b(m, n_4, n_6, n_8+1, n_{10}-1) \\
+16(n_6+1)\,b(m, n_4, n_6+1, n_8-1, n_{10}) \\
+12(n_4+1)\,b(m, n_4+1, n_6-1, n_8, n_{10}) \\
-\frac{24}{5}(n_6+1)\,b(m, n_4-2, n_6+1, n_8, n_{10}) \\
+\frac{3}{5}(-k+3)(-k+2)\,b(m, n_4-1, n_6, n_8, n_{10}) \\
-\frac{8}{5}(n_{10}+1)\,b(m, n_4-1, n_6, n_8-1, n_{10}+1) \\
-\frac{16}{5}(n_8+1)\,b(m, n_4-1, n_6-1, n_8+1, n_{10}) \\
+2(-k+2)\,b(m+1, n_4, n_6, n_8, n_{10}) \\
-3m(m-1)\,b(m-2, n_4, n_6, n_8-1, n_{10}) \\
+\frac{4}{5}m(m-1)\,b(m-2, n_4-2, n_6, n_8, n_{10}) \\
+\frac{8}{5}m\,b(m-1, n_4-1, n_6, n_8, n_{10} & (\text{if } k > 1 \text{ and } m \geqq 0), \\
10(n_6+1)\,b(m-1, n_4, n_6+1, n_8, n_{10}-1) + \cdots & (\text{if } k = 1 \text{ and } m > 0), \\
-\frac{16}{5}(1+n_{10})\,b(m-2, n_4, n_6, n_8-2, n_{10}+1) + \cdots & (\text{if } k = 0 \text{ and } m > 1).
\end{cases}
$$

The Expansion of $(2,5)$-Sigma Function

From these, we see the expansion of $\sigma(u)$ is Hurwitz integral over $\mathbb{Z}_{(5)}$:

$$\sigma(u_3,u_1) = u_3 - 2\frac{{u_1}^3}{3!} - 4\mu_4\frac{{u_1}^7}{7!} - 2\mu_4\frac{u_3{u_1}^4}{4!} + 64\mu_6\frac{{u_1}^9}{9!} - 8\mu_6\frac{u_3{u_1}^6}{6!} - 2\mu_6\frac{{u_3}^2{u_1}^3}{2!3!} + \mu_6\frac{{u_3}^3}{3!}$$
$$+ (-408{\mu_4}^2 + 1600\mu_8)\frac{{u_1}^{11}}{11!} - (4{\mu_4}^2 + 32\mu_8)\frac{u_3{u_1}^8}{8!} - 8\mu_8\frac{{u_3}^2{u_1}^5}{2!5!} - 2\mu_8\frac{{u_3}^3{u_1}^2}{3!2!} + \cdots .$$

The recursion in $(3,4)$-case

We have 6 heat equations $(L_j - H_j)\sigma(u) = 0$ for $j = 0,\ 3,\ 4,\ 6,\ 7, 10$.

Thus, 6 recursion relations.

The first equation $(L_0 - H_0)\sigma(u) = 0$ implies the sigma is of the form

$$\sigma(u_5, u_3, u_1) = \sum_{\substack{\ell,m,n_4,n_6,n_8,\\ n_{10},n_{12},n_{14}}} b(\ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14})$$

$$\cdot\, {u_1}^6\Big(\frac{u_5}{{u_1}^5}\Big)^{\ell}\Big(\frac{u_3}{{u_1}^3}\Big)^{m} \cdot \big(\mu_4 {u_1}^4\big)^{n_4} \big(\mu_6 {u_1}^6\big)^{n_6} \big(\mu_8 {u_1}^8\big)^{n_8} \big(\mu_{10} {u_1}^{10}\big)^{n_{10}} \big(\mu_{12} {u_1}^{12}\big)^{n_{12}} \big(\mu_{14} {u_1}^{14}\big)^{n_{14}}$$

$$\Big/ (6 - 5\ell - 3m + 4n_4 + 6n_6 + 8n_8 + 10n_{10} + 12n_{12} + 14n_{14})!\,\ell!\,m!\ .$$

The set of rest 5 recursion relations indeed gives the sigma function.

However, we need a kind of “switch back” on weight.

Expansion of the $(3,4)$-Sigma Function

ここでは

$$f(x,y) = y^3 + (\mu_1 x + \mu_4)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12})$$

に対するものを書いておく. $\mu_1 = \mu_4 = \mu_2 = \mu_5 = \mu_8 = 0$ としたものが今回の結果となる.

$$\sigma(u) = C_5 + C_6 + C_7 + C_8 + C_9 + \cdots$$

と u に関する weight ごとに分けると

$$C_5 = u_5 - u_1 {u_2}^2 + 6\frac{{u_1}^5}{5!},$$

$$C_6 = 2\mu_1 \frac{{u_1}^4}{4!}\frac{u_2}{1!} - 2\mu_1 \frac{{u_2}^3}{3!},$$

$$C_7 = 10({\mu_1}^2 - 3\mu_2)\frac{{u_1}^7}{7!} + 2\mu_2 \frac{{u_1}^3}{3!}\frac{{u_2}^2}{2!},$$

$$C_8 = 2({\mu_1}^3 + 9\mu_3 - 2\mu_1\mu_2)\frac{{u_1}^6}{6!}\frac{u_2}{1!} - 6\mu_3 \frac{{u_1}^2}{2!}\frac{{u_2}^3}{3!},$$

$$C_9 = 14({\mu_1}^2 - 3\mu_2)^2 \frac{{u_1}^9}{9!} + 2(2\mu_4 - {\mu_2}^2 + {\mu_1}^2\mu_2 + 6\mu_1\mu_3)\frac{{u_1}^5}{5!}\frac{{u_2}^2}{2!}$$

$$- 2(4\mu_1\mu_3 + 4\mu_4 + {\mu_2}^2)\frac{u_1}{1!}\frac{{u_2}^4}{4!} + 2\mu_4 \frac{{u_1}^4}{4!}\frac{u_5}{1!},$$

$$\cdots\cdots\cdots.$$

Problem

We have a proof of the one-dimensionality of the solution space of the system of heat equations only for any plane telescopic curves of genus 3 or smaller. Our proof of the one-dimensionality is a consequence of expected good behalf of the recursion system. However, the $(3,4)$-recursion is so complicated. So, we shall pose the following

Problem

Can we prove one-dimensionality of the solution space of the system of heat equations

$$\left\{ (L_{v_j} - H_{v_j})\, \sigma(u) = 0 \,\middle|\, j = 0,\ 1,\ \cdots,\ 2g \right\}$$

for any telescopic curve of modality one?

It might be a hint for this problem that
there is the following closed form of entries of the matrix V for $(2,q)$-case, which is given by JCE.

Lemma

$$\begin{aligned} V_{ij} &= -\frac{2i(q-j)}{q}\, \mu_{2i}\mu_{2j} + \sum_{m=1}^{m_0} 2(j-i+2m)\, \mu_{2(i-m)}\, \mu_{2(j+m)} \\ &= -\frac{2i(q-j)}{q}\, \mu_{2i}\mu_{2j} + \sum_{\ell=\ell_0}^{i-1 \text{ or } j} 2(i+j-2\ell)\, \mu_{2\ell}\, \mu_{2(i+j-\ell)}, \end{aligned}$$

where $\mu_0 = 1$, $\mu_2 = 0$, $m_0 = \min\{i,\ q-j\}$, *and* $\ell_0 = \max\{0, i+j-q\}$.