A Recursion System on the Expansion Coefficients of the Sigma Function for a Higher Genus Curve

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A Recursion System on the Sigma Function

Main References

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Classical theory of elliptic functions (of Weierstrass)

Let Λ be a lattice in the complex plane $\mathbb{C}$.

$$\wp(u) = \frac{1}{u^2} + \sum_{u \in \Lambda - \{0\}} \left(\frac{1}{(u-\ell)^2} - \frac{1}{\ell^2} \right),$$

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\,\wp(u) - g_3,$$

where

$$g_2 = 60 \sum_{\ell \in \Lambda - \{0\}} \frac{1}{\ell^4}, \quad g_3 = 140 \sum_{\ell \in \Lambda - \{0\}} \frac{1}{\ell^6},$$

$$\zeta(u) = \frac{1}{u} - \int_0^u \left(\wp(u) - \frac{1}{u^2} \right) du,$$

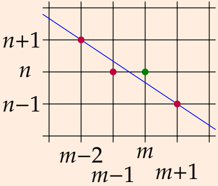
$$\sigma(u) = u \exp\left(\int_0^u \int_0^u \left(\frac{1}{u^2} - \wp(u) \right) du\, du \right) = u \prod_{\ell \in \Lambda - \{0\}} \left(1 - \frac{u}{\ell} \right) e^{\frac{u}{\ell} + \frac{u^2}{2\ell^2}}.$$

The recursion in the $(2,3)$-case (Weierstrass' work)

[Weierstrass 1882] Regarding $\wp'^2 = 4\wp^3 + 4a_4\wp + 4a_6$ as a differential equation for $\sigma(u)$, by highly technical feat he got the heat eq. of $\sigma(u)$:

$$\text{(W1)}\quad \left(4a_4\frac{\partial}{\partial a_4} + 6a_6\frac{\partial}{\partial a_6} - u\frac{\partial}{\partial u} + 1\right)\sigma(u) = 0,$$

$$\text{(W2)}\quad \left(6a_6\frac{\partial}{\partial a_4} - \frac{4}{3}{a_4}^2\frac{\partial}{\partial a_6} - \frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{6}a_4u^2\right)\sigma(u) = 0.$$



(W1) shows

$$\sigma(u) = \sum_{n_4,n_6\geqq 0} b(n_4,n_6)\,\frac{u\,(a_4u^4)^{n_4}(a_6u^6)^{n_6}}{(1+4n_4+6n_6)!}.$$

By (W2) (by rewriting $m = n_4$, $n = n_6$),

$$b(m,n) = \tfrac{2}{3}(4m+6n-1)(2m+3n-1)\,b(m-1,n) - \tfrac{8}{3}(n+1)\,b(m-2,n+1) + 12(m+1)\,b(m+1,n-1),$$

$$b(m,n) = 0 \quad \text{if } m<0 \text{ or } n<0.$$

From this, we get

$$\sigma(u) = u + 2\,a_4\,\frac{u^5}{5!} + 24\,a_6\,\frac{u^7}{7!} - 36\,{a_4}^2\,\frac{u^9}{9!} - 288\,a_4a_6\,\frac{u^{11}}{11!} + \cdots.$$

Modern approach to elliptic functions (1)

However, we shall start at the (elliptic) curve :

$$\mathscr{C} \;:\; y^2 = x^3 + a_4 x + a_6.$$

Associating to $\mathscr{C}$, we define

$$\sigma(u) = \Big(\frac{2\pi}{\omega'}\Big)^{1/2} \Delta^{-\frac{1}{8}} \,\exp\Big(-\tfrac{1}{2}{\omega'}^{-1}\eta' u^2\Big)\cdot \vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}({\omega'}^{-1}u \mid \omega''/\omega'),$$

where

$$\Delta = -16(4\,{a_4}^3 + 27{a_6}^2) = \text{the discriminant},$$

$$\begin{bmatrix}\omega' & \omega'' \\ \eta' & \eta''\end{bmatrix} = \begin{bmatrix}\int_{\alpha_1}\omega & \int_{\beta_1}\omega \\ \int_{\alpha_1}\eta & \int_{\beta_1}\eta\end{bmatrix} \quad \text{with} \quad \omega_1 = \frac{dx}{2y}\,, \quad \eta_{-1} = \frac{xdx}{2y}$$

and (α_1, β_1) is a symplectic basis of $H_1(\mathscr{C}, \mathbb{Z})$, and

$$\vartheta\begin{bmatrix}b\\ a\end{bmatrix}(z \mid \tau) = \sum_{n\in\mathbb{Z}} \exp 2\pi\boldsymbol{i}\Big(\tfrac{1}{2}\tau(n+b)^2 + (n+b)(z+a)\Big) \quad (a,\ b \in \mathbb{R})$$

is Jacobi's theta series.

Modern approach to elliptic functions (2)

$$\sigma(u) = \left(\frac{2\pi}{\omega'}\right)^{1/2} \Delta^{-\frac{1}{8}} \exp\left(- \tfrac{1}{2}\omega'^{-1}\eta' u^2\right) \cdot \vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\omega^{-1}u \mid \omega''/\omega'),$$

$$\zeta(u) = -\frac{\partial}{\partial u} \log \sigma(u), \quad \wp(u) = -\frac{\partial^2}{\partial^2 u} \log \sigma(u),$$

$$\wp'(u)^2 = 4\,\wp(u)^3 + 4a_4\,\wp(u) + 4a_6, \qquad -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma(u)^2\,\sigma(v)^2} = \wp(u) - \wp(v).$$

Kiepert's n-plication formula :

$$(-1)^{n-1}\,(1!2!\cdots(n-1)!)^2\ \frac{\sigma(nu)}{\sigma(u)^{n^2}} = \begin{vmatrix} \wp'(u) & \wp''(u) & \cdots & \wp^{(n-1)}(u) \\ \wp''(u) & \wp'''(u) & \cdots & \wp^{(n)}(u) \\ \vdots & \vdots & \ddots & \vdots \\ \wp^{(n-1)}(u) & \wp^{(n)}(u) & \cdots & \wp^{(2n-3)}(u) \end{vmatrix}.$$

These are nicely generalized to higher genus case. (Motivation!)

Some remarks on the definition of $\sigma(u)$

$$\sigma(u) = \Big(\frac{2\pi}{\omega'}\Big)^{1/2} \varDelta^{-\frac{1}{8}} \exp\Big(- \tfrac{1}{2}\omega'^{-1}\eta' u^2\Big) \cdot \vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\omega^{-1}u \mid \omega'^{-1}\omega'')$$

—. It is not clear that $\sigma(u)$ is independent of the choice of α_1 and β_1.

(This is obvious if we adapt the classical definition.)

—. However, eventually the changes of the factors cancel!

—. Using the Dedekind eta function

$$\boldsymbol{\eta}(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}),$$

the blue part above is explicitly written as

$$\Big(\frac{2\pi}{\omega'}\Big)^{1/2} \varDelta^{-\frac{1}{8}} = -\frac{\omega'}{2\pi} \boldsymbol{\eta}\Big(\omega'^{-1}\omega''\Big)^{-3}.$$

⟶ Jacobi's derivative formula.

Characterization of σ in genus 1 case

$\Lambda := \left\{ \oint \omega \right\}$, where, $\omega = \dfrac{dx}{2y}$.

For $u \in \mathbb{C}$, we define $u', u'' \in \mathbb{R}$ by $u = u'\omega' + u''\omega''$.

For $\ell \in \Lambda$, we define $\ell', \ell'' \in \mathbb{Z}$ by $\ell = \ell'\omega' + \ell''\omega''$.

$L(u, v) := u(v'\eta' + v''\eta'')$, $\chi(\ell) := \exp \pi \boldsymbol{i}(\ell' + \ell'' + \ell'\ell'')$.

Proposition (Characterization of σ of genus 1)

The sigma function for $\mathscr{C}$ is characterize by the following 5 properties :

(S1) $\sigma(u)$ is an entire odd function on $\mathbb{C}$;

(S2) $\sigma(u + \ell) = \chi(\ell)\,\sigma(u) \exp L(u + \frac{1}{2}\ell, \ell)$ for any $u \in \mathbb{C}$ and $\ell \in \Lambda$;

(S3) $\sigma(u)$ is expanded as a power series around the origin with coefficients in $\mathbb{Q}[\boldsymbol{a}]$ of homogeneous weight $1\left(= \frac{(2^2-1)(3^2-1)}{24}\right)$; with $\mathrm{wt}(a_j) = -j$, $\mathrm{wt}(u) = 1$;

(S4) $\sigma(u)|_{\boldsymbol{a}=\boldsymbol{0}} = u$ (the Schur polynomial $s_{2,3}(u) = u$ for genus one) ;

(S5) $\sigma(u) = 0 \iff u \in \Lambda$. Zeroes of order 1.

These properties might be not independent each other.

BL-Theory finds $\Delta^{-\frac{1}{8}}$

Genus 1 case:

$$\sigma(u) = \left(\frac{2\pi}{\omega'}\right)^{\frac{1}{2}} \Delta^{-\frac{1}{8}} \exp\left(-\tfrac{1}{2}\omega'^{-1}\eta' u^2\right) \cdot \vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\omega'^{-1}u \mid \omega''/\omega').$$

Genus > 1 case: Expect also for a plane telescopic curve $\mathscr{C}$ of genus g,

$$\sigma(u) \overset{?}{=} \Delta^{-\frac{1}{8}} \left(\frac{(2\pi)^g}{|\omega'|}\right)^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}\,{}^t u\,\omega'^{-1}\eta'\,u\right) \vartheta\begin{bmatrix}\delta'' \\ \delta'\end{bmatrix}(\omega'^{-1}u \mid \omega'^{-1}\omega'')$$

$$=:\ \hat{\sigma}(u)\ =:\ \Delta^{-\frac{1}{8}}\widetilde{\sigma}(u).$$

Here, Δ is the discriminant of $\mathscr{C}$ i.e. $\Delta \in \mathbb{Z}[a]$ which is irreducible and "$\Delta \neq 0 \iff \mathscr{C}$ is smooth".
$\begin{bmatrix}\delta'' \\ \delta'\end{bmatrix} \in \left(\frac{1}{2}\mathbb{Z}\right)^{2g}$ is determined by the Riemann constant vector of $\mathscr{C}$.

Fact : $\sigma(u)$ is $\widetilde{\sigma}(u)$ times a function independent of u.	Hope to show $\sigma(u) = \hat{\sigma}(u)$.

There is a proof for genus 2 case (by D.Grant using Thomae's formula)).

Theorem (Buchstaber-Leykin + EGÔY)

If the (plane telescopic) curve is of genus 3 or smaller, we have up to non-zero multiplicative absolute constant that

$$\sigma(u) = \hat{\sigma}(u).$$

Work of Frobenius-Stickelberger

$$-4a_4 = g_2 = 60 \sum_{n',n''}{}' \frac{1}{(n'\omega' + n''\omega'')^4}, \quad -4a_6 = g_3 = 140 \sum_{n',n''}{}' \frac{1}{(n'\omega' + n''\omega'')^6},$$

$$\wp(u) = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + \frac{{g_2}^2}{1200}u^6 + \cdots, \quad \zeta(u + n'\omega' + n''\omega'') = \zeta(u) + n'\eta' + n''\eta'',$$

and got the following :

$$\begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix} \begin{bmatrix} \frac{\partial g_2}{\partial \omega'} & \frac{\partial g_3}{\partial \omega'} \\ \frac{\partial g_2}{\partial \omega''} & \frac{\partial g_3}{\partial \omega''} \end{bmatrix} = \begin{bmatrix} -4g_2 & -6g_3 \\ -6g_3 & -\frac{1}{3}{g_2}^2 \end{bmatrix}.$$

Multiplying $\begin{bmatrix} \frac{\partial}{\partial g_2} \\ \frac{\partial}{\partial g_3} \end{bmatrix}$, we have

$$\begin{cases} \omega' \frac{\partial}{\partial \omega'} + \omega'' \frac{\partial}{\partial \omega''} = -4g_2 \frac{\partial}{\partial g_2} - 6g_3 \frac{\partial}{\partial g_3} & \text{(former part of (W1)),} \\ \eta' \frac{\partial}{\partial \omega'} + \eta'' \frac{\partial}{\partial \omega''} = -6g_3 \frac{\partial}{\partial g_2} - \frac{1}{3}{g_2}^2 \frac{\partial}{\partial g_3} & \text{(former part of (W2).} \end{cases}$$

Analytic side	=	Algebraic side

—. Want similar fourmulae for $g > 1$. But no such Eisenstein series!

—. Note that the RHSs of the operators are tangent to ${g_2}^3 - 27{g_3}^2$.

Plane Telescopic Curves

Let $e < q$, $\gcd(e,q) = 1$, and

$$f(x,y) = y^e + p_1(x)\,y^{e-1} + \cdots + p_{e-1}(x)\,y - p_e(x),$$

$$p_j(x) = \sum_{k\,:\,jq-ek>0} a_{jq-ek}\,x^k \quad (1 \le j \le e-1), \ \ \deg(p_j) = \left\lceil \frac{jq}{e} \right\rceil, \quad p_e(x) = x^q + a_{e(q-1)}x^{q-1} + \cdots + a_{eq}.$$

For simplicity, we assume a_i are constants in $\mathbb{C}$ over $\mathbb{Q}$.

Definition

We denote by $\mathscr{C}$ the (non-singular) curve defined by $f(x,y) = 0$ which is added unique point ∞ at infinity, which is called (e,q)-curve, or plane telescopic curve.

We introduce a weight defined by

$$\mathrm{wt}(a_j) = -j, \quad \mathrm{wt}(x) = -e, \quad \mathrm{wt}(y) = -q.$$

Examples :

$(2,3)$-curve : $f = y^2 + (a_1x + a_3)y - (x^3 + a_2x^2 + a_4 + a_6)$.

$(2,5)$-curve : $f = y^2 + (a_1x^2 + a_3x + a_5)y - (x^5 + a_2x^4 + a_4x^3 + a_6x^2 + a_8x + a_{10})$.

$(2,7)$-curve : $f = y^2 + (a_1x^3 + a_3x^2 + a_5x + a_7)y - (x^7 + a_2x^6 + a_4x^5 + a_6x^4 + a_8x^3 + a_{10}x^2 + a_{12}x + a_{14})$.

$(3,4)$-curve : $f = y^3 + (a_1x + a_4)y^2 + (a_2x^2 + a_5x + a_8)y - (x^4 + a_3x^3 + a_6x^2 + a_9x + a_{12})$.

Discriminant of the Curve

A discriminant Δ of the curve $\mathscr{C}$:

$$\Delta \in \mathbb{Z}[\boldsymbol{a}] \text{ is irreducible, and "} \Delta \neq 0 \iff \mathscr{C} \text{ is smooth".}$$

Conjecture

Supposing all a_j indeterminates, we define

$$R_1 = \mathrm{rslt}_x\Big(\mathrm{rslt}_y\Big(f(x,y), \tfrac{\partial}{\partial x} f(x,y)\Big), \mathrm{rslt}_y\Big(f(x,y), \tfrac{\partial}{\partial y} f(x,y)\Big)\Big),$$

$$R_2 = \mathrm{rslt}_y\Big(\mathrm{rslt}_x\Big(f(x,y), \tfrac{\partial}{\partial x} f(x,y)\Big), \mathrm{rslt}_x\Big(f(x,y), \tfrac{\partial}{\partial y} f(x,y)\Big)\Big),$$

$$R = \gcd(R_1, R_2) \quad \text{in } \mathbb{Z}[\boldsymbol{a}].$$

rslt_z : Sylvester's resultant w. r. t. z.

Then $R = \Delta^2$.

This is OK for our cases.

The first degree de Rham cohomology

We introduce

$$H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\boldsymbol{a}]) = \frac{\left\{ h(x,y)\,\dfrac{dx}{\frac{\partial}{\partial y} f(x,y)} \;\middle|\; h(x,y) \in \mathbb{Q}[\boldsymbol{a},x,y]/(f) \right\}}{d\Big(\mathbb{Q}[\boldsymbol{a},x,y]/(f)\Big)}$$

equipped with the inner product $(\omega,\eta) \mapsto \operatorname*{Res}_{P=\infty}\left(\int_{\infty}^{P} \omega\right)\eta(P)$.

Overview of [BL]

(1) The primary heat equation.

(2) Hypothesis $\sigma(u) \overset{?}{=} \hat{\sigma}(u) := \Delta^{-\frac{1}{8}} \cdot \widetilde{\sigma}$ (clue is in genus one case).

(3) Find a basis $L_0 = L_{v_1}$, ... , $L_{v_{2g}}$ of the tangent space of $\Delta = 0$.
using a method known in *the singularity theory* (number theorists may not know).

(4) Using the primary heat equation, we get algebraic heat operators :
$L_{v_j} - H^{L_{v_j}} - (L_{v_j} \log \Delta)$.

(5) Solve them and show that the solution space is of dimension 1.
This implies the standard solution is no other than $\sigma(u)$: $\sigma(u) = \hat{\sigma}(u)$.

We express the whole theory in one breath : Weierstrass' heat equation system

$$\left(4a_4\frac{\partial}{\partial a_4} + 6a_6\frac{\partial}{\partial a_6} - u\frac{\partial}{\partial u} - \frac{1}{2} + \frac{3}{2}\right)\sigma(u) = 0,$$

$$\left(6a_6\frac{\partial}{\partial a_4} - \frac{4}{3}{a_4}^2\frac{\partial}{\partial a_6} - \frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{6}a_4u^2 + 0 + 0\right)\sigma(u) = 0$$

is generalized to the system $\left(L_j - H^{L_j} + \frac{1}{8}L_j(\log \Delta)\right)\sigma(u) = 0,$
where (j runs certain $2g$ integers in $\{1, 2, \cdots, 4g-2\}$).
The $\{L_j\}$ form a basis of the tangent space of Δ, and
the $\{H^{L_j}\}$ are determined by the action of L_js on $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\boldsymbol{a}])$.

Characterization of the Sigma Functions

Proposition (F.Klein, H.F.Baker, ... , Nakayashiki)

Assume all $\{a_j\}$s are complex numbers, and $\Delta \neq 0$.

There is unique function $\sigma(u)$ satisfying the following :

(S1) $\sigma(u) = \sigma(u_{w_g}, \cdots, u_{w_1})$ is an entire, odd or even, function on $\mathbb{C}^g$;

(S2) $\sigma(u+\ell) = \chi(\ell)\,\sigma(u)\exp L(u+\frac{1}{2}\ell, \ell)$ for any $u \in \mathbb{C}^g$ and any $\ell \in \Lambda$;

(S3) $\sigma(u)$ is expanded around the origin as a power series with coefficients in $\mathbb{Q}[\boldsymbol{a}]$ and is homoge. weight of $(e^2-1)(q^2-1)/24$;

(S4) $\sigma(u)\,|_{\boldsymbol{a}=\boldsymbol{0}} = s_{e,q}(u)$ (the Schur polynomial) ;

(S5) $\sigma(u) = 0 \iff u \in \kappa^{-1}(\Theta)$, and the order of zeroes along $\kappa^{-1}(\Theta)$ is 1.

Here $[\,\delta''\ \delta'\,] \in \left(\frac{1}{2}\mathbb{Z}\right)^{2g}$ gives the Riemann constant vector of $\mathscr{C}$,

$$\chi(\ell) := \exp\left(2\pi\boldsymbol{i}\left({}^t\ell'\delta'' + {}^t\ell''\delta' + \tfrac{1}{2}{}^t\ell'\ell''\right)\right), \quad L(u,v) := {}^tu(v'\eta' + v''\eta''),$$

where η' and η'' are period matices for $\boldsymbol{\eta}$ of a symplectic basis

$$(\boldsymbol{\omega}\ \ \boldsymbol{\eta}) = (\omega_{w_g}, \cdots, \omega_{w_1}, \eta_{-w_1}, \cdots, \eta_{-w_g}) \text{ of } H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\boldsymbol{a}]).$$

$$\boldsymbol{\omega} := (\omega_{w_g}\ \cdots\ \omega_{w_1}) \ \left(\text{e.g. if } (e,q) = (2,5),\ \text{then } \omega_3 = \frac{dx}{\frac{\partial}{\partial y}f},\ \omega_1 = \frac{xdx}{\frac{\partial}{\partial y}f}\right),\ \Lambda := \left\{\oint \boldsymbol{\omega}\right\} \subset \mathbb{C}^g,$$

Θ := Abel-Jacobi image of $\mathrm{Sym}^{g-1}(\mathscr{C})$ with the base ∞, $\kappa : \mathbb{C}^g \longrightarrow \mathrm{Jac}(\mathscr{C})(\mathbb{C}) = \mathbb{C}^g/\Lambda$,

$$s_{2,3} = u_1,\ s_{2,5} = u_3 - 2\frac{{u_1}^3}{3!},\quad s_{2,7} = u_1u_5 - 2\frac{{u_3}^2}{2!} - 2\frac{{u_1}^3u_3}{3!} + 16\frac{{u_1}^6}{6!},\quad s_{3,4} = u_5 - u_1{u_2}^2 + 6\frac{{u_1}^5}{5!},\ \cdots.$$

The Primary Heat Equation (1)

For any $L \in \bigoplus_j \mathbb{Q}[a]\frac{\partial}{\partial a_j}$, which is an object of Algebraic Side,

we want to find the corresponding operator in Analytic Side.

However, it is sufficient to know only $L(\Omega)$ for $\Omega = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}$.

By a lemma due to Chevalley, the operator L acts on $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[a])$.

Define $\Gamma = \Gamma^L$ by $L((\boldsymbol{\omega} \ \ \boldsymbol{\eta})) = (\boldsymbol{\omega} \ \ \boldsymbol{\eta})\,{}^t\Gamma$.

Integrating this along closed paths, we have $L(\Omega) = \Gamma\Omega$, and operating L to the Legendre relation

$$ {}^t\Omega J \Omega = 2\pi i J, \quad \text{where} \quad J = \begin{bmatrix} & 1_g \\ -1_g & \end{bmatrix}, $$

we get ${}^t\Gamma J + J\Gamma = 0$. This means that $K := \Gamma J$ is a symmetric matrix. Denoting

$$ K = \begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix} \ \ (\,\alpha,\ \gamma \ \text{ is symmetric }), $$

we have

$$ \Gamma = -KJ = -K\begin{bmatrix} & 1_g \\ -1_g & \end{bmatrix} = \begin{bmatrix} \beta & -\alpha \\ \gamma & -{}^t\beta \end{bmatrix}, $$

$$ \Gamma\Omega = \begin{bmatrix} \beta\omega' - \alpha\eta' & \beta\omega'' - \alpha\eta'' \\ \gamma\omega' - {}^t\beta\eta' & \gamma\omega'' - {}^t\beta\eta'' \end{bmatrix} = L(\Omega), \quad (\text{all the entries of } \alpha,\ \beta,\ \gamma \text{ are } \in \mathbb{Q}[a]). $$

The Primary Heat Equation (2) (作用素 L and H^L)

We have seen how to get $L(\Omega)$. Analytic Side $\leftsquigarrow$ Algebraic Side.

Taking the blue part L_0 and L_2 of Weierstrass' operators L as

$$L_0 - H^{L_0} = 4a_4\frac{\partial}{\partial a_4} + 6a_6\frac{\partial}{\partial a_6} - u\frac{\partial}{\partial u} - \frac{1}{2} + \frac{3}{2},$$

$$L_2 - H^{L_2} = 6a_6\frac{\partial}{\partial a_4} - \frac{4}{3}{a_4}^2\frac{\partial}{\partial a_6} - \frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{6}a_4u^2 + 0 + 0,$$

$$\Gamma^{L_0} = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \quad \Gamma^{L_2} = \begin{bmatrix} & 1 \\ \frac{a_4}{3} & \end{bmatrix}.$$

The higher genus generalization of H^{L_0} と H^{L_2}. 先に L から得られた対称行列

$$K = \begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix} = \Gamma^L J \quad (\alpha,\ \gamma \ \text{は対称行列}), \quad \left(\Gamma^L = \begin{bmatrix} \beta & -\alpha \\ \gamma & -{}^t\beta \end{bmatrix} \right)$$

に対して, 作用素 H^L を次の様に定義する([BL]).

$$H^L = \frac{1}{2}\begin{bmatrix} {}^t\frac{\partial}{\partial u} & {}^tu \end{bmatrix} K \begin{bmatrix} \frac{\partial}{\partial u} \\ u \end{bmatrix} + \frac{1}{2}\mathrm{Tr}(\beta)$$

$$= \frac{1}{2}\begin{bmatrix} \frac{\partial}{\partial u_{w_g}} & \cdots & \frac{\partial}{\partial u_{w_1}} & u_{w_g} & \cdots & u_{w_1} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial u_{w_g}} \\ \vdots \\ \frac{\partial}{\partial u_{w_1}} \\ u_{w_g} \\ \vdots \\ u_{w_1} \end{bmatrix} + \frac{1}{2}\mathrm{Tr}(\beta)$$

The Primary Heat Equation (3)

始めに characteristic $b = {}^t[b'\ b'']$ の theta の項を掛けた函数をを考へる：

$$G(b,u,\Omega) = \left(\frac{(2\pi)^g}{\det(\omega')}\right)^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}\,{}^t u\,\omega'^{-1}\eta'\,u\right)\cdot\exp\left(2\pi i\left(\tfrac{1}{2}{}^t b''\omega'^{-1}\omega'' b'' + {}^t b''(\omega'^{-1}u + b')\right)\right)$$

これは $\sigma(u)$ の定義式の第 n 項で ${}^t[n+\delta''\ \delta'] = b = {}^t[b''\ b']$ としたもの.

これについても次の定理が成り立つ.

([BL] の Thm.13 の修正版)

Theorem (Primary heat equation)

For the function $G(b,u,\Omega)$ above, one has

$$(L - H^L)\,G(b,u,\Omega) = 0.$$

これは自明ではなく, かなりの計算を要する.

ここでも，もちろん $L(\Omega) = \Gamma\Omega$ を使つて計算する.

The Primary Heat Equation (4)

(*Proof of the Primary Heat Equation*)

最初は $L\Big(G(b,u,\Omega)\Big)$ と $H^L\Big(G(b,u,\Omega)\Big)$ を straight forward に計算して証明してゐた.
J. Gibbons 氏がある程度, 見通しのよい証明をこしらへたが, B-L がどうやつて見抜いたかは不明.

ともかく $\dfrac{L\Big(G(b,u,\Omega)\Big)}{G(b,u,\Omega)}$ と $\dfrac{H^L\Big(G(b,u,\Omega)\Big)}{G(b,u,\Omega)}$ を計算すれば, どちらも

$$\frac{1}{2}{}^t u\,{}^t\omega'^{-1}\,{}^t\eta'\alpha\eta'\omega'^{-1}u - 2\pi\boldsymbol{i}\,{}^t b''\omega'^{-1}\alpha\eta'\omega'^{-1}u - 2\pi^2\,{}^t b''\,{}^t\omega'^{-1}\alpha\,{}^t\omega'^{-1}b'' - \frac{1}{2}\sum_{i,j}\alpha_{ij}(\eta'\omega'^{-1})_{ij}$$
$$+ 2\pi\boldsymbol{i}\,{}^t b''\omega'^{-1}\beta u - {}^t u\omega'^{-1}\eta'\beta u + \frac{1}{2}\mathrm{tr}\beta + \frac{1}{2}{}^t u\gamma u$$

となる. (QED)

The Primary Heat Equation (6)

Therefore, for

$$\widetilde{\sigma}(u) := \left(\frac{(2\pi)^g}{|\omega'|}\right)^{\frac{1}{2}} \cancel{\Delta^{-\frac{1}{8}}} \exp\left(-\tfrac{1}{2}\,{}^t u\,\omega'^{-1}\eta'\,u\right)\cdot\vartheta\begin{bmatrix}\delta''\\ \delta'\end{bmatrix}(\omega^{-1}u \mid \omega'^{-1}\omega''),$$

we have

$$(L - H^L)\,\widetilde{\sigma}(u) = 0.$$

Hope to find a nice

$$L \in \bigoplus_j \mathbb{Q}[a]\frac{\partial}{\partial a_j}$$

such that (some modification of) $L - H^L$ kills $\hat{\sigma}(u)$.

Algebraic Heat Operators L and H^L

予備的考察　先の様に $L \in \bigoplus_j \mathbb{Q}[\boldsymbol{a}]\frac{\partial}{\partial a_j}$ をとり H^L を作る.

いま Φ を $\{a_j\}$ のみの（$\{u_j\}$ に依存しない）函数とすれば,

$$L\,(\Phi\,\widetilde{\sigma}(u)) = (L\Phi)\,\widetilde{\sigma}(u) + \Phi\cdot L\,\widetilde{\sigma}(u),\quad H^L\,\Phi\,\widetilde{\sigma}(u) = \Phi\cdot H^L\,\widetilde{\sigma}(u),\quad (L - H^L)\big(\widetilde{\sigma}(u)\big) = 0.$$

従つて, $\Phi\,\widetilde{\sigma}(u)$ について

$$(L - H^L)(\Phi\,\widetilde{\sigma}(u)) = (L\Phi)\,\widetilde{\sigma}(u) = \frac{L\Phi}{\Phi}\,\Phi\,\widetilde{\sigma}(u) = (L\log\Phi)\,\Phi\,\widetilde{\sigma}(u),$$

つまり $\big(L - H^L - L\,(\log\,\Phi)\big)\big(\Phi\,\widetilde{\sigma}(u)\big) = 0.$

もし $\Phi\,\widetilde{\sigma}(u)$ が求める $\sigma(u) \in \mathbb{Q}[a][[u]]$ であるならば, $L\,(\log\,\Phi)\cdot\Phi\,\widetilde{\sigma}(u) \in \mathbb{Q}[a][[u]]$ である.

ゆゑに, $\sigma(u) = \Phi\cdot\widetilde{\sigma}(u)$ であると期待するなら, $L(\log\Phi) \in \mathbb{Q}[a]$ なる L を探すべきである.

帰 結　以上より,

$L(\log\Delta) \in \mathbb{Q}[a]$ なる $L \in \bigoplus_j \mathbb{Q}[\boldsymbol{a}]\dfrac{\partial}{\partial a_j}$ について

$$\big(L - H^L - L(\log\Delta^{-\frac{1}{8}})\big)\,\hat{\sigma}(u) = 0.$$

従つて, L を探すためには $\Delta = 0$ の tangent vector fields を調べればよい.

Modality, Weierstrass Form

(e,q)-curve の方程式 $f(x,y)=0$ に対して $\mathbb{Q}[a]$ 上の Tschirnhaus 変換で x^{q-1} と y^{e-1} の項を落した形のものを Weierstrass form と呼ぶ.
Weierstrass form に変換したのち a_j の名前を元の付け方に戻したものを考へる.
例へば $(e,q)=(3,4)$ ならば

$$f = y^3 \cancel{+(a_1x+a_4)y^2} + (a_2x^2+a_5x+a_8)y - (x^4 \cancel{+a_3x^3} + a_6x^2+a_9x+a_{12}).$$

その結果残る a_j の個数は $2g$ 以下になるが, この差を modality と呼ぶ.
Modality は

$$(e-3)(q-3)+\left\lfloor \frac{q}{e} \right\rfloor - 1$$

で与へられる.　特に

$$\text{modality}=0 \iff (e,q)=(2,2g+1),\ (3,4),\ (3,5).$$

The Operator Space which is tangent to the Discriminant Variety (1)

How to find L? (2,5)-case を例にして説明する.

$\mathbb{Q}[\boldsymbol{a}]$ 上の加群 $\mathbb{Q}[\boldsymbol{a}][x,y]/(\frac{\partial}{\partial x}f, \frac{\partial}{\partial y}f)$ （modality 0 ゆゑ階数は $2g=4$）から自身への $-eq\cdot f(x,y)$ 倍写像 を考へて,

基底 $\check{M}(x,y) = [\, x^3 \;\; x^2 \;\; x \;\; 1\,]$ に関する表現行列を $T = [\,T_{ij}\,]$ （$2g=4$ 次正方行列）とおくと

$$\det(T) = \text{“non-zero rational”}\cdot \varDelta.$$

しかし, この T より, 以下の様に簡単に計算される対称行列 V の方がはるかに有用である.

$$H := \frac{1}{2}\begin{vmatrix} \dfrac{f_1(x,y)-f_1(z,w)}{x-z} & \dfrac{f_2(x,y)-f_2(z,w)}{x-z} \\ \dfrac{f_1(z,y)-f_1(x,w)}{y-w} & \dfrac{f_2(z,y)-f_2(x,w)}{y-w} \end{vmatrix}, \quad f_1(x,y) = \frac{\partial}{\partial x}f(x,y), \quad f_2(x,y) = \frac{\partial}{\partial y}f(x,y),$$

$$\check{M} = [\, x^3 \;\; x^2 \;\; x \;\; 1\,] \quad (4 = 2g)$$

とおき, $\mathbb{Q}\big[x,y,z,w\big]\big/\big(f_1(x,y), f_2(x,y), f_1(z,w), f_2(z,w)\big)$ の中で

$${}^t\check{M}(x,y)\, V\, \check{M}(z,w) = f(x,y)\, H$$

で $V = [V_{ij}] \in \mathrm{Mat}\,(2g, \mathbb{Q}[\boldsymbol{a}])$ を定義する.

このとき V は T に簡単な左上三角行列を掛けたものになることが容易にわかり,

$$\det(V) = \det(T) = \text{“non-zero rational”}\cdot \varDelta$$

となる.

The Operator Space which is tangent to the Discriminant Variety (2)

さうして $L_{v_i} = \sum_{j=1}^{2g} V_{ij} \dfrac{\partial}{\partial a_{eq-v_j}}$ とおく.

ここに $j \mapsto v_j$ は $\mathrm{wt}\,(L_{v_j}) = -v_j$ となる様に仕組んだ函数.

これらの L_{v_j} が丁度 $\Delta = 0$ を極大積分多様体とする operators の空間を張る.

（齋藤恭司氏の理論）

例へば $(e, q) = (2, 3)$ の場合

$$[\,x \;\; 1\,]\begin{bmatrix} 4a_4 & 6a_6 \\ 6a_6 & -\frac{4}{3}{a_4}^2 \end{bmatrix}\begin{bmatrix} z \\ 1 \end{bmatrix} \equiv \left((y^2 - (x^3 + a_4 x + a_6)\right) \cdot 6(x+z)$$

$$\mod (2y,\ 3x^2 + a_4,\ 2w,\ z^2 + a_4 z)$$

であり, $\det(V) = \frac{1}{12}\Delta$ である. また $v_1 = 0$, $v_2 = 2$ であつて, L_0 と L_2 は

$$L_0 = 4a_4 \frac{\partial}{\partial a_4} + 6a_6 \frac{\partial}{\partial a_6},$$

$$L_2 = 6a_6 \frac{\partial}{\partial a_4} - \frac{4}{3}{a_4}^2 \frac{\partial}{\partial a_6}$$

と定義される.

Values $L\log(\Delta) = L(\Delta)/\Delta$

Let $(\boldsymbol{\omega} \;\; \boldsymbol{\eta}) = M(x,y)\,\omega_1$ be the canonical symplectic basis of $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\boldsymbol{a}])$.

For example, if $\mathscr{C}$ is the $(2, 2g+1)$-curve, we have

$$M(x,y) = [\,1 \;\; x \;\; \cdots \;\; x^{2g-1}\,], \qquad \omega_1 = \frac{1}{2y}\,dx.$$

Then we have ([BL2005], proved by S. Yasuda)

Proposition

In the module $\mathbb{Q}[a][x,y]/(\frac{\partial}{\partial x}f, \frac{\partial}{\partial y}f)$, *we have*

$$\check{M}(x,y)\;{}^t[\,L_{v_1} \;\; \cdots \;\; L_{v_{2g}}\,](\Delta) = \mathrm{Hess}\,f(x,y)\cdot\Delta.$$

(これはかなり有用な公式！ しかしあまり知られてゐない？ 齋藤恭司氏の理論にある ?)

この式から, 明らかに

$$L_{v_j}(\log\Delta) = \frac{L(\Delta)}{\Delta} \in \mathbb{Q}[a].$$

Theorem (E-G-Ô-Y)

級数 $\varphi(u) \in \mathbb{Q}[\boldsymbol{a}][[u]]$ $(u = (u_{w_g}, \cdots, u_{w_1}))$ についての方程式系

$$\left(L_{v_j} - H^{L_{v_j}} - L_{v_j}(\log \Delta^{-\frac{1}{8}})\right) \varphi(u) = 0 \quad (j = 0, \cdots, 2g)$$

の解空間は, 種数 3 以下の場合

$$(e, q) = (2, 3), \ (2, 5), \ (2, 7), \ (3, 4)$$

のすべてにおいて 1 次元, つまり $\hat{\sigma}(u) = \Delta^{-\frac{1}{8}} \widetilde{\sigma}(u)$ の絶対定数倍の全体である.

Proof.

We can explicitly construct a recursion system on the coefficients, and check uniqueness of the solution once the initial coefficient is given. It is easy to check the solution is independent of the choice of such a recursion system. □

—. It would be very nice if one has find a general proof which reveals intrinsic structure of the heat equations for any plane telescopic curve.

—. A generalization of Jacobi's derivative formula.

Sample calculation in the $(2,3)$ case (1)

$(e,q)=(2,3)$ の場合： $f(x,y)=y^2-(x^3+a_4x+a_6)$.

$$\check{M}=[\ x\ \ 1\ ],\quad f_1(x,y)=\frac{\partial}{\partial x}f(x,y)=-3x^2-a_4,\quad f_2(x,y)=\frac{\partial}{\partial y}f(x,y)=2y.$$

$$H:=\frac{1}{2}\begin{vmatrix} \dfrac{f_1(x,y)-f_1(z,w)}{x-z} & \dfrac{f_2(x,y)-f_2(z,w)}{x-z} \\ \dfrac{f_1(z,y)-f_1(x,w)}{y-w} & \dfrac{f_2(z,y)-f_2(x,w)}{y-w} \end{vmatrix}=6(x+z)$$

となり，$\mathbb{Q}[a][x,y,z,w]\,/\,(\,f_1(x,y),\ f_2(x,y),\ f_1(z,w),\ f_2(z,w)\,)$ の中で

$$ {}^t\check{M}(x,y)\,V\,\check{M}(z,w)=f(x,y)\,H=4a_4xz+6a_6z+6a_6x-\frac{4}{3}{a_4}^2.$$

よって
$$V=\begin{bmatrix} 4a_4 & 6a_6 \\ 6a_6 & -\frac{4}{3}{a_4}^2 \end{bmatrix}$$

であり，確かに $\det(V)=\frac{1}{12}\Delta$ である.

以上により L_0 と L_2 は

$$L_0=4a_4\frac{\partial}{\partial a_4}+6a_6\frac{\partial}{\partial a_6},\quad L_2=6a_6\frac{\partial}{\partial a_4}-\frac{4}{3}{a_4}^2\frac{\partial}{\partial a_6}.$$

Sample calculation in the $(2,3)$ case (2)

Choose the differential forms and the local parameter by

$$(\,\boldsymbol{\omega}\ \ \boldsymbol{\eta}\,) = (\,\omega_1,\ \eta_{-1}\,) = \left(\frac{dx}{2y},\ \ \frac{xdx}{2y}\right), \qquad t = x^{-\frac{1}{2}},$$

and suppose $\frac{\partial}{\partial a_j} t = 0$ for any j.

So, we have $\frac{\partial}{\partial a_j} x = 0$ for $j = 4, 6$, and we compute the matrix Γ as follows.

Using $f(x,y) = y^2 - (x^3 + a_4 x + a_6)$, we see $2y\frac{\partial}{\partial a_4} y = x$ and $2y\frac{\partial}{\partial a_6} y = 1$, so that

$$\frac{\partial}{\partial a_4} y = \frac{x}{2y}, \quad \frac{\partial}{\partial a_6} y = \frac{1}{2y}.$$

Therefore, we have

$$\frac{\partial}{\partial a_6}\omega_1 = -\frac{1}{4y^3}dx, \quad \frac{\partial}{\partial a_4}\omega_1 = \frac{\partial}{\partial a_6}\eta_{-1} = \frac{x}{4y^3}dx, \quad \frac{\partial}{\partial a_4}\eta_{-1} = \frac{x^2}{4y^3}dx.$$

By computing $d\left(\frac{1}{y}\right)$, $d\left(\frac{x}{y}\right)$, $d\left(\frac{x^2}{y}\right)$, we get

$$L_0(\omega_1) = -\omega_1 + d\left(\frac{1}{y}\right), \qquad L_0(\eta_{-1}) = \eta_1 - d\left(\frac{x^2}{y}\right),$$

$$L_2(\omega_1) = \eta_{-1} - d\left(\frac{x^2}{y}\right) - \frac{2}{3}\, d\left(\frac{1}{y}\right), \quad L_2(\eta_{-1}) = \frac{a_4}{3}\,\omega_1 + a_6\, d\left(\frac{1}{y}\right).$$

Sample calculation in the $(2,3)$ case (3)

We have got

$$L_0(\omega_1) = -\omega_1 + d\Big(\frac{1}{y}\Big), \qquad L_0(\eta_{-1}) = \eta_{-1} - d\Big(\frac{x^2}{y}\Big),$$

$$L_2(\omega_1) = \eta_{-1} - d\Big(\frac{x^2}{y}\Big) - \frac{2}{3}\, d\Big(\frac{1}{y}\Big), \quad L_2(\eta_{-1}) = \frac{a_4}{3}\,\omega_1 + a_6\, d\Big(\frac{1}{y}\Big).$$

Therefore, we have on $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\boldsymbol{a}])$ that

$$L_0\,((\omega_1\ \ \eta_{-1})) = (\omega_1\ \ \eta_{-1})\,{}^t\Gamma_0,\ \ L_2\,((\omega_1\ \ \eta_{-1})) = (\omega_1\ \ \eta_{-1})\,{}^t\Gamma_2,\ \ \text{where}\ \ \Gamma_0 = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix},\ \ \Gamma_2 = \begin{bmatrix} & 1 \\ \frac{a_4}{3} & \end{bmatrix}.$$

Since $\operatorname{Hess} f = -12x$, we have

$$[\,L_0\ \ L_2\,](\Delta) = [\,12\ \ 0\,]\,\Delta,$$

we have arrived at

$$(L_0 - H_0)\,\hat{\sigma}(u) = \Big(4a_4\frac{\partial}{\partial a_4} + 6a_6\frac{\partial}{\partial a_6} - u\frac{\partial}{\partial u} + 1\Big)\hat{\sigma}(u) = 0,$$

$$(L_2 - H_2)\,\hat{\sigma}(u) = \Big(6a_6\frac{\partial}{\partial a_4} - \frac{4}{3}{a_4}^2\frac{\partial}{\partial a_6} - \frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{6}a_4u^2\Big)\hat{\sigma}(u) = 0,$$

where $H_j = H^{L_j} + \frac{1}{8}L_j \log\Delta$ for $j = 0$ and 2.

Explicit Expression of the Heat Operators (1)

For $(2,5)$-curve $y^2 = x^5 + a_2x^4 + a_4x^3 + a_6x^2 + a_8x + a_{10}$, we have

$$\omega_3 = \frac{dx}{2y},\ \ \omega_1 = \frac{xdx}{2y},\ \ \eta_{-1} = \frac{x^2dx}{2y},\ \ \eta_{-3} = \frac{(3x^3 + 2a_2x^2 + a_4x)dx}{2y},$$

$$\begin{aligned}
L_0 &= 4a_4\tfrac{\partial}{\partial a_4} + 6a_6\tfrac{\partial}{\partial a_6} + 8a_8\tfrac{\partial}{\partial a_8} + 10a_{10}\tfrac{\partial}{\partial a_{10}},\\
L_2 &= 6a_6\tfrac{\partial}{\partial a_4} + \tfrac{4}{5}(10a_8 - 3{a_4}^2)\tfrac{\partial}{\partial a_6} + \tfrac{2}{5}(25a_{10} - 4a_6a_4)\tfrac{\partial}{\partial a_8} - \tfrac{4}{5}a_8a_4\tfrac{\partial}{\partial a_{10}},\\
L_4 &= 8a_8\tfrac{\partial}{\partial a_4} + \tfrac{2}{5}(25a_{10} - 4a_4a_6)\tfrac{\partial}{\partial a_6} + \tfrac{4}{5}(5a_4a_8 - 3{a_6}^2)\tfrac{\partial}{\partial a_8} + \tfrac{6}{5}(5a_4a_{10} - a_6a_8)\tfrac{\partial}{\partial a_{10}},\\
L_6 &= 10a_{10}\tfrac{\partial}{\partial a_4} - \tfrac{4}{5}a_8a_4\tfrac{\partial}{\partial a_6} + \tfrac{6}{5}(5a_{10}a_4 - a_8a_6)\tfrac{\partial}{\partial a_8} + \tfrac{4}{5}(5a_{10}a_6 - 2{a_8}^2)\tfrac{\partial}{\partial a_{10}}.
\end{aligned}$$

The action of these operators on the discriminant Δ is as follows:

$$[\,L_0\ \ L_2\ \ L_4\ \ L_6\,]\Delta = [\,40\quad 0\quad 12\,a_4\ \ 4\,a_6\,]\Delta.$$

The representation matrices Γ_j for L_j's acting on the space $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\boldsymbol{a}])$ are

$$\Gamma_0 = \begin{bmatrix} -3 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix},\quad \Gamma_2 = \begin{bmatrix} 0 & -1 & 0 & 0\\ \frac{4}{5}\,a_4 & 0 & 0 & 1\\ \frac{4}{5}\,{a_4}^2 - 3\,a_8 & 0 & 0 & -\frac{4}{5}\,a_4\\ 0 & \frac{3}{5}\,a_4 & 1 & 0 \end{bmatrix},$$

$$\Gamma_4 = \begin{bmatrix} -a_4 & 0 & 0 & 1\\ \frac{6}{5}\,a_6 & 0 & 1 & 0\\ \frac{6}{5}\,a_4a_6 - 6\,a_{10} & -a_8 & a_4 & -\frac{6}{5}\,a_6\\ -a_8 & \frac{2}{5}\,a_6 & 0 & 0 \end{bmatrix},\quad \Gamma_6 = \begin{bmatrix} 0 & 0 & 1 & 0\\ \frac{3}{5}\,a_8 & 0 & 0 & 0\\ \frac{3}{5}\,a_4a_8 & -2\,a_{10} & 0 & -\frac{3}{5}\,a_8\\ -2\,a_{10} & \frac{1}{5}\,a_8 & 0 & 0 \end{bmatrix}.$$

Explicit Expression of the Heat Operators (2)

Therefore, we have the operators $H_j = H^{L_j} + \frac{1}{8} L_j \log \Delta$ as follows:

$$H_0 = 3u_3 \frac{\partial}{\partial u_3} + u_1 \frac{\partial}{\partial u_1} + 3,$$

$$H_2 = \frac{1}{2} \frac{\partial^2}{\partial {u_1}^2} + u_1 \frac{\partial}{\partial u_3} - \frac{4}{5} a_4 u_3 \frac{\partial}{\partial u_1} - \frac{3}{10} a_4 {u_1}^2 - \left(\frac{3}{2} a_8 - \frac{2}{5} {a_4}^2\right) {u_3}^2,$$

$$H_4 = \frac{\partial^2}{\partial u_1 \partial u_3} - \frac{6}{5} a_6 u_3 \frac{\partial}{\partial u_1} + a_4 u_3 \frac{\partial}{\partial u_3} - \frac{1}{5} a_6 {u_1}^2 + a_8 u_1 u_3 - \left(\frac{3}{5} a_4 a_6 + 2a_{10}\right) {u_3}^2 + a_4,$$

$$H_6 = \frac{1}{2} \frac{\partial^2}{\partial {u_3}^2} - \frac{3}{5} a_8 u_3 \frac{\partial}{\partial u_1} - \frac{1}{10} a_8 {u_1}^2 + 2a_{10} u_3 u_1 - \frac{3}{10} a_8 a_4 {u_3}^2 - \frac{1}{2} a_6.$$

By the first heat equation $(L_0 - H_0)\,\sigma(u) = 0$, where

$$L_0 - H_0 = \left(4a_4 \tfrac{\partial}{\partial a_4} + 6a_6 \tfrac{\partial}{\partial a_6} + 8a_8 \tfrac{\partial}{\partial a_8} + 10a_{10} \tfrac{\partial}{\partial a_{10}}\right) - \left(3u_3 \tfrac{\partial}{\partial u_3} - u_1 \tfrac{\partial}{\partial u_1} + 3\right),$$

the sigma function should be of the form

$$\sigma(u_3, u_1) = \sum_{\substack{m, n_4, n_6, n_8, n_{10} \geq 0 \\ 3-3m+4n_4+6n_6+8n_8+10n_{10} \geq 0}} b(m, n_1, n_4, n_6, n_8, n_{10}) \frac{{u_1}^3 \left(\frac{u_3}{{u_1}^3}\right)^m \left(a_4 {u_1}^4\right)^{n_4} \left(a_6 {u_1}^6\right)^{n_6} \left(a_8 {u_1}^8\right)^{n_8} \left(a_{10} {u_1}^{10}\right)^{n_{10}}}{m!\,(3 - 3m + 4n_4 + 6n_6 + 8n_8 + 10n_{10})!}.$$

Recursion in the $(2,5)$-case [BL2005]

Let $k = 3 - 3m + 4n_4 + 6n_6 + 8n_8 + 10n_{10}$.

Then the other heat equations $(L_j - H_j)\,\sigma(u) = 0$ $(j = 2, 4, 6)$ imply the following recursion relations:

$$
\begin{aligned}
&b(m, n_4, n_6, n_8, n_{10}) \\
&= \begin{cases}
20(n_8+1)\,b(m, n_4, n_6, n_8+1, n_{10}-1) \\
+16(n_6+1)\,b(m, n_4, n_6+1, n_8-1, n_{10}) \\
+12(n_4+1)\,b(m, n_4+1, n_6-1, n_8, n_{10}) \\
-\frac{24}{5}(n_6+1)\,b(m, n_4-2, n_6+1, n_8, n_{10}) \\
+\frac{3}{5}(-k+3)(-k+2)\,b(m, n_4-1, n_6, n_8, n_{10}) \\
-\frac{8}{5}(n_{10}+1)\,b(m, n_4-1, n_6, n_8-1, n_{10}+1) \\
-\frac{16}{5}(n_8+1)\,b(m, n_4-1, n_6-1, n_8+1, n_{10}) \\
+2(-k+2)\,b(m+1, n_4, n_6, n_8, n_{10}) \\
-3m(m-1)\,b(m-2, n_4, n_6, n_8-1, n_{10}) \\
+\frac{4}{5}m(m-1)\,b(m-2, n_4-2, n_6, n_8, n_{10}) \\
+\frac{8}{5}m\,b(m-1, n_4-1, n_6, n_8, n_{10} & (\text{if } k > 1 \text{ and } m \geqq 0), \\
10(n_6+1)\,b(m-1, n_4, n_6+1, n_8, n_{10}-1) + \cdots & (\text{if } k = 1 \text{ and } m > 0), \\
-\frac{16}{5}(1+n_{10})\,b(m-2, n_4, n_6, n_8-2, n_{10}+1) + \cdots & (\text{if } k = 0 \text{ and } m > 1).
\end{cases}
\end{aligned}
$$

The Expansion of $(2,5)$-Sigma Function

From these, we see the expansion of $\sigma(u)$ is Hurwitz integral over $\mathbb{Z}_{(5)}$:

$$\sigma(u_3, u_1) = u_3 - 2\frac{{u_1}^3}{3!} - 4a_4\frac{{u_1}^7}{7!} - 2a_4\frac{u_3{u_1}^4}{4!} + 64a_6\frac{{u_1}^9}{9!} - 8a_6\frac{u_3{u_1}^6}{6!} - 2a_6\frac{{u_3}^2{u_1}^3}{2!3!} + a_6\frac{{u_3}^3}{3!}$$
$$+ (-408{a_4}^2 + 1600a_8)\frac{{u_1}^{11}}{11!} - (4{a_4}^2 + 32a_8)\frac{u_3{u_1}^8}{8!} - 8a_8\frac{{u_3}^2{u_1}^5}{2!5!} - 2a_8\frac{{u_3}^3{u_1}^2}{3!2!} + \cdots .$$

The recursion in $(3,4)$-case

We have 6 heat equations $(L_j - H_j)\sigma(u) = 0$ for $j = 0,\ 3,\ 4,\ 6,\ 7, 10.$

Thus, 6 recursion relations.

The first equation $(L_0 - H_0)\sigma(u) = 0$ implies the sigma is of the form

$$\sigma(u_5, u_3, u_1) = \sum_{\substack{\ell, m, n_4, n_6, n_8, \\ n_{10}, n_{12}, n_{14}}} b(\ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14})$$

$$\cdot\, {u_1}^6 \Big(\frac{u_5}{{u_1}^5}\Big)^{\ell} \Big(\frac{u_3}{{u_1}^3}\Big)^{m} \cdot \big(a_4 {u_1}^4\big)^{n_4} \big(a_6 {u_1}^6\big)^{n_6} \big(a_8 {u_1}^8\big)^{n_8} \big(a_{10} {u_1}^{10}\big)^{n_{10}} \big(a_{12} {u_1}^{12}\big)^{n_{12}} \big(a_{14} {u_1}^{14}\big)^{n_{14}}$$

$$\Big/ (6 - 5\ell - 3m + 4n_4 + 6n_6 + 8n_8 + 10n_{10} + 12n_{12} + 14n_{14})!\, \ell!\, m!\ .$$

The set of rest 5 recursion relations indeed gives the sigma function.

However, we need a kind of “switch back” on weight.

Expansion of the $(3, 4)$-Sigma Function

ここでは

$$f(x, y) = y^3 + (a_1x + a_4)y^2 + (a_2x^2 + a_5x + a_8)y - (x^4 + a_3x^3 + a_6x^2 + a_9x + a_{12})$$

に対するものを書いておく. $a_1 = a_4 = a_2 = a_5 = a_8 = 0$ とすれば, 今回の結果となる.

$$\sigma(u) = C_5 + C_6 + C_7 + C_8 + C_9 + \cdots$$

と u に関する weight ごとに分けると

$$C_5 = u_5 - u_1{u_2}^2 + 6\frac{{u_1}^5}{5!},$$

$$C_6 = 2a_1\frac{{u_1}^4}{4!}\frac{u_2}{1!} - 2a_1\frac{{u_2}^3}{3!},$$

$$C_7 = 10({a_1}^2 - 3a_2)\frac{{u_1}^7}{7!} + 2a_2\frac{{u_1}^3}{3!}\frac{{u_2}^2}{2!},$$

$$C_8 = 2({a_1}^3 + 9a_3 - 2a_1a_2)\frac{{u_1}^6}{6!}\frac{u_2}{1!} - 6a_3\frac{{u_1}^2}{2!}\frac{{u_2}^3}{3!},$$

$$C_9 = 14({a_1}^2 - 3a_2)^2\frac{{u_1}^9}{9!} + 2(2a_4 - {a_2}^2 + {a_1}^2a_2 + 6a_1a_3)\frac{{u_1}^5}{5!}\frac{{u_2}^2}{2!}$$

$$- 2(4a_1a_3 + 4a_4 + {a_2}^2)\frac{u_1}{1!}\frac{{u_2}^4}{4!} + 2a_4\frac{{u_1}^4}{4!}\frac{u_5}{1!},$$

$$\cdots\cdots\cdots.$$

Problem

We have a proof of the one-dimensionality of the solution space of the system of heat equations only for any plane telescopic curves of genus 3 or smaller. Our proof of the one-dimensionality is a consequence of expected good behalf of the recursion system. However, the $(3,4)$-recursion is so complicated. So, we shall pose the following

Problem

Can we prove one-dimensionality of the solution space of the system of heat equations

$$\left\{ (L_{v_j} - H_{v_j})\,\sigma(u) = 0 \,\middle|\, j = 0,\ 1,\ \cdots,\ 2g \right\}$$

for any telescopic curve of modality one?

It might be a hint for this problem that
there is the following closed form of entries of the matrix V for $(2,q)$-case, which is given by JCE.

Lemma

$$\begin{aligned} V_{ij} &= -\frac{2i(q-j)}{q}\,a_{2i}a_{2j} + \sum_{m=1}^{m_0} 2(j-i+2m)\,a_{2(i-m)}\,a_{2(j+m)} \\ &= -\frac{2i(q-j)}{q}\,a_{2i}a_{2j} + \sum_{\ell=\ell_0}^{i-1 \text{ or } j} 2(i+j-2\ell)\,a_{2\ell}\,a_{2(i+j-\ell)}, \end{aligned}$$

where $a_0 = 1$, $a_2 = 0$, $m_0 = \min\{i,\ q-j\}$, *and* $\ell_0 = \max\{0, i+j-q\}$.