# Further generalization of the addition formula of Frobenius-Stickelberger to higher genus Abelian functions

*( joint work with John Christopher Eilbeck and Matthew England )*

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# Contents

## The Double Gamma function and the Weierstrass $\sigma$ function

From E.W. Barnes : *The Theory of Double Gamma Function* (p.310).

$$\sigma(z) = e^{-\mu z - \nu \frac{z^2}{2}} \cdot z \cdot \frac{\Pi \Gamma_2^{-1}(z \mid \pm \omega_1, \pm \omega_2)}{\Pi \Gamma_1^{-1}(z \mid \pm \omega_1) \Pi \Gamma_1^{-1}(z \mid \pm \omega_2)},$$

# Main references


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# Introduction

Let $\wp(u)$ and $\sigma(u)$ be the Weierstrass functions satisfying

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\,\wp(u) - g_3,$$

$$\sigma(u) = u \exp\left\{ \int_0^u \int_0^u \left( \wp(u) - \frac{1}{u^2} \right) du du \right\}, \quad \wp(u) = -\frac{d^2}{du^2} \log \sigma(u).$$

Then we have ((Hermite and) Frobenius-Stickelberger, 1877)

$$\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma(u)^2\,\sigma(v)^2} = \wp(v) - \wp(u) \quad \left( = \begin{vmatrix} 1 & \wp(u) \\ 1 & \wp(v) \end{vmatrix} \right),$$

$$\frac{\sigma\big(u^{(1)} + u^{(2)} + \cdots + u^{(n)}\big) \prod_{i<j} \sigma\big(u^{(i)} - u^{(j)}\big)}{\prod_{j=1}^{n} \sigma\big(u^{(j)}\big)^n} = \frac{1}{\prod_j j!} \begin{vmatrix} 1 & \wp(u^{(1)}) & \wp'(u^{(1)}) & \cdots & \wp^{(n-2)}(u^{(1)}) \\ 1 & \wp(u^{(2)}) & \wp'(u^{(2)}) & \cdots & \wp^{(n-2)}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(u^{(n)}) & \wp'(u^{(n)}) & \cdots & \wp^{(n-2)}(u^{(n)}) \end{vmatrix}.$$

These formulae correspond to the canonical involution $v \mapsto -v$.

Today I will talk on an extreme and elaborate generalization of these addition formulae.

# I-1. The most general genus one curve

To step up higher genus cases smoothly, we reformulate the equalities for genus 1 case.
We start with the most general genus one curve $\mathscr{C} : f(x,y) = 0$ (not with $\wp(u)$), where

$$f(x,y) = y^2 + (\mu_1 x + \mu_3)y - (x^3 + \mu_2 x^2 + \mu_4 x + \mu_6),$$
$$\mathbf{wt}(x) = -2, \ \mathbf{wt}(y) = -3, \ \ \mathbf{wt}(\mu_j) = -j,$$

with the point $\infty$ at infinity. Then

$$H^1_{\mathrm{dR}}(\mathscr{C}/\mathbf{Q}[\mu]) \cong \left\{ \frac{h(x,y)\,dx}{f_y(x,y)} \,\middle|\, h(x,y) \in \mathbf{Q}[\mu][x,y] \right\} \Big/ d\mathbf{Q}[\mu][x,y]$$
$$= \mathbf{Q}[\mu]\,\frac{dx}{f_y} + \mathbf{Q}[\mu]\,\frac{xdx}{f_y} \ \left( = \mathbf{Q}[\mu]\,\omega + \mathbf{Q}[\mu]\,\eta. \right)$$

(Note that $f_x(x,y)\,dx + f_y(x,y)\,dy = 0$. )
Let $x(u)$ and $y(u)$ be the inverse functions defined by

$$u = \int_\infty^{(x(u),y(u))} \omega.$$

Then

$$x(u) = \frac{1}{u^2} + \cdots, \quad y(u) = -\frac{1}{u^3} + \cdots.$$

# I-2. Sigma function for the most general genus 1 curve

The sigma function $\sigma(u)$ associate to the genus 1 curve is

$$\sigma(u) = \left(\frac{2\pi}{\omega'}\right)^{1/2} \Delta^{-\frac{1}{8}} \cdot \exp\left(-\tfrac{1}{2}\omega'^{-1}\eta' u^2\right) \cdot \vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\omega'^{-1}u,\ \omega''/\omega'),$$

where $\Delta =$ the discriminant of $\mathscr{C}$,

$$\begin{bmatrix}\omega' & \omega'' \\ \eta' & \eta''\end{bmatrix} = \begin{bmatrix}\int_{\alpha_1}\omega & \int_{\beta_1}\omega \\ \int_{\alpha_1}\eta & \int_{\beta_1}\eta\end{bmatrix} \quad \text{with} \quad \omega = \frac{dx}{f_y}, \quad \eta = \frac{xdx}{f_y}$$

and $\{\alpha_1, \beta_1\}$ is a symplectic basis of $H_1(\mathscr{C}^{\mathrm{an}}, \mathbf{Z})$.

However, $\sigma(u)$ is modular invariant. Indeed we have more tightly

$$\sigma(u) = u + \left(\left(\frac{\mu_1}{2}\right)^2 + \mu_2\right)\frac{u^3}{3!} + \cdots \in \mathbf{Z}[\mu, \tfrac{\mu_1}{2}]\langle\langle u \rangle\rangle \qquad \text{( Hurwitz-integral series ).}$$

We define

$$\wp(u) := -\frac{d^2}{du^2}\log\sigma(u).$$

Then, we have the solution to Jacobi's Umkehr problem

$$\wp(u) = x(u), \quad \wp'(u) = 2y(u) + \mu_1 x(u) + \mu_3.$$

Then we have

$$\sigma(u^{(1)}+u^{(2)}+\cdots+u^{(n)})\prod_{i<j}\sigma(u^{(i)}-u^{(j)}) \Big/ \prod_{j}\sigma(u^{(j)})^n$$

$$= \frac{1}{\prod_j j!}\begin{vmatrix} 1 & \wp(u^{(1)}) & \wp'(u^{(1)}) & \cdots & \wp^{(n-2)}(u^{(1)}) \\ 1 & \wp(u^{(2)}) & \wp'(u^{(2)}) & \cdots & \wp^{(n-2)}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(u^{(n)}) & \wp'(u^{(n)}) & \cdots & \wp^{(n-2)}(u^{(n)}) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & yx(u^{(1)}) & x^3(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & yx(u^{(2)}) & x^3(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & yx(u^{(n)}) & x^3(u^{(n)}) & \cdots \end{vmatrix}.$$

## II-1. Sample of the main results  $(3,4)$-curve, genus $g = 3$

Suppose we have defined the multivariate $\sigma(u) = \sigma(u_5, u_2, u_1)$.

The $n$-variable case (Here $n \geq 3$ for simplicity) for

$$\mathscr{C} \ : \ y^3 - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}) = 0.$$

Theorem. [Ô, 2011] Let $[\zeta]$ be the natural action of $\zeta = \exp\frac{2\pi i}{3}$. Then

$$\sigma\left(u^{(1)} + \cdots + u^{(n)}\right) \prod_{i<j} \sigma_1\left(u^{(i)} + [\zeta]u^{(j)}\right) \sigma_1\left(u^{(i)} + [\zeta]^2 u^{(j)}\right) \Big/ \prod_j \sigma_2\left(u^{(j)}\right)^{2n-1}$$

$$= \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & yx(u^{(1)}) & y^2(u^{(1)}) & x^3(u^{(1)}) & yx^2(u^{(1)}) & y^2x(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & yx(u^{(2)}) & y^2(u^{(2)}) & x^3(u^{(2)}) & yx^2(u^{(2)}) & y^2x(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & yx(u^{(n)}) & y^2(u^{(n)}) & x^3(u^{(n)}) & yx^2(u^{(n)}) & y^2x(u^{(n)}) & \cdots \end{vmatrix}$$

$$\cdot \begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & \cdots & x^{n-1}(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & \cdots & x^{n-1}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^{n-1}(u^{(n)}) \end{vmatrix}.$$

Here $u^{(j)} = (\,{u^{(j)}}_5,\ {u^{(j)}}_2,\ {u^{(j)}}_1\,)$'s are variables on the 1st stratum.

# II-2. Another result $\boxed{(3,4)\text{-curve, } g=3}$

Suppose we have defined the multivariate $\sigma(u) = \sigma(u_5, u_2, u_1)$.

We define $\wp$-functions by

$$\wp_{ij}(u) := -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u), \quad \wp_{ijk}(u) := \frac{\partial}{\partial u_k} \wp_{ij}(u), \text{ etc.}$$

Then, we have a beautiful solution (explained later) to **Jacobi's Umkehr Problem**, and
$\wp_{ij}(u) \in \Gamma\Big(\mathrm{Jac}(\mathscr{C}), \mathscr{O}(2\Theta^{[g-1]})\Big)$, $\quad \wp_{ijk}(u) \in \Gamma\Big(\mathrm{Jac}(\mathscr{C}), \mathscr{O}(3\Theta^{[g-1]})\Big)$, etc.

The case of the $(3,4)$-curve on the largest stratum in $2$ variables :

Theorem. [EEMÔP, 2008] For $u,\ v \in \mathbf{C}^3 = \kappa^{-1}(W^{[3]})$, we have

$$\begin{aligned}\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} &= -\wp_{55}(u) + \wp_{55}(v) - \wp_{52}(u)\wp_{21}(v) + \wp_{52}(v)\wp_{21}(u)\\ &- \wp_{51}(u)\wp_{22}(v) + \wp_{51}(v)\wp_{22}(u) - \tfrac{1}{3}\big(\wp_{11}(u)\,Q_{5111}(v) - \wp_{11}(v)\,Q_{5111}(u)\big)\\ &+ \tfrac{1}{3}\mu_1\big(\wp_{52}(u)\wp_{11}(v) - \wp_{52}(v)\wp_{11}(u)\big) + \mu_1\big(\wp_{51}(u)\wp_{21}(v) - \wp_{51}(v)\wp_{21}(u)\big)\\ &- \tfrac{1}{3}({\mu_1}^2 - \mu_2)\big(\wp_{51}(u)\wp_{11}(v) - \wp_{51}(v)\wp_{11}(u)\big) - \tfrac{1}{3}\mu_8\big(\wp_{11}(u) - \wp_{11}(v)\big),\end{aligned}$$

where $\quad Q_{5111} = \wp_{5111} - 6\wp_{51}\wp_{11}$.

# II-3. One more example $\boxed{(3,4)\text{-curve, } g=3}$

Theorem. [EEMÔP] (2008)

$$\frac{\sigma(u+v)\,\sigma(u+[\zeta]v)\,\sigma(u+[\zeta^2]v)}{\sigma(u)^3\,\sigma(v)^3} = R(u,v)+R(v,u),$$

where

$$\begin{aligned} R(u,v) = &-\tfrac{1}{3}\wp_{51}(u)\,\tfrac{\partial}{\partial u_1}Q_{5111}(v) - \tfrac{3}{4}\wp_{21}(u)\wp_{552}(v) - \tfrac{1}{2}\wp_{555}(u) \\ &+ \tfrac{1}{4}\wp_{522}(u)\wp^{[55]}(v) - \tfrac{1}{4}\wp_{222}(u)\wp^{[52]}(v) + \tfrac{1}{12}\,\tfrac{\partial}{\partial u_1}Q_{5111}(u)\wp^{[55]}(v) \\ &+ \tfrac{1}{2}\wp_{111}(u)\wp^{[22]}(v) - \tfrac{1}{4}\mu_1\wp_{111}(u)\wp^{[52]}(v) \\ &+ \tfrac{1}{2}\mu_6\wp_{51}(u)\wp_{111}(v) - \tfrac{1}{4}\mu_9\wp_{21}(u)\wp_{111}(v) - \tfrac{1}{2}\mu_{52}\wp_{111}(u) \end{aligned}$$

with

$$\wp^{[ij]} = \text{“the determinant of the } (i,j)\text{-(complementary) minor of } [\wp_{ij}]_{3\times 3}\text{”}.$$

# Meta-mathematics on the generalization

In order to generalize the classical Frobenius-Stickelberger formula there are following three "Linearly Independent Directions" :

(1) Going to higher genus case;

(2) Involving Galois conjugates, especially involving an automorphism;

(3) Changing the strata on which the formula is alive;

There are various "Linear Combinations" of them.

The theory which I will talk about today is special for functions on Jacobian varieties, but not on Abelian varieties in general.

# III-1. Warming up via genus 2

We define the sigma function $\sigma(u)$ for $\mathscr{C} : y^2 = x^5 + \mu_4 x^3 + \mu_6 x^2 + \mu_8 x + \mu_{10}$.

$$H^1_{\mathrm{dR}}\Big(\mathscr{C}/\mathbf{Q}[\mu]\Big) \cong \mathbf{Q}[\mu]\,\frac{dx}{f_y} + \mathbf{Q}[\mu]\,\frac{xdx}{f_y} + \mathbf{Q}[\mu]\,\frac{(3x^3+\mu_4 x)\,dx}{f_y} + \mathbf{Q}[\mu]\,\frac{x^2dx}{f_y}$$
$$= \mathbf{Q}[\mu]\,\omega_3 + \mathbf{Q}[\mu]\,\omega_1 + \mathbf{Q}[\mu]\,\eta_3 + \mathbf{Q}[\mu]\,\eta_1.$$

Let $\omega'$, $\omega''$, $\eta'$ and $\eta''$ be the period matrices of size $2\times 2$ with respect to the basis $\omega_3,\ \omega_1,\ \eta_3,\ \eta_1$ and any symplectic basis of $H_1(\mathscr{C}^{\mathrm{an}}, \mathbf{Z})$.

The sigma function $\sigma(u)$ is defined by

$$\sigma(u) = \sigma(u_3, u_1) = \left(\frac{2\pi}{\omega'}\right)^{2/2} \Delta^{-\frac{1}{8}}\ \exp\big(-\tfrac{1}{2}\,{}^t u\omega'^{-1}\eta' u\big)\cdot\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ 0\\ \frac{1}{2}\end{bmatrix}(\omega'^{-1}u, \omega''/\omega'),$$

which is a modular invariant entire function on $\mathbf{C}^2$ and a quite natural generalization of Weierstrass sigma function.

$$\sigma(u_3,u_1) = u_3 - 2\frac{{u_1}^3}{3!} - 4\mu_4\frac{{u_1}^7}{7!} - 2\mu_4\frac{u_3{u_1}^4}{4!} + 64\mu_6\frac{{u_1}^9}{9!} - 8\mu_6\frac{u_3{u_1}^6}{6!}$$
$$- 2\mu_6\frac{{u_3}^2{u_1}^3}{2!3!} + \mu_6\frac{{u_3}^3}{3!} + \cdots . \in \mathbf{Z}[\mu]\langle\langle u_3,\, u_1\rangle\rangle$$

## III-2. Characterization of the $\sigma(u)$ for genus 2

We define $\mathbf{R}$-bilinear form $L(\ ,\ ) : \mathbf{C}^2 \times \mathbf{C}^2 \to \mathbf{C}$ by

$$L(u,v) = u^{\,t}(\eta' v' + \eta'' v'), \quad \text{where } v = \omega' v' + \omega'' v'' \text{ with } v,\ v'' \in \mathbf{R}^2,$$

which is $\mathbf{C}$-linear on the 1st space and
the map $(\ell,k) \mapsto L(\ell,k) - L(k,\ell)$ on $\Lambda \times \Lambda$ is $2\pi i\,\mathbf{Z}$-valued,

The function $\sigma(u) = \sigma(u_3, u_1)$ is characterized (up to a multiplicative constant) by the following properties :

(i) $\sigma(u+\ell) = \chi(\ell)\,\sigma(u)\,\mathbf{exp}\,L(u+\frac{1}{2}\ell,\ell)$, $u \in \mathbf{C}^2$, $\ell \in \Lambda$,
with $\chi(\ell) \in \{\pm 1\}$ satisfying
$\chi(\ell+k) = \chi(\ell)\chi(k)\,\mathbf{exp}\,\frac{1}{2}[L(\ell,k) - L(k,\ell)]$ ;

(ii) The set of zeroes of $u \mapsto \sigma(u)$ is exactly the canonical image $\Theta^{[2-1]}$ of $\mathscr{C} = \mathbf{Sym}^{2-1}\mathscr{C}$, which is of order 1.

# III-3. Frobenius-Stickelberger in genus 2 (1/2)

$$\begin{array}{ccc} \kappa^{-1}\iota(\mathscr{C}) & \longrightarrow & \mathbf{C}^2 \\ \big\downarrow & & \big\downarrow \kappa \\ \mathscr{C} & \xrightarrow{\ \iota\ } & \mathbf{C}^2/\Lambda \end{array}$$

$$\iota\,:\,(x,y) \longmapsto u = \int_\infty^{(x(u),\,y(u))} (\omega_3,\,\omega_1) \ \ \mathbf{mod}\ \Lambda.$$

$$\wp_{11}(u+v) = -x(u) - x(v), \quad \wp_{13}(u+v) = x(u)\,x(v)$$

for $u$, $v \in \kappa^{-1}(\iota(\mathscr{C}))$ (The solu. to Jacobi's Umkehr Problem).

**Theorem.** [Ô, 2012] Let $\sigma_1(u) = \frac{\partial}{\partial u_1}\sigma(u_3, u_1)$.
Let $n \geq 2$ and $u^{(1)}, \cdots, u^{(n)}$ be variables on $\kappa^{-1}\iota(\mathscr{C})$. Then we have

$$\sigma(u^{(1)} + u^{(2)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} - u^{(j)}) \Big/ \prod_j \sigma_1(u^{(j)})^n$$

$$= -\begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & y(u^{(1)}) & yx(u^{(1)}) & x^3(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & y(u^{(2)}) & yx(u^{(2)}) & x^3(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & y(u^{(n)}) & yx(u^{(n)}) & x^3(u^{(n)}) & \cdots \end{vmatrix}.$$

Proof.

$$0 = \sigma(v) = v_3 - \frac{1}{3}{v_1}^3 + \cdots,$$

$$\sigma(u+v) = \sigma_1(u)v_1 + \sigma_3(u)v_3 + \sigma_{11}(u){v_1}^2 + \cdots.$$

# III-3. Frobenius-Stickelberger in genus 2 (2/2)

Define

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u).$$

I realized the last formula from H.F. Baker's formulae :

$$-\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma(u)^2\,\sigma(v)^2} = \wp_{33}(u) - \wp_{33}(v) + \wp_{13}(u)\,\wp_{11}(v) - \wp_{11}(u)\,\wp_{13}(v).$$

Bringing $v \to$ a point $\in \kappa^{-1}\big(\iota(\mathscr{C})\big)$ after multiplying $\dfrac{\sigma(v)^2}{\sigma_1(v)^2}$,

$$\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma(u)^2\,\sigma_1(v)^2} = -x(v)^2 + \wp_{13}(u) - x(v)\,\wp_{11}(u). \qquad \text{( D. Grant )}$$

Bringing $u \to$ a point $\in \kappa^{-1}\big(\iota(\mathscr{C})\big)$ after multiplying $\dfrac{\sigma(u)^2}{\sigma_1(u)^2}$,

$$\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma_1(u)^2\,\sigma_1(v)^2} = x(u) - x(v).$$

This is the initial case of the formula in the last page.

# IV-1. Higher genus curves

For coprime positive integers $q > d$, let $\mathscr{C}$ be the curve defined by

$$f(x,y) = 0$$

with

$$f(x,y) = y^d - x^q + \sum_{i,j\,:\,dq>iq+jd} (\text{some coeff.})\, x^i y^j, \quad (\mathbf{wt}(x) = -d, \ \ \mathbf{wt}(y) = -q)$$

adjoining unique point $\infty$ at infinity. Call this $(d,q)$-curve.

If $\mathscr{C}$ is non-singular, then its genus is given by $g = \frac{(d-1)(q-1)}{2}$.

For example,

$$\begin{cases} f(x,y) = y^2 + (\mu_1 x + \mu_3)y - (x^3 + \mu_2 x^2 + \mu_4 x + \mu_6), \\ \qquad\qquad \mathbf{wt}(x) = -2,\ \mathbf{wt}(y) = -3,\ \ \mathbf{wt}(\mu_j) = -j. \end{cases}$$

$$\begin{cases} f(x,y) = y^3 + (\mu_1 x + \mu_4)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}) \\ \qquad\qquad \mathbf{wt}(x) = -3,\ \ \mathbf{wt}(y) = -4,\ \ \mathbf{wt}(\mu_j) = -j. \end{cases}$$

.......

# IV-2. Weierstrass gaps at $\infty$ of the curve $\mathscr{C}$

Let $w_1, \cdots, w_g$ be the Weierstrass gap sequence at $\infty$ of

$$\mathscr{C} \;:\; y^d + \cdots = x^q + \cdots .$$

For example,

$(2,3)$-curve ........... $w_1 = 1$.

$(2,2g+1)$-curve ... $(w_1,\ w_2,\ \cdots,\ w_g) = (1,\ 3,\ \cdots,\ 2g+1)$.

$(3,4)$-curve ........... $(w_1,\ w_2,\ w_3) = (1,\ 2,\ 5)$.

$(3,5)$-curve ........... $(w_1,\ w_2,\ w_3,\ w_4) = (1,\ 2,\ 4,\ 7)$.

Let us fix a vector $\overrightarrow{\omega} = (\omega_{w_g}, \omega_{w_{g-1}}, \cdots, \omega_{w_1})$ consists of the "natural" basis of $\Gamma(\mathscr{C}, \Omega^1)$ with $\mathbf{wt}(\omega_{w_j}) = w_j$.

**Example**. For the $(2,7)$-curve

$$f(x,y) = y^2 + (\mu_1 x^3 + \mu_3 x^2 + \mu_5) y$$
$$- (x^7 + \mu_2 x^6 + \mu_4 x^5 + \mu_6 x^4 + \mu_8 x^3 + \mu_{10} x^2 + \mu_{12} x + \mu_{14}) = 0,$$

the vector $\overrightarrow{\omega}$ consists of $\omega_5 = \dfrac{dx}{f_y(x,y)}$, $\omega_3 = \dfrac{x\,dx}{f_y(x,y)}$, $\omega_1 = \dfrac{x^2 dx}{f_y(x,y)}$.

**Example**. For the $(3,4)$-curve

$$f(x,y) = y^3 + (\mu_1 x + \mu_4)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}) = 0,$$

the vector $\vec{\omega}$ consists of $\omega_5 = \dfrac{dx}{f_y(x,y)}$, $\omega_2 = \dfrac{xdx}{f_y(x,y)}$, $\omega_1 = \dfrac{ydx}{f_y(x,y)}$.

Using $\vec{\omega} = (\omega_{w_g}, \omega_{w_{g-1}}, \cdots, \omega_{w_1})$, define the period lattice $\Lambda = \left\{ \oint \vec{\omega} \right\} \subset \mathbf{C}^g$.
We define, for each integer $k \geq 0$,

$$\iota : \mathrm{Sym}^k(\mathscr{C}) \to \mathbf{C}^g/\Lambda = \mathrm{Jac}(\mathscr{C})$$
$$(\mathrm{P}_1, \cdots, \mathrm{P}_k) \mapsto \sum_{j=1}^{k} \int_{\infty}^{\mathrm{P}_j} \vec{\omega} \bmod \Lambda.$$

We denote the mod $\Lambda$ map by $\kappa : \mathbf{C}^g \to \mathbf{C}^g/\Lambda$.
We denote $W^{[k]} = \iota\big(\mathrm{Sym}^k(\mathscr{C})\big)$. Then $W^{[1]} \cong \mathscr{C}$. Let

$$\Theta^{[k]} = [-1]W^{[k]} \cup W^{[k]}.$$

# IV-4. The stratification

Summing up, we have the following stratification:

$$\begin{array}{ccccccccc}
\Lambda & \subset & \kappa^{-1}(\Theta^{[1]}) & \subset & \kappa^{-1}(\Theta^{[2]}) & \subset\cdots\subset & \kappa^{-1}(\Theta^{[g-1]}) & \subset & \kappa^{-1}(\Theta^{[g]}) = \mathbf{C}^g. \\
\downarrow\kappa & & \downarrow\kappa & & \downarrow\kappa & & \downarrow\kappa & & \downarrow\kappa \\
0 & \in & \Theta^{[1]} & \subset & \Theta^{[2]} & \subset\cdots\subset & \Theta^{[g-1]} & \subset & \Theta^{[g]} = \mathbf{C}^g/\Lambda \\
\| & & \cup & & \cup & & \| & & \| \\
0 & \in & \iota(\mathscr{C}) = W^{[1]} & \subset & W^{[2]} & \subset\cdots\subset & W^{[g-1]} & \subset & W^{[g]} \\
\uparrow\iota & & \uparrow\iota & & \uparrow\iota & & \uparrow\iota & & \uparrow\iota \\
\infty & \in & \mathscr{C} = \mathrm{Sym}^1\mathscr{C} & \subset & \mathrm{Sym}^2\mathscr{C} & \subset\cdots\subset & \mathrm{Sym}^{g-1}\mathscr{C} & \subset & \mathrm{Sym}^g\mathscr{C}
\end{array}$$

We note that Jacobi's theorem implies

$$\Theta^{[g-1]} = W^{[g-1]}.$$

We shall define afterward an important function $\sigma_{\natural^k}(u)$ (a higher derivative of $\sigma(u)$), which is useful on the $k$-th stratum $\kappa^{-1}(\Theta^{[k]})$.

## VI-5. de Rham cohomology and its symplectic structure

On the 1st de Rham cohomology

$$H^1_{\mathrm{dR}}(\mathscr{C}/\mathbf{Q}[\mu]) \;=\; \left\{ \frac{h(x,y)\,dx}{f_y(x,y)} \,\middle|\, h(x,y)\in\mathbf{Q}[\mu][x,y] \right\} \Big/ \, d\mathbf{Q}[\mu][x,y],$$

$$\Big( \supset \Gamma(\mathscr{C},\Omega^1) \Big)$$

we have the following symplectic product $\star$:

For $\omega,\ \eta \in H^1_{\mathrm{dR}}(\mathscr{C}/\mathbf{Q}[\mu])$,

$$\omega\star\eta = \sum_{\mathrm{P}} \operatorname*{Res}_{\mathrm{P}} \left( \int_\infty^{\mathrm{P}} \omega \right) \eta(\mathrm{P}) \quad \left( = \operatorname*{Res}_{\mathrm{P}=\infty} \left( \int_\infty^{\mathrm{P}} \omega \right) \eta(\mathrm{P}) \right).$$

There is a "concise" symplectic basis of $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbf{Q}[\mu])$ :

$$\omega_{w_g},\ \omega_{w_{g-1}},\ \cdots,\ \omega_{w_1},\ \ \eta_{w_g},\ \eta_{w_{g-1}},\ \cdots,\ \eta_{w_1},$$

where $w_j$ stands for the weight (or the negative of weight).

# IV-6. The sigma function for a higher genus curve

The sigma function $\sigma(u)$ for $\mathscr{C}$ is defined by using the symplectic basis $\{\omega_{w_g}, \omega_{w_{g-1}}, \cdots, \omega_{w_1}\} \cup \{\eta_{w_g}, \eta_{w_{g-1}}, \cdots, \eta_{w_1}\}$ of $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbf{Q}[\mu])$ and any symplectic basis of $H_1(\mathscr{C}^{\mathrm{an}}, \mathbf{Z})$. It is an entire function on $\mathbf{C}^g$ with $g$ variables $u = (u_{w_g}, \cdots, u_{w_1})$, and it is a quite natural generalization of the Weierstrass sigma function.

**Example**. If $\mathscr{C}$ is $(3,4)$-curve,then

$$\sigma(u) = \sigma(u_5, u_2, u_1) = \left(u_5 - u_1 u_2{}^2 + \frac{1}{20} u_1{}^5\right) + \left(\frac{1}{12}\mu_1 u_1{}^4 u_2 - \frac{1}{3}\mu_1 u_2{}^3\right) + \cdots.$$

We define $\mathbf{R}$-bilinear form $L(\ ,\ ) : \mathbf{C}^2 \times \mathbf{C}^2 \to \mathbf{C}$ by

$$L(u,v) = u^{\,t}(\eta' v' + \eta'' v'), \quad \text{where } v = \omega' v' + \omega'' v'' \text{ with } v,\ v'' \in \mathbf{R}^g,$$

which is $\mathbf{C}$-linear on the 1st space and
the map $(\ell, k) \mapsto L(\ell, k) - L(k, \ell)$ on $\Lambda \times \Lambda$ is $2\pi i\,\mathbf{Z}$-valued,

The function $\sigma(u) = \sigma(u_{w_g}, \cdots, u_{w_1})$ is characterized ( up to non-zero multiplicative constant ) by

(i) $\sigma(u + \ell) = \chi(\ell)\sigma(u) \exp L(u + \frac{1}{2}\ell, \ell)$ for $u \in \mathbf{C}^g$, $\ell \in \Lambda$,
with $\chi(\ell) \in \{\pm 1\}$ satisfying $\chi(\ell + k) = \chi(\ell)\chi(k) \exp \frac{1}{2}[L(\ell,k) - L(k,\ell)]$ ;

(ii) The set of zeroes of $u \mapsto \sigma(u)$ is exactly on $\Theta^{[g-1]} \cup [-1]\Theta^{[g-1]}$,
which is of order 1. Here $\Theta^{[g-1]}$ is the canonical image of $\mathrm{Sym}^{g-1}\mathscr{C}$.

# V-1. On the largest stratum $\boxed{(3,4)\text{-curve}, \; g=3}$

We define $\wp$-functions by

$$\wp_{ij}(u) := -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u), \quad \wp_{ijk}(u) := \frac{\partial}{\partial u_k} \wp_{ij}(u), \text{ etc.}$$

Then $\wp_{ij}(u) \in \Gamma\big(\mathrm{Jac}(\mathscr{C}), \mathscr{O}(2\Theta^{[g-1]})\big)$, $\wp_{ijk}(u) \in \Gamma\big(\mathrm{Jac}(\mathscr{C}), \mathscr{O}(3\Theta^{[g-1]})\big)$, etc.

The case of the $(3,4)$-curve on the largest stratum in $2$ variables :

**Theorem.** [EEMÔP] (2008)
For $u, v \in \mathbf{C}^3 = \kappa^{-1}(W^{[3]})$, we have

$$\begin{aligned}
\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma(u)^2\,\sigma(v)^2} &= -\wp_{55}(u) + \wp_{55}(v) - \wp_{52}(u)\wp_{21}(v) + \wp_{52}(v)\wp_{21}(u) \\
&- \wp_{51}(u)\wp_{22}(v) + \wp_{51}(v)\wp_{22}(u) - \tfrac{1}{3}\big(\wp_{11}(u)\,Q_{5111}(v) - \wp_{11}(v)\,Q_{5111}(u)\big) \\
&+ \tfrac{1}{3}\mu_1\left(\wp_{52}(u)\wp_{11}(v) - \wp_{52}(v)\wp_{11}(u)\right) + \mu_1\left(\wp_{51}(u)\wp_{21}(v) - \wp_{51}(v)\wp_{21}(u)\right) \\
&- \tfrac{1}{3}\left({\mu_1}^2 - \mu_2\right)\left(\wp_{51}(u)\wp_{11}(v) - \wp_{51}(v)\wp_{11}(u)\right) - \tfrac{1}{3}\mu_8\left(\wp_{11}(u) - \wp_{11}(v)\right),
\end{aligned}$$

where

$$Q_{5111} = \wp_{5111} - 6\wp_{51}\wp_{11}.$$

## V-2. On the largest stratum for the purely trigonal curve

**Theorem.** [EEMÔP] (2008)

For $\mathscr{C}: f(x,y) = y^3 - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}) = 0$ with the canonical automorphism $[\zeta] : (x,y) \mapsto (x, \zeta y)$ of $\zeta = \exp(2\pi i/3)$, we have

$$\frac{\sigma(u+v)\,\sigma(u+[\zeta]v)\,\sigma(u+[\zeta]^2 v)}{\sigma(u)^3\,\sigma(v)^3} = R(u,v) + R(v,u),$$

where

$$\begin{aligned} R(u,v) = &-\tfrac{1}{3}\wp_{51}(u)\,\tfrac{\partial}{\partial u_1}Q_{5111}(v) - \tfrac{3}{4}\wp_{21}(u)\wp_{552}(v) - \tfrac{1}{2}\wp_{555}(u) \\ &+ \tfrac{1}{4}\wp_{522}(u)\wp^{[55]}(v) - \tfrac{1}{4}\wp_{222}(u)\wp^{[52]}(v) + \tfrac{1}{12}\tfrac{\partial}{\partial u_1}Q_{5111}(u)\wp^{[55]}(v) \\ &+ \tfrac{1}{2}\wp_{111}(u)\wp^{[22]}(v) - \tfrac{1}{4}\mu_1\wp_{111}(u)\wp^{[52]}(v) \\ &+ \tfrac{1}{2}\mu_6\wp_{51}(u)\wp_{111}(v) - \tfrac{1}{4}\mu_9\wp_{21}(u)\wp_{111}(v) - \tfrac{1}{2}\mu_{52}\wp_{111}(u) \end{aligned}$$

with

$\wp^{[ij]} =$ "the determinant of the $(i,j)$-(complementary) minor of $[\wp_{ij}]_{3\times 3}$".

# VI-1. Higher derivatives of the sigma function

We define, for the multi-index $I = \natural^n$ with respect to $\{w_g, \cdots, w_1\}$ defined in the next page, or for arbitrary multi-index $I$,

$$\sigma_I(u) = \left(\prod_{j \in I} \frac{\partial}{\partial u_j}\right) \sigma(u).$$

**Examples**. If $(d, q) = (3, 4)$ then $\flat = \natural^2 = \{1\}$ and $\sharp = \natural^1 = \{2\}$, and

$$\sigma_\flat(u) = \sigma_1(u) = \frac{\partial}{\partial u_1} \sigma(u_5, u_2, u_1),$$

$$\sigma_\sharp(u) = \sigma_2(u) = \frac{\partial}{\partial u_2} \sigma(u_5, u_2, u_1).$$

We define $\sigma_{\natural^0}(u) = 1$, a constant function.

# VI-2. Table of $\natural^n$

| $(d,p)$ | $g$ | $\sharp = \natural^1$ | $\flat = \natural^2$ | $\natural^3$ | $\natural^4$ | $\natural^5$ | $\natural^6$ | $\natural^7$ | $\natural^8$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $(2,3)$ | 1 | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\cdots$ |
| $(2,5)$ | 2 | $\{1\}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\cdots$ |
| $(2,7)$ | 3 | $\{3\}$ | $\{1\}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\cdots$ |
| $(2,9)$ | 4 | $\{1,5\}$ | $\{3\}$ | $\{1\}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\cdots$ |
| $(2,11)$ | 5 | $\{3,7\}$ | $\{1,5\}$ | $\{3\}$ | $\{1\}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\cdots$ |
| $(2,13)$ | 6 | $\{1,5,9\}$ | $\{3,7\}$ | $\{1,5\}$ | $\{3\}$ | $\{1\}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\cdots$ |
| $(2,15)$ | 7 | $\{3,7,11\}$ | $\{1,5,9\}$ | $\{3,7\}$ | $\{1,5\}$ | $\{3\}$ | $\{1\}$ | $\{\ \}$ | $\{\ \}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |
| $(3,4)$ | 3 | $\{2\}$ | $\{1\}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\cdots$ |
| $(3,5)$ | 4 | $\{4\}$ | $\{2\}$ | $\{1\}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\cdots$ |
| $(3,7)$ | 6 | $\{1,6\}$ | $\{1,5\}$ | $\{4\}$ | $\{2\}$ | $\{1\}$ | $\{\ \}$ | $\{\ \}$ | $\{\ \}$ | $\cdots$ |
| $(3,9)$ | 7 | $\{4,10\}$ | $\{2,7\}$ | $\{1,5\}$ | $\{4\}$ | $\{2\}$ | $\{1\}$ | $\{\ \}$ | $\{\ \}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

# VI-3. Table of $\natural^n$

We explain by an example : $(d,q) = (3,7)$, $g = 6$. Write a $g \times g = 6 \times 6$ table as follows. We first write the Weierstrass gap sequence with respect to $(d,q)$ on the last column, namely,

| | | | | | |
|---|---|---|---|---|---|
| | | | | | **11** |
| | | | | | **8** |
| | | | | | **5** |
| | | | | | **4** |
| | | | | | **2** |
| | | | | | **1** |

# VI-4. Table of $\natural^n$

Then, put into other boxes naturally increasing non-negative integers as follows:

| | | | | | |
|---|---|---|---|---|---|
| **6** | **7** | **8** | **9** | **10** | **11** |
| **3** | **4** | **5** | **6** | **7** | **8** |
| **0** | **1** | **2** | **3** | **4** | **5** |
| | **0** | **1** | **2** | **3** | **4** |
| | | | **0** | **1** | **2** |
| | | | | **0** | **1** |

# VI-5. Table of $\natural^n$

If we wish to get $\natural^n = \natural^2$, extract $(g-n)\times(g-n) = \mathbf{4}\times\mathbf{4}$ minor on the lower right corner. and Remove all rows and columns including $\mathbf{0}$.

| | | | | | |
|---|---|---|---|---|---|
| 6 | 7 | 8 | 9 | 10 | 11 |
| 3 | 4 | 5 | 6 | 7 | 8 |
| 0 | 1 | 2 | 3 | 4 | 5 |
| | 0 | 1 | 2 | 3 | 4 |
| | | | 0 | 1 | 2 |
| | | | | 0 | 1 |

$\rightarrow$

| | | | |
|---|---|---|---|
| 2 | 3 | 4 | 5 |
| 1 | 2 | 3 | 4 |
| | 0 | 1 | 2 |
| | | 0 | 1 |

$\rightarrow$

| | |
|---|---|
| 2 | 5 |
| 1 | 4 |

Finally, by reading the numbers on the off-diagonal, we have

$$\natural^2 = \{1, 5\} \quad \text{and} \quad \sigma_{\natural^2}(u) = \sigma_{1,5}(u) = \frac{\partial^2}{\partial u_1 \partial u_5}\sigma(u).$$

## VI-3. Properties of the satellite sigma functions (The most important page!)

The set of higher derivatives of the $\sigma(u)$

$$\{\, \kappa^{-1}(\Theta^{[n]}) \ni u \mapsto \sigma_{\natural^n}(u) \mid 0 \leqq n \leqq g-1 \,\}$$

the satellite sigma functions for $\mathscr{C}$. They have the following very nice properties:

(i) $\sigma_{\natural^n}(u+\ell) = \chi(\ell)\, \sigma_{\natural^n}(u)\, L(u+\frac{1}{2}\ell, \ell)$, $u \in \kappa^{-1}(\Theta^{[n]})$, $\ell \in \Lambda$.

(ii) If $u \in \kappa^{-1}(W^{[n]} \setminus W^{[n-1]})$, then the function $\kappa^{-1}(W^{[1]}) \ni v \mapsto \sigma_{\natural^{n+1}}(u+v)$ has a zero at $\Lambda$ of order $w_{g-n} - g + n + 1$,

and other $g - (w_{g-n} - g + n + 1)$ zeroes elsewhere mod $\Lambda$.

Moreover, $\sigma_{\natural^{n+1}}(u+v) = \pm \sigma_{\natural^n}(u)\, {v_1}^{w_{g-n}-g+n+1} +$ "higher terms in $v_1$".

The exact places of all zeroes of $v \mapsto \sigma_\flat(u+v) := \sigma_{\natural^2}(u+v)$ are known.

$\sigma_\sharp(u) := \sigma_{\natural^1}(u) = \pm\, {v_1}^g + \cdots$ and this has only zero at $\Lambda$.

(iii) The set of zeroes of the function $\kappa^{-1}(W^{[n+1]}) \ni u \mapsto \sigma_{\natural^{n+1}}(u)$ is $\kappa^{-1}(\Theta^{[n]})$, which is of order 1.

(iv) For an index $I$, if $\mathbf{wt}(I) < \mathbf{wt}(\natural^n)$, then $\sigma_I(u) = 0$ on $\kappa^{-1}(\Theta^{[n]})$.

(v) If $\mathbf{wt}(I) = \mathbf{wt}(\natural^n)$, then the function $\sigma_I(u) =$ "an integer"$\cdot\, \sigma_{\natural^n}(u)$ on $\kappa^{-1}(\Theta^{[n]})$.

Proof : By certain expression of $\sigma(u)$ as the determinant of a matrix of size $\mathbf{N} \times \mathbf{N}$

(or by precise observation of power series expansions).

# VII-1. Guide Function

We may extend this class of addition formulae by considering more general map

$$\varphi : \mathscr{C} \longrightarrow \mathbf{P}^1$$

which belongs to $\mathbf{Z}[\mu_1, \mu_2, \cdots, \mu_6][x(u), y(u)]$, and of homogeneous weight.
We suppose the coefficient of the lowest weight term w. r. t. $x(u)$ and $y(u)$ is $1$.
Let $m \geq 2$ be the order of unique pole of $\varphi$, and $u$ be the analytic variable of $\varphi$ regarding $\mathscr{C}$ as a complex torus. Then there exist

$$u,\ u^{\star},\ u^{\star^2},\ u^{\star^3},\ \cdots,\ u^{\star^{m-1}} \quad \in \mathbf{C}$$

such that these $m$ variables are generically different, vary continuously, and satisfy

$$\varphi(u) = \varphi(u^{\star}) = \cdots = \varphi(u^{\star^{m-1}}).$$

Moreover, we may choose them as

$$u + u^{\star} + \cdots + u^{\star^{m-1}} = 0.$$

Indeed $d(u + u^{\star} + \cdots + u^{\star^{m-1}})$ can be regarded as a holomorphic 1-form on $\mathbf{P}^1$.

# VII-2. An example of new addition formula

**Example**. ([Eilbeck-England-Ô, 2014]) We take the $(2,3)$-curve and $y(u)$ as a guide function. ( $y(u) = y(u^\star) = y(u^{\star\star})$ ) Let $u = u^{(1)}$ and $v = u^{(2)}$ (two variable case). Then we have the addition formula

$$-\frac{\sigma(u+v)\,\sigma(u+v^\star)\,\sigma(u+v^{\star\star})}{\sigma(u)^3\,\sigma(v)\,\sigma(v^\star)\,\sigma(v^{\star\star})} = y(v) - y(-u)$$

$$= y(u) + y(v) + \mu_1 x(u) + \mu_3$$

$$= \left.\frac{f(x(u), Y) - f(x(u), W)}{Y - W}\right|_{Y=y(u),\, W=y(v)}.$$

**Proof**. Use the following : As a function of $u$,

$$y(v) - y(-u) = 0 \iff u = -v,\ -v^\star, \text{ or } -v^{\star\star};$$

$$y(v) - y(-u) = \infty \iff u = 0;$$

$$u + u^\star + u^{\star\star} = 0;$$

$$\sigma(u) = 0 \iff u \in \Lambda.$$

**Remark**. The RHS is defined over $\mathbf{Z}[\mu] = \mathbf{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$.

**Remark**. There is [Eilbeck-S.Matsutani-Ô, 2011] for $y^2 + \mu_3 y = x^3 + \mu_6$.

## VII-3. Second example of new addition formulae

We take the $(2,3)$-curve and $x^2(u)$ as a guide function.

Let $u = u^{(1)}$ and $v = u^{(2)}$ (two variable case). Then

**Example**. We have the addition formula

$$\frac{\sigma(u+v)\,\sigma(u+v^{\star})\,\sigma(u+v^{\star\star})\,\sigma(u+v^{\star\star\star})}{\sigma(u)^4\,\sigma(v)\,\sigma(v^{\star})\,\sigma(v^{\star\star})\,\sigma(v^{\star\star\star})} = x^2(u) - x^2(v).$$

**Remark**. The RHS is defined over $\mathbf{Z}[\mu] = \mathbf{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$.

## VII-4. On the first stratum in two variables $(3,4)$-curve, $g=3$

We define the functions $\kappa^{-1}(W^{[1]}) \ni u \mapsto x(u)$, $\kappa^{-1}(W^{[1]}) \ni u \mapsto y(u)$ by

$$u = (u_{w_g}, \cdots, u_{w_1}) = \int_{\infty}^{(x(u),y(u))} \vec{\omega}.$$

Let us take $x(u)$ be the guide function.

For a variable $v \in \kappa^{-1}(W^{[1]})$, let $\{v, v', v''\}$ be a complete representative modulo $\Lambda$ of the inverse image of the map $v \mapsto x(v)$ such that $v'$ and $v''$ vary continuously with respect to $v$ and $v' = v'' = 0$ when $v = 0$.

Of course, $y(v)$, $y(v')$, $y(v'')$ are the three roots of $f(x(v), Y) = 0$.

**Lemma.** [Ô] (2011) Then, for $u$, $v \in \kappa^{-1}(W^{[1]})$, we have

$$\frac{\sigma_\flat(u+v)\,\sigma_\flat(u+v')\,\sigma_\flat(u+v'')}{\sigma_\sharp(u)^3\,\sigma_\sharp(v)\,\sigma_\sharp(v')\,\sigma_\sharp(v'')} = \begin{vmatrix} 1 & x(u) \\ 1 & x(v) \end{vmatrix}^2.$$

Here we recall that

$$\sigma_\flat(u) = \sigma_{\natural^2}(u) = \sigma_2(u) = \tfrac{\partial}{\partial u_2}\sigma(u), \quad \sigma_\sharp(u) = \sigma_{\natural^1}(u) = \sigma_1(u) = \tfrac{\partial}{\partial u_1}\sigma(u).$$

Theorem. [Ô] (2011) In $n$-variable case (Here $n \geq 3$ for simplicity):

$$\frac{\sigma(u^{(1)} + \cdots + u^{(n)}) \prod_{i<j} \sigma_1(u^{(i)} + u^{(j)\prime}) \sigma_1(u^{(i)} + u^{(j)\prime\prime})}{\prod_j \sigma_2(u^{(j)})^{2n-2j+1} \sigma_2(u^{(j)\prime})^{j-1} \sigma_2(u^{(j)\prime\prime})^{j-1}}$$

$$= \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & yx(u^{(1)}) & y^2(u^{(1)}) & x^3(u^{(1)}) & yx^2(u^{(1)}) & y^2x(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & yx(u^{(2)}) & y^2(u^{(2)}) & x^3(u^{(2)}) & yx^2(u^{(2)}) & y^2x(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & yx(u^{(n)}) & y^2(u^{(n)}) & x^3(u^{(n)}) & yx^2(u^{(n)}) & y^2x(u^{(n)}) & \cdots \end{vmatrix}$$

$$\cdot \begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & \cdots & x^{n-1}(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & \cdots & x^{n-1}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^{n-1}(u^{(n)}) \end{vmatrix}.$$

# VII-6. Using guide fct. $y$, on the 1st Stratum  $(3,4)$-curve, $g=3$

**Theorem.** [EEÔ] (2014)

On the $1$st stratum in $2$-variables $u$ and $v$ with guide function $y$ (order 4), we have

$$\frac{\sigma_1(u+v)\sigma_1(u+v^\star)\sigma_1(u+v^{\star\star})\sigma_1(u+v^{\star\star\star})}{\sigma_2(u)^4\sigma_2(v)\sigma_2(v^\star)\sigma_2(v^{\star\star})\sigma_2(v^{\star\star\star})}$$

$$= y(u)^2+y(u)y(v)+y(v)^2+\big(\mu_1 x(u)+\mu_4\big)\big(y(u)+y(v)\big)+\mu_2 x(u)^2+\mu_5\, x(u)+\mu_8$$

$$= \frac{f(x(u),Y)-f(x(u),W)}{Y-W}\bigg|_{Y=y(u),W=y(v)} = \big(y(v)-y(u')\big)\big(y(v)-y(u'')\big).$$

**Remark.** Of course, $y(u)=y(u^\star)=y(u^{\star\star})=y(u^{\star\star\star})$,

$y(u')=y(u^{\star\prime})=y(u^{\star\star\prime})=y(u^{\star\star\star\prime})$,

$y(u'')=y(u^{\star\prime\prime})=y(u^{\star\star\prime\prime})=y(u^{\star\star\star\prime\prime})$.

**Keys of the proof.** For a fixed $u\in\kappa^{-1}(\Theta^{[1]})$, the map

$$v\mapsto\sigma_\flat(u+v)$$

has a zero at $v=0$, $u'$, $u''$ modulo $\Lambda$ of order $1$, and the map

$$u\mapsto\sigma_\sharp(u)$$

has only zero at $u=0$ modulo $\Lambda$ of order $(g=)3$, and no zeroes elsewhere.

# Connection with multiple Gamma functions?

Recall the famous infinite product expression for the Weierstrass sigma:

$$\sigma(u) = u \prod_{\substack{\ell \in \Lambda \\ \ell \neq 0}} \left(1 - \frac{u}{\ell}\right) \exp\left(\frac{u}{\ell} + \frac{u^2}{2\ell^2}\right).$$

This implies the connection with the double Gamma functions:

$$\sigma(z) = e^{-\mu z - \nu \frac{z^2}{2}} \cdot z \cdot \frac{\Pi \Gamma_2^{-1}(z \mid \pm \omega_1, \pm \omega_2)}{\Pi \Gamma_1^{-1}(z \mid \pm \omega_1) \Pi \Gamma_1^{-1}(z \mid \pm \omega_2)},$$

In the higher genus case,

$\sigma_\sharp(u) = \sigma_\sharp(u_{w_g}, \cdots, u_{w_1})$ on $\kappa^{-1}\iota(\mathscr{C})$ has zeroes of order $g$, and no zeroes elsewhere.

Does it has some infinite product expression?

The speaker has a dream on existence of

(1) an infinite product expression of $\sigma_\sharp(u)$ and

(2) an infinite product expression of multivariate multiple $\Gamma$ functions,

and their connection.

# VIII. Summary and Some Questions

For each curve $\mathscr{C}$ and for each the following setting, we have an addition formula of F-S type :

(1) $k \cdots$ the stratum : on the 1st stratum $\longrightarrow$ by using $x(u)$ and $y(u)$;

on the largest stratum $\longrightarrow$ by using $\wp$-functions,

(2) $n \cdots$ the number of variables,

(3) $\varphi \cdots$ the guide function.

Some Questions:

Q1 Is there further natural generalization?

Q2 Why the coefficients of RHS belong to $\mathbf{Z}[\mu]$? (It is obvious they belong to $\mathbf{Q}[\mu]$.)

(If the order of the guide function is small Q2 is OK because the RHS is a determinant, etc. )

Q3 How do these formulae link with other existing mathematical world? Or some applications?

Q4 Can the general RHS be regarded as a sort of higher generalization of the concept of "determinant"?

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Please check

`http://www2.meijo-u.ac.jp/~yonishi/`

Thank you very much for your attention!