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Determinant Formulae in Abelian Functions for a General Trigonal Curve of Degree Five

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Abstract. This paper shows a natural generalization of the Frobenius-Stickelberger formula and the Kiepert formula for elliptic functions ([4] and [5]) to the curve of genus four defined by $y^3 + (\mu_2 x + \mu_5)y^2 + (\mu_1 x^3 + \mu_4 x^2 + \mu_7 x + \mu_{10})y = x^5 + \mu_3 x^4 + \mu_6 x^3 + \mu_9 x^2 + \mu_{12} x + \mu_{15}$, where μ_j are constants.

Keywords. Curves of high genus, Abelian functions, Jacobian varieties.

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Introduction

The classical Frobenius-Stickelberger formula ([4]) establishes an equality between a product of Weierstrass sigma functions and a determinant of $\wp$-functions of Weierstrass and their higher derivatives. The Kiepert formula ([5]) is a formula that expresses n-plication formula for any positive integer n as a Wronskian determinant with repsect to the derivative $\wp'$ of $\wp$, which is regarded as a certain limit of the former formula.

In this paper, we give a Frobenius-Stickelberger-type determinant formula (Proposition 11.1 and Theorem 11.4) and a Kiepert-type formula (Theorem 13.6) for the curve

$$\begin{aligned} y^3 + (\mu_2 x + \mu_5)y^2 + (\mu_1 x^3 + \mu_4 x^2 + \mu_7 x + \mu_{10})y \\ = x^5 + \mu_3 x^4 + \mu_6 x^3 + \mu_9 x^2 + \mu_{12} x + \mu_{15} \quad (\mu_j \text{ are constants}). \end{aligned}$$

There are generalizations of the Frobenius-Stickelberger formula and the Kiepert formula to hyperelliptic curves [9], trigonal curve of genus three [10] and pentagonal curves of genus ten [6]. For wider family of plane curves, the reader should be referred to recent work [7]. However, each of those curves has a *geometric* automorphism, whose defining equations are of the form $y^d =$ "a monic polynomial of x of degree coprime to d".

Our result in the present paper is the first step in further generalization of [9], [10] and [6] to general curves *without geometric automorphism*. The results of this paper might be generalized to any d-gonal curve with unique point at infinity. Indeed, the author has detailed formulations and their proofs of such the formula

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for several curves. However, since he does not have a proof that works in the fully general case, we treat only the curve above as an example. (A slight improvement of the method in [7] might give rise to a proof for the most general case.)

The results in [9] and [7] are obtained by using Riemann's singularity theorem. The method of this paper is quite different from those papers. *We never use* (*any extension of*) *Riemann's singularity theorem.* We require a small amount of background from the theory of Abelian functions. The computation in Section 10 of the present paper is small in size but the idea is the same as in [6] and [10].

The author hopes even after getting a general proof, the down-to-earth method in this paper will also be useful for other researches.

Notation. As usual, we denote by $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ the ring of rational integers, the field of real numbers, the field of complex numbers. The transpose of a vector or a matrix A is denoted by ${}^tA$. The imaginary unit is denoted by $\boldsymbol{i}$.

Contents

1. The curve

Let

$$f(x,y) = y^3 + (\mu_2 x + \mu_5)y^2 + (\mu_1 x^3 + \mu_4 x^2 + \mu_7 x + \mu_{10})y - (x^5 + \mu_3 x^4 + \mu_6 x^3 + \mu_9 x^2 + \mu_{12} x + \mu_{15}), \tag{1.1}$$

where μ_j's are constants. We treat mainly the curve defined by

$$\mathscr{C} \ : \ f(x,y) = 0. \tag{1.2}$$

We are regarding this as a projective curve with unique point ∞ at infinity. Although the coefficients μ_j are usually complex numbers, on many occasions they will be elements in a quite general commutative ring. Any variables, coordinates, and coefficients in this paper have weight denoted by wt. We set

$$\mathrm{wt}(x) = -3, \quad \mathrm{wt}(y) = -5, \quad \mathrm{wt}(\mu_j) = -j. \tag{1.3}$$

Then all the formulae in this paper are of homogeneous weight. We denote simply

$$\mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_9, \mu_{10}, \mu_{12}, \mu_{15}] = \mathbb{Z}[\boldsymbol{\mu}].$$

2. The discriminant of the curve

We are going to define the discriminant of $\mathscr{C}$. We use the notation $f_x = \frac{\partial}{\partial x} f$ and $f_y = \frac{\partial}{\partial y} f$.

Definition 2.1. Assume all the μ_j's are *indeterminates*. For the polynomial f in (1.2), let

$$\begin{aligned} R_1 &= \mathrm{rslt}_x\Big(\mathrm{rslt}_y\big(f(x,y), f_x(x,y)\big), \mathrm{rslt}_y\big(f(x,y), f_y(x,y)\big)\Big), \\ R_2 &= \mathrm{rslt}_y\Big(\mathrm{rslt}_x\big(f(x,y), f_x(x,y)\big), \mathrm{rslt}_x\big(f(x,y), f_y(x,y)\big)\Big), \\ R &= \gcd(R_1, R_2) \ \text{ in } \ \mathbb{Z}[\boldsymbol{\mu}], \end{aligned} \tag{2.2}$$

where rslt_z is Sylvester's resultant with respect to z. Then it is very plausible[1] that R is a perfect square in the ring $\mathbb{Z}[\boldsymbol{\mu}]$. We denote by $D \in \mathbb{Z}[\boldsymbol{\mu}]$ a square root of R. We fix the choice of the square root just after 7.5.

Now we suppose a set of values μ_j's is given, i.e. a specific curve $\mathscr{C}$ is given at the beginning. Then the *discriminant* of $\mathscr{C}$ is the value of D above given by substituting the specific values to the indeterminates μ_j's, respectively.

Although we usually assume $D \neq 0$ in this paper, many formulae are valid without this assumption. If $D \neq 0$, then the curve $\mathscr{C}$ is a non-singular projective curve. Moreover, if the base ring is $\mathbb{C}$, it can be regarded as a closed Riemann surface.

[1] The author does not have proof of this. However, many numerical computations suggest that this will be true.

3. Arithmetic local parameter

We introduce a nice local parameter at ∞, which should be called the arithmetic local parameter. Let

$$t = \frac{y}{x^2}, \quad s = \frac{x}{y}.$$

Then t is a local parameter at ∞ on $\mathscr{C}$. While another choice of local parameter will be sufficient to the purpose of this paper, we shall use t in this paper for convenience to compare with further (arithmetical) investigations of $\mathscr{C}$ in near future. Multiplying by s^6t^5 the equation

$$f\left(\tfrac{1}{st}, \tfrac{1}{ts^2}\right) = 0, \tag{3.1}$$

we get the relation

$$\begin{aligned} s = &-\mu_3 s^2 t + (-\mu_6 s^3 + \mu_4 s^2 + \mu_2 s + 1)\, t^2 \\ &+ (-\mu_9 s^4 + \mu_7 s^3 + \mu_5 s^2)\, t^3 + (-\mu_{12} s^5 + \mu_{10} s^4)\, t^4 - \mu_{15} s^6 t^5. \end{aligned}$$

Using this recursively, we have

$$\begin{aligned} s = {} & t^2 + \mu_2\, t^4 - \mu_3\, t^5 + ({\mu_2}^2 + \mu_4)\, t^6 + (-3\mu_3\mu_2 + \mu_5)\, t^7 \\ &+ ({\mu_2}^3 + 3\mu_4\mu_2 + 2{\mu_3}^2 - \mu_6)\, t^8 + (-6\mu_3{\mu_2}^2 + 3\mu_5\mu_2 - 4\mu_3\mu_4 + \mu_7)\, t^9 \\ &+ ({\mu_2}^4 + 6\mu_4{\mu_2}^2 + 10{\mu_3}^2\mu_2 - 4\mu_6\mu_2 - 4\mu_3\mu_5 + 2{\mu_4}^2)\, t^{10} + \cdots \in \mathbb{Z}[\boldsymbol{\mu}][[\, t\,]]. \end{aligned}$$

This implies that

$$\begin{cases} x = t^{-3} - \mu_2\, t^{-1} + \mu_3 - \mu_4\, t + (\mu_3\mu_2 - \mu_5)\, t^2 + (-\mu_4\mu_2 - {\mu_3}^2 + \mu_6)\, t^3 \\ \qquad + (\mu_3{\mu_2}^2 - \mu_5\mu_2 + 2\mu_3\mu_4 - \mu_7)\, t^4 \\ \qquad + (-\mu_4{\mu_2}^2 - 3{\mu_3}^2\mu_2 + 2\mu_6\mu_2 + 2\mu_3\mu_5 - {\mu_4}^2)\, t^5 + \cdots \\ y = t^{-5} - 2\mu_2\, t^{-3} + 2\mu_3\, t^{-2} + ({\mu_2}^2 - 2\mu_4)\, t^{-1} - 2\mu_5 + (-{\mu_3}^2 + 2\mu_6)\, t \\ \qquad + (2\mu_3\mu_4 - 2\mu_7)\, t^2 + (-2{\mu_3}^2\mu_2 + 2\mu_6\mu_2 + 2\mu_3\, \mu_5 - {\mu_4}^2)\, t^3 + \cdots \end{cases}$$

The coefficients of these expansions belong to $\mathbb{Z}[\boldsymbol{\mu}]$.

4. Differential forms of the first kind

By

$$\frac{dx}{f_y(x,y)} = -\frac{dy}{f_x(x,y)}, \tag{4.1}$$

we see that the set of

$$\omega_1 = \frac{dx}{f_x(x,y)}, \quad \omega_2 = \frac{x\,dx}{f_x(x,y)}, \quad \omega_3 = \frac{y\,dx}{f_x(x,y)}, \quad \omega_4 = \frac{x^2dx}{f_x(x,y)} \tag{4.2}$$

is a basis of the space of differential forms on $\mathscr{C}$ of the first kind. The differential forms of (4.2) are expanded with respect to t as follows:

$$
\begin{aligned}
&\frac{dx}{f_y(x,y)} = \big(-t^6 - 3\mu_2\, t^8 + \cdots\big)dt,\\
&\frac{xdx}{f_y(x,y)} = \begin{cases} \big(-t^3 - 2\mu_2 t^5 + 3\mu_3 t^6 + (-3{\mu_2}^2 - 3\mu_4)t^7 \\ \quad + (12\mu_3\mu_2 - 3\mu_5)t^8 + \cdots\big)dt, \end{cases}\\
&\frac{ydx}{f_y(x,y)} = \begin{cases} \big(-t - \mu_2 t^3 + 2\mu_3 t^4 + (-{\mu_2}^2 - 2\mu_4)t^5 + (6\mu_3\mu_2 - 2\mu_5)t^6 \\ \quad - {\mu_2}^3 - 6\mu_4\mu_2 - 6{\mu_3}^2 + 3\mu_6)\, t^7 \\ \quad + (12\mu_3{\mu_2}^2 - 6\mu_5\mu_2 + 12\mu_3\mu_4 - 3\mu_7)t^8 + \cdots\big)dt, \end{cases}\\
&\frac{x^2dx}{f_y(x,y)} = \begin{cases} \big(-1 - \mu_2 t^2 + 2\mu_3 t^3 + (-{\mu_2}^2 - 2\mu_4)t^4 + (6\mu_3\mu_2 - 2\mu_5)t^5 \\ \quad + (-{\mu_2}^3 - 6\mu_4\mu_2 - 6{\mu_3}^2 + 3\mu_6)t^6 \\ \quad + (12\mu_3{\mu_2}^2 - 6\mu_5\mu_2 + 12\mu_3\mu_4 - 3\mu_7)t^7 \\ \quad + (-{\mu_2}^4 - 12\mu_4{\mu_2}^2 - 30{\mu_3}^2\mu_2 + 12\mu_6\mu_2 \\ \qquad + 12\mu_3\mu_5 - 6{\mu_4}^2)t^8 + \cdots\big)dt. \end{cases}
\end{aligned}
\tag{4.3}
$$

The relation (4.1) shows that all the coefficients of these expansions belong to $\mathbb{Z}[\boldsymbol{\mu}]$.

5. The stratification

Let choose and fix closed paths α_j, β_j ($j = 1, \cdots, 4$) on $\mathscr{C}$ such that the images of them in $H_1(\mathscr{C}, \mathbb{Z})$ form a symplectic base of the space obtained by tensoring $\mathbb{C}$ to this space. We define matrices given by integrating the differential forms of (4.3) along the paths above by

$$\Omega' = \left[\int_{\alpha_j} \omega_i\right]_{i,j=1,2,3,4}, \quad \Omega'' = \left[\int_{\beta_j} \omega_i\right]_{i,j=1,2,3,4}.$$

Then the $\mathbb{Z}$-module

$$\Lambda = \Omega'\mathbb{Z}^4 + \Omega''\mathbb{Z}^4$$

is a lattice in $\mathbb{C}^4$. The factor variety $J = \mathbb{C}^4/\Lambda$ is the Jacobian variety of $\mathscr{C}$. Now, for $k \geqq 1$, let $\mathrm{Sym}^k\mathscr{C}$ be the symmetric product of $\mathscr{C}$ (i.e. k-tuple of points in $\mathscr{C}$ neglecting their order). The image of the Abel-Jacobi map which is defined by

$$\iota : \mathrm{Sym}^k\mathscr{C} \to J, \quad (P_1, \cdots, P_k) \mapsto \sum_{j=1}^{k} \int_\infty^{P_j} \omega \mod \Lambda \tag{5.1}$$

is denoted by $W^{[k]}$. We denote the point obtained by (-1)-multiplication of a point $u \in \mathbb{C}^4$ by $u \mapsto [-1]u$. Let

$$\Theta^{[k]} = [-1]W^{[k]} \cup W^{[k]}.$$

We denote the quotient map modulo Λ by

$$\kappa : \mathbb{C}^4 \to J = \mathbb{C}^4/\Lambda.$$

Summing up, we have the following stratification:

$$\begin{array}{ccccccccc}
\infty \in & \mathscr{C} = \mathrm{Sym}^1\mathscr{C} & \subset & \mathrm{Sym}^2\mathscr{C} & \subset & \mathrm{Sym}^3\mathscr{C} & \subset & \mathrm{Sym}^4\mathscr{C} & \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
O \in & \iota(\mathscr{C}) = W^{[1]} & \subset & W^{[2]} & \subset & W^{[3]} & \subset & W^{[4]} & \\
\| & \cap & & \cap & & \| & & \| & \\
O \in & \Theta^{[1]} & \subset & \Theta^{[2]} & \subset & \Theta^{[3]} & \subset & \Theta^{[4]} = J = \mathbf{C}^4/\Lambda & \\
\uparrow & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\Lambda \subset & \kappa^{-1}(\Theta^{[1]}) & \subset & \kappa^{-1}(\Theta^{[2]}) & \subset & \kappa^{-1}(\Theta^{[3]}) & \subset & \kappa^{-1}(\Theta^{[4]}) = \mathbf{C}^4. &
\end{array}$$

We note that Jacobi's theorem implies

$$\Theta^{[3]} = W^{[3]}. \tag{5.2}$$

For each point $P \in \mathscr{C}$ and each path from ∞ to P, the point

$$u = \int_\infty^P \omega$$

in $\mathbb{C}^4$ is determined. According to this integral, the weights of the coordinates of the variable u on $\mathbb{C}^4$ are determined by the weights (1.3) of ω_1, ω_2, ω_3, ω_4. So it is convenient to denote the coordinates of the variable as

$$u = {}^t[u_7,\, u_4,\, u_2,\, u_1].$$

Namely, the weight of u_j is induced from ι of (5.1) and given by

$$\mathrm{wt}(u_j) = j.$$

If we restrict the domain where u varies to the set $\kappa^{-1}(\Theta^{[1]})$, we see by (4.3) that

$$u_7 = \tfrac{1}{7}\,t^7 + O(t^8), \quad u_4 = \tfrac{1}{4}\,t^4 + O(t^5), \quad u_2 = \tfrac{1}{2}\,t^2 + O(t^3), \quad u_1 = t + O(t^2).$$

Therefore, we have

$$u_7 = \tfrac{1}{7}\,{u_1}^7 + O({u_1}^8), \quad u_4 = \tfrac{1}{4}\,{u_1}^4 + O({u_1}^5), \quad u_2 = \tfrac{1}{2}\,{u_1}^2 + O({u_1}^3). \tag{5.3}$$

For $u \in \kappa^{-1}\iota(\mathscr{C})$, we denote by $x(u)$ and $y(u)$ the coordinates x and y of uniquely determined point of $\mathscr{C}$ by

$$u = {}^t[u_7, u_4, u_2, u_1] = {}^t\left[\int_\infty^{(x,y)} \omega_1, \int_\infty^{(x,y)} \omega_2, \int_\infty^{(x,y)} \omega_3, \int_\infty^{(x,y)} \omega_4\right].$$

Then we have expansions with respect to u_1

$$x(u) = \tfrac{1}{{u_1}^3} + \cdots, \qquad y(u) = \tfrac{1}{{u_1}^5} + \cdots. \tag{5.4}$$

6. Differential forms of the second kind

The exact sequence of sheaves

$$0 \to \mathbb{C} \to \varinjlim_n \mathscr{O}(n\cdot\infty) \xrightarrow{d} d\varinjlim_n \mathscr{O}(n\cdot\infty) \to 0$$

on $\mathscr{C}$ gives rise to the following long exact sequence

$$0 \to H^0(\mathscr{C}, \mathbb{C}) \to H^0(\mathscr{C}, \varinjlim_n \mathscr{O}(n\cdot\infty)) \xrightarrow{d} H^0(\mathscr{C}, d \varinjlim_n \mathscr{O}(n\cdot\infty))$$
$$\to H^1(\mathscr{C}, \mathbb{C}) \to H^1(\mathscr{C}, \varinjlim_n \mathscr{O}(n\cdot\infty)) \to \cdots .$$

We know that

$$H^1(\mathscr{C}, \mathscr{O}(k\cdot\infty)) \cong H^0(\mathscr{C}, \Omega(-k\cdot\infty))$$

by Kodaira-Serre duality,

$$H^0(\mathscr{C}, \varinjlim_k \mathscr{O}(k\cdot\infty)) = \varinjlim_k H^0(\mathscr{C}, \mathscr{O}(k\cdot\infty)),$$

and

$$H^0(\mathscr{C}, \Omega(-k\cdot\infty)) = 0$$

for $k > 2\cdot 4 - 2 = 2$ (here the 4 is the genus of $\mathscr{C}$). Hence we have a canonical isomorphism of vector spaces

$$H^1(\mathscr{C}, \mathbb{C}) \simeq \varinjlim_n H^0(\mathscr{C}, d\mathscr{O}(n\cdot\infty))/d \varinjlim_n H^0(\mathscr{C}, \mathscr{O}(n\cdot\infty)).$$

So that each element in $H^1(\mathscr{C}, \mathbb{C})$ is represented by a differential form of the second kind. For any two forms ω and $\eta \in \varinjlim_n H^0(\mathscr{C}, \mathscr{O}(n\cdot\infty))$, we define a product by

$$\omega \star \eta = \sum_{P\in\mathscr{C}} \mathrm{Res}_P \Big(\int_\infty^P \omega \Big) \eta(P) = \frac{1}{2\pi \boldsymbol{i}} \sum_{j=1}^{g} \Big(\int_{\alpha_j} \omega \int_{\beta_j} \eta - \int_{\alpha_j} \eta \int_{\beta_j} \omega \Big).$$

This[2] is just the product induced on $H^1(\mathscr{C}, \mathbb{C})$ by the usual intersection product in the homology group $H_1(\mathscr{C}, \mathbb{Z}) \otimes \mathbb{C}$ with respect to Pontryagin duality. We will find a symplectic basis $\{\, \omega_j,\ \eta_j \,|\, j = 1,\ \cdots,\ 4 \,\}$ of $H^1(\mathscr{C}, \mathbb{C})$ with respect to $\star$ extending $\{\omega_j\}$. To do so, we introduce Klein's fundamental 2-form

$$\boldsymbol{\xi}(x, y; z, w) = \omega_1(x, y)\, \frac{d}{dz} \frac{1}{(x - z)} \frac{f(Z, y) - f(Z, w)}{y - w} \Big|_{Z=z} dz$$
$$- \sum_{j=1}^{g} \omega_j(x, y)\, \eta_j(z, w),$$

on $\mathscr{C} \times \mathscr{C}$ which includes unknown forms $\{\, \eta_j \,\}$ of the second kind. If we find a set $\{\, \eta_j \,\}$ such that

$$\boldsymbol{\xi}(x, y; z, w) = \boldsymbol{\xi}(z, w; x, y) \ \text{ for all } (x, y),\, (z, w) \in \mathscr{C},$$
$$\boldsymbol{\xi}(x, y; z, w) - \frac{dt_1 dt_2}{(t_2 - t_1)^2} \in \mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2]] dt_1 dt_2,$$

[2] If we regard $\mathscr{C}$ as a 2-dimensional real manifold and ω and η are locally closed C^1-class forms, then the product $\omega \star \eta$ coincides with $\frac{1}{2\pi} \int_{\mathscr{C}} \omega \wedge {}^* \eta$.

where t_1 and t_2 are values of the arithmetic parameter t at (x, y), $(z, w) \in \mathscr{C}$, respectively, then $\{\,\omega_j,\ \eta_j\,|\,j = 1,\ \cdots,\ 4\}$ is a required basis (see [8], Prop. 3). After some computation, we find the following simplest ones:

$$\begin{cases} \eta_1 = \big(\mu_4 y^2 + 7x^3 y + 3\mu_2\mu_3 x^3 + 5\mu_3 x^2 y + (3\mu_6 + \mu_2\mu_4)xy + \mu_5\mu_4 y + 3\mu_5 x^3 \\ \qquad + 4\mu_2 x^4 + (2\mu_5\mu_3 + 2\mu_6\mu_2 + \mu_4^2)x^2 + (\mu_2\mu_9 + \mu_4\mu_7 + \mu_5\mu_6)x\big)\dfrac{dx}{f_y(x,y)}, \\ \eta_2 = \dfrac{4x^2 y + 2\mu_3 xy + \mu_5 x^2 + 2\mu_2 x^3 + \mu_2\mu_3 x^2}{f_y(x,y)}\,dx, \\ \eta_3 = \dfrac{2x^3 + \mu_3 x^2 - \mu_9}{f_y(x,y)}\,dx, \\ \eta_4 = \dfrac{xy}{f_y(x,y)}\,dx. \end{cases}$$

Because the η_j's have poles only at ∞, it is not so difficult to check

$$\omega_i \star \omega_j = \eta_i \star \eta_j = 0, \quad \omega_i \star \eta_j = \delta_{ij}.$$

7. The sigma function

In this section, we define the function $\sigma(u)$. First of all we define

$$H' = \left[\int_{\alpha_j} \eta_i\right]_{i,j=1,2,3,4}, \quad H'' = \left[\int_{\beta_j} \eta_i\right]_{i,j=1,2,3,4},$$

and we set

$$M = \begin{bmatrix} \Omega' & \Omega'' \\ H' & H'' \end{bmatrix}. \tag{7.1}$$

This matrix M satisfies the general Legendre relation (Weierstrass relation)

$$M\begin{bmatrix} & -1_g \\ 1_g & \end{bmatrix}{}^t M = -2\pi\boldsymbol{i}\begin{bmatrix} & -1_g \\ 1_g & \end{bmatrix} \tag{7.2}$$

and

$${}^t M\begin{bmatrix} & -1_g \\ 1_g & \end{bmatrix}M = -2\pi\boldsymbol{i}\begin{bmatrix} & -1_g \\ 1_g & \end{bmatrix}. \tag{7.3}$$

A proof of the first relation is given in [1], p.197. The second one is shown by multiplying by ${}^t M$ the left and M on the right of the relation given by taking the inverse of the first relation. Especially, because of (7.2), we see that the matrix $H'\Omega'^{-1}$ is symmetric and

$$H''\,{}^t\Omega' - H'\,{}^t\Omega'' = -2\pi\boldsymbol{i}\,1_g.$$

For two column vectors a and $b \in \mathbb{R}^4$, a 4×4 symmetric matrix T with positive definite imaginary part, we define a function of column vector $z \in \mathbb{C}^4$ by

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z; T) = \sum_{n \in \mathbb{Z}^4} \exp\left[2\pi i\left\{\tfrac{1}{2}\,{}^t(n+a)T(n+a) + {}^t(n+a)(z+b)\right\}\right]. \tag{7.4}$$

Definition 7.5. Under the convention on the symbols above for the curve $\mathscr{C}$, we define

$$\begin{aligned}\sigma(u) &= \sigma(u_7, u_4, u_2, u_1)\\ &= C\ \exp\big(-\tfrac{1}{2}{}^t u H' {\Omega'}^{-1} u\big)\,\vartheta\begin{bmatrix}\delta''\\ \delta'\end{bmatrix}({\Omega'}^{-1}u; {\Omega'}^{-1}\Omega''),\end{aligned}$$

where

$$C = \frac{1}{D^{1/8}}\frac{\sqrt{\det(\Omega')}}{4\pi^2}. \tag{7.6}$$

This is called the *sigma function* attached to $\mathscr{C}$.

We can prove that this function is independent of the choice of $\{\alpha_j\}$ and $\{\beta_j\}$ (Section 5) by using the transformation property of multi-variable theta series with (7.2) and (7.3).

The paper [2] (or [8]) states that the leading terms of the expansion of $\sigma(u)$ around $u = \mathbf{0}$ are given as follows. Namely, there exists a non zero constant C' such that

$$\lim_{\forall \mu_j \to 0} \sigma(u) = C' \cdot (\tfrac{1}{448}{u_1}^8 - \tfrac{1}{8}{u_2}^2{u_1}^4 + u_4u_2{u_1}^2 - u_7u_1 - \tfrac{1}{4}{u_2}^4 + {u_4}^2).$$

It is plausible that $C' = 1$ if we take suitable roots in the definition (7.6) of C. However, the author has no proof of this. So, if necessary, we redefine the constant C such that $C' = 1$. Namely, the function $\sigma(u)$ has the property that

$$\lim_{\forall \mu_j \to 0} \sigma(u) = \tfrac{1}{448}{u_1}^8 - \tfrac{1}{8}{u_2}^2{u_1}^4 + u_4u_2{u_1}^2 - u_7u_1 - \tfrac{1}{4}{u_2}^4 + {u_4}^2. \tag{7.7}$$

The function $\sigma(u)$ is of homogeneous weight $(3^2-1)(5^2-1)/24 = 8$ by [8], p.193.

8. Properties of the sigma function

For each $u \in \mathbb{C}^4$, we denote by u' and u'' the uniquely determined elements in $\mathbb{R}^4$ such that

$$u = \Omega' u' + \Omega'' u''.$$

Moreover, we define

$$L(u,v) = {}^t u(H'v' + H''v'').$$

If we take the point ∞ as a base point, by looking at

$$\omega_1 = \frac{dx}{f_y(x,y)} = -\frac{dy}{f_x(x,y)},$$

we see the canonical class of the divisor class group of $\mathscr{C}$ is represented by $6{\cdot}\infty$. So, if we write the Riemann constant as

$$\Omega'\delta' + \Omega''\delta''$$

using the notational rule above, then δ', $\delta'' \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^4$. By using these symbols, we define

$$\chi(\ell) = \exp 2\pi \boldsymbol{i}({}^t\ell'\delta'' - {}^t\ell''\delta' + \tfrac{1}{2}{}^t\ell'\ell''),$$

where $\ell = \Omega'\ell + \Omega''\ell'' \in \Lambda$. Then we have the following formula.

Lemma 8.1. (translational relation) *For any $u \in \mathbb{C}^4$ and for any $\ell \in \Lambda$, one has*

$$\sigma(u+\ell) = \chi(\ell)\,\sigma(u)\exp L(u + \tfrac{1}{2}\ell, \ell). \tag{8.2}$$

Proof. For the sigma functions for hyperelliptic curve, the relation of type (8.2) is described in several articles. For the general case (hence, for our case), a proof is given in [8]. This property is equivalent to the usual translational relation of multi-variable Riemann theta series and is checked by using (7.2) and (7.3). ■

Lemma 8.3. *The function $\sigma(u)$ vanishes only on the set $\kappa^{-1}(\Theta^{[3]})$, and does not vanish elsewhere. The order of this vanishing is 1. This means that any entire function $G(u)$ on the whole space $\mathbb{C}^4$ vanishing on $\kappa^{-1}(\Theta^{[3]})$ is $\sigma(u)$ times some entire function.*

Proof. Choose a regular polygon associated to the Riemann surface of $\mathscr{C}$. Then, as usual, taking logarithm of (8.2) and integrating it around the boundary of the regular polygon, we get the statement. For detail, we refer to the reader to [1], p.252, for instance. ■

The two Lemmas above, the symmetry $[-1]\Theta^{[3]} = \Theta^{[3]}$ obtained from (5.2), and (7.7) show that $u \mapsto \sigma(u)$ is an even function:

$$\sigma([-1]u) = \sigma(u).$$

We shall give an important relation between the sigma function and Klein's fundamental 2-form (8.4) below. Klein's fundamental 2-form is explicitly computed as

$$\boldsymbol{\xi}(x,y;z,w) = \frac{F(x,y;z,w)dzdx}{(x-z)^2 f_y(x,y) f_y(z,w)},$$

where

$$
\begin{aligned}
&F(x,y;z,w) = (-2x^6+3zx^5-z^3x^3+3z^5x-2z^6){\mu_1}^2\\
&+(-2x^4y+3zx^3y+3wz^3x-2wz^4)\mu_1\mu_2\\
&+(-2x^3y+4zx^2y-2z^2xy+z^3y+wx^3-2wzx^2+4wz^2x-2wz^3)\mu_1\mu_5\\
&+(-3x^5+5zx^4-z^2x^3-z^3x^2+5z^4x-3z^5)\mu_1\mu_4\\
&+(-2x^4+4zx^3-2z^2x^2+4z^3x-2z^4)\mu_1\mu_7\\
&+(-x^3+2zx^2+2z^2x-z^3)\mu_1\mu_{10}\\
&+(-2x^3y^2+3zx^2y^2+wx^3y-2wzx^2y-2wz^2xy+wz^3y+3w^2z^2x-2w^2z^3)\mu_1\\
&+2wzxy{\mu_2}^2+(2wx+2wz)y\mu_2\mu_5+(-x^3y+2zx^2y+2wz^2x-wz^3)\mu_2\mu_4\\
&+(zxy+wzx)\mu_2\mu_7+(xy+wz)\mu_2\mu_{10}\\
&+(-3x^5+5zx^4-z^2x^3-z^3x^2+5z^4x-3z^5)\mu_2\mu_3\\
&+(-2x^4+4zx^3-2z^2x^2+4z^3x-2z^4)\mu_2\mu_6\\
&+(-x^3+2zx^2+2z^2x-z^3)\mu_2\mu_9+2zx\mu_2\mu_{12}+(x+z)\mu_2\mu_{15}\\
&+(2wzy^2+2w^2xy-4x^6+6zx^5-2z^3x^3+6z^5x-4z^6)\mu_2+2wy{\mu_5}^2\\
&+(-x^2y+2zxy+2wzx-wz^2)\mu_4\mu_5+(zy+wx)\mu_5\mu_7+(y+w)\mu_5\mu_{10}\\
&+(-2x^4+4zx^3-2z^2x^2+4z^3x-2z^4)\mu_3\mu_5+(-x^3+2zx^2+2z^2x-z^3)\mu_5\mu_6\\
&+2zx\mu_5\mu_9+(x+z)\mu_5\mu_{12}+2\mu_5\mu_{15}\\
&+(2wy^2+2w^2y-3x^5+5zx^4-z^2x^3-z^3x^2+5z^4x-3z^5)\mu_5\\
&+(-x^4+2zx^3-z^2x^2+2z^3x-z^4){\mu_4}^2+(-x^3+2zx^2+2z^2x-z^3)\mu_4\mu_7\\
&+2zx\mu_4\mu_{10}+(x^2y^2-2zxy^2+2wzxy-2w^2zx+w^2z^2)\mu_4+zx{\mu_7}^2\\
&+(x+z)\mu_7\mu_{10}+(zy^2-wxy-wzy+w^2x)\mu_7+{\mu_{10}}^2+(y^2-2wy+w^2)\mu_{10}\\
&+(5x^4y-8zx^3y+z^2x^2y-2z^3xy+z^4y+wx^4-2wzx^3+wz^2x^2-8wz^3x+5wz^4)\mu_3\\
&+(-3x^3y+6zx^2y-6wz^2x+3wz^3)\mu_6+(2zxy+z^2y+wx^2+2wzx)\mu_9\\
&+(xy+2zy+2wx+wz)\mu_{12}+(3y+3w)\mu_{15}\\
&+(3w^2y^2-7x^5y+10zx^4y-2z^3x^2y+4z^4xy-2z^5y-2wx^5\\
&\qquad\qquad+4wzx^4-2wz^2x^3+10wz^4x-7wz^5)
\end{aligned}
$$

is an element in $\mathbb{Z}[\boldsymbol{\mu}][x,y,z,w]$. The sigma function is very important function for beyond the result of this paper also. So that, for convenience to reference, we quote several terms of expansion $\boldsymbol{\xi}$ in terms of t_1 and t_2, the values of the arithmetic local parameter t at (x,y), $(z,w)\in\mathscr{C}$, respectively:

$$
\begin{aligned}
\boldsymbol{\xi}(x,y;z,w) = \Big\{&\tfrac{1}{(t_1-t_2)^2}-\mu_3(t_1+t_2)+(\mu_4-2\mu_3\mu_1)({t_1}^2+{t_2}^2)\\
&+(2\mu_4-3\mu_3\mu_1)t_1t_2+(2\mu_4\mu_1+\mu_5-3\mu_3{\mu_1}^2-2\mu_3\mu_2)({t_1}^3+{t_2}^3)\\
&+(4\mu_4\mu_1+2\mu_5-5\mu_3{\mu_1}^2-3\mu_3\mu_2)(t_1{t_2}^2+{t_1}^2t_2)\\
&+\text{“higher terms w.r.t. } t_1 \text{ and } t_2\text{”}\Big\}dt_1dt_2.
\end{aligned}
$$

It is not difficult to show that

$$\boldsymbol{\xi}(x,y;z,w)\in\tfrac{1}{(t_1-t_2)^2}\,dt_1dt_2+\mathbb{Z}[\boldsymbol{\mu}][[t_1,t_2]]dt_1dt_2.$$

The property (8.2) implies Riemann's fundamental formula

$$\tag{8.4} \begin{aligned} &\frac{\sigma\Big(\displaystyle\int_\infty^{(x,y)}\omega-\sum_{j=1}^4\int_\infty^{P_j}\omega\Big)\,\sigma\Big(\displaystyle\int_\infty^{(z,w)}\omega-\sum_{j=1}^4\int_\infty^{Q_j}\omega\Big)}{\sigma\Big(\displaystyle\int_\infty^{(x,y)}\omega-\sum_{j=1}^4\int_\infty^{Q_j}\omega\Big)\,\sigma\Big(\displaystyle\int_\infty^{(z,w)}\omega-\sum_{j=1}^4\int_\infty^{P_j}\omega\Big)}\\ &=\exp\Big(\sum_{j=1}^4\int_{P_j}^{(x,y)}\int_{Q_j}^{(z,w)}\boldsymbol{\xi}(x,y;z,w)\Big). \end{aligned}$$

9. Conjugate points

For each point (x,y) on $\mathscr{C}$, the symbols

$$\tag{9.1} (x,y^\star),\quad (x,y^{\star\star})$$

always denote the other conjugate roots of the equation $f(x,y)=0$ with the same x-coordinate. In particular $\infty=\infty^\star=\infty^{\star\star}$. Similarly, for a point $v\in\mathbb{C}^4$ given by

$$\tag{9.2} v=\int_\infty^{(x,y)}\omega,$$

the symbols

$$\tag{9.3} v^\star=\int_\infty^{(x,y^\star)}\omega,\quad v^{\star\star}=\int_\infty^{(x,y^{\star\star})}\omega$$

denote the points obtained by continuously transforming each point on the path of integration in (9.2) to its conjugate point. We frequently use the notations *, ** in this paper.

Lemma 9.4. *Under the usage of symbols above, one has*

$$v+v^\star+v^{\star\star}=\mathbf{0}.$$

Proof. Let (x,y) be a variable point on $\mathscr{C}$. Then, we have

$$\tag{9.5} f(x,Y)=(Y-y)(Y-y^\star)(Y-y^{\star\star}).$$

It suffices to show

$$\frac{dx}{f_y(x,y)}+\frac{dx}{f_y(x,y^\star)}+\frac{dx}{f_y(x,y^{\star\star})}=0.$$

The left hand side of this is written as

$$\frac{dx}{(y-y^\star)(y-y^{\star\star})}+\frac{dx}{(y^\star-y^{\star\star})(y^\star-y)}+\frac{dx}{(y^{\star\star}-y)(y^{\star\star}-y^\star)},$$

and, by (9.5), this equals to 0. ∎

10. Higher derivatives of the sigma function

We denote partial derivatives of $\sigma(u)$ by

$$\sigma_{ij\cdots k}(u) = \frac{\partial}{\partial u_k}\cdots\frac{\partial}{\partial u_j}\frac{\partial}{\partial u_i}\sigma(u).$$

Extending Lemmas 8.4 and 8.3, we will show in this Section the following.

Proposition 10.1. (1) (On the whole space) *The function* $u \mapsto \sigma(u)$ *on* $\mathbb{C}^4$ *vanishes only on* $\kappa^{-1}(\Theta^{[3]})$. *For any* $u \in \mathbb{C}^4$ *and* $\ell \in \Lambda$, *we have*

$$\sigma(u+\ell) = \chi(\ell)\sigma(u)\exp L(u+\tfrac{1}{2}\ell,\ell).$$

Let $u \in \kappa^{-1}(W^{[3]})$ *be a fixed point not belonging to* $\kappa^{-1}(W^{[2]})$, *and let* $v^{(1)}$, $v^{(2)}$, $v^{(3)} \in \kappa^{-1}(W^{[1]})$ *be three points determined by*

$$u = -v^{(1)} - v^{(2)} - v^{(3)}.$$

Then the function

$$\kappa^{-1}(W^{[1]}) \ni v \mapsto \sigma(u+v) \tag{10.2}$$

has zeroes modulo Λ *only at* $u = 0$, $v^{(1)}$, $v^{(2)}$, $v^{(3)}$, *which are of order* 1, *and no other zeroes. Moreover, for a fixed* $u \in \kappa^{-1}(W^{[3]})$, *the power series expansion of* (10.2) *at the origin with respect to* v_1 *is of the form*

$$\sigma(u+v) = \sigma_1(u)v_1 + O({v_1}^2). \tag{10.3}$$

(2) (On the 3rd stratum) *The function* $u \mapsto \sigma_1(u)$ *on* $\kappa^{-1}(\Theta^{[3]})$ *vanishes only on* $\kappa^{-1}(\Theta^{[2]})$:

$$\sigma_1(u) = 0 \iff u \in \kappa^{-1}(\Theta^{[2]}) \quad (\textit{ for } u \in \kappa^{-1}(\Theta^{[3]})\ ). \tag{10.4}$$

For any $u \in \kappa^{-1}(\Theta^{[3]})$ *and* $\ell \in \Lambda$, *we have*

$$\sigma_1(u+\ell) = \chi(\ell)\sigma_1(u)\exp L(u+\tfrac{1}{2}\ell,\ell). \tag{10.5}$$

Let $u \in \kappa^{-1}(W^{[2]})$ *be a fixed point not belonging to* $\kappa^{-1}(W^{[1]})$, *and let* $v^{(1)}$, $v^{(2)}$, $v^{(3)} \in \kappa^{-1}(W^{[1]})$ *be three points determined by*

$$u = -v^{(1)} - v^{(2)} - v^{(3)}. \tag{10.6}$$

Then the function

$$\kappa^{-1}(W^{[1]}) \ni v \mapsto \sigma_1(u+v) \tag{10.7}$$

has zeroes modulo Λ *only at* $u = 0$, $v^{(1)}$, $v^{(2)}$, $v^{(3)}$, *which are of order* 1, *and no other zeroes. Moreover, for a fixed* $u \in \kappa^{-1}(W^{[2]})$, *the power series expansion of* (10.7) *at the origin with respect to* v_1 *is of the form*

$$\sigma_1(u+v) = -\sigma_2(u)v_1 + O({v_1}^2). \tag{10.8}$$

(3) (On the 2nd stratum) *The function* $u \mapsto \sigma_2(u)$ *on* $\kappa^{-1}(\Theta^{[2]})$ *vanishes only on* $\kappa^{-1}(\Theta^{[1]})$:

$$\sigma_2(u) = 0 \iff u \in \kappa^{-1}(\Theta^{[1]}) \quad (\textit{ for } u \in \kappa^{-1}(\Theta^{[2]})\ ). \tag{10.9}$$

For any $u \in \kappa^{-1}(\Theta^{[2]})$ *and* $\ell \in \Lambda$, *we have*

$$\sigma_2(u+\ell) = \chi(\ell)\sigma_2(u)\exp L(u+\tfrac{1}{2}\ell,\ell). \tag{10.10}$$

Let $u \in \kappa^{-1}(W^{[1]})$ *be a fixed point not belonging to* Λ. *Then the function*

$$\kappa^{-1}(W^{[1]}) \ni v \mapsto \sigma_2(u+v) \tag{10.11}$$

has zeroes modulo Λ *at* $u = 0$ *which is of order* 2, *at* $u = v^\star$ *and* $v^{\star\star}$, *which are of order* 1, *and no other zeroes. Moreover, for a fixed* $u \in \kappa^{-1}(W^{[1]})$, *the power series expansion of* (10.11) *at the origin with respect to* v_1 *is of the form*

$$\sigma_2(u+v) = -\sigma_4(u){v_1}^2 + O({v_1}^3). \tag{10.12}$$

(4) (On the 1st stratum) *The function* $u \mapsto \sigma_4(u)$ *on* $\kappa^{-1}(\Theta^{[1]})$ *vanishes only on* Λ:

$$\sigma_4(u) = 0 \iff u \in \Lambda \quad (\text{ for } u \in \kappa^{-1}(\Theta^{[1]})). \tag{10.13}$$

For any $u \in \kappa^{-1}(\Theta^{[1]})$ *and* $\ell \in \Lambda$, *we have*

$$\sigma_4(u+\ell) = \chi(\ell)\sigma_4(u)\exp L(u+\tfrac{1}{2}\ell,\ell). \tag{10.14}$$

Let $u \in \kappa^{-1}(W^{[1]})$ *be a fixed point not belonging to* Λ. *Then the function*

$$\kappa^{-1}(W^{[1]}) \ni v \mapsto \sigma_4(v) \tag{10.15}$$

has zeroe at each point in Λ *which is of order* 4, *and no other zeroes. Moreover, the power series expansion of* (10.15) *at the origin with respect to* v_1 *is of the form*

$$\sigma_4(v) = {v_1}^4 + O({v_1}^5). \tag{10.16}$$

Proof. After differentiating (8.2) with respect to u_1, if we restrict u to $\kappa^{-1}(\Theta^{[3]}) = \kappa^{-1}(W^{[3]})$, then we have

$$\sigma_1(u+\ell) = \chi(\ell)\sigma_1(u)\exp L(u+\tfrac{1}{2}\ell,\ell) \quad \text{for } u \in \kappa^{-1}(\Theta^{[3]})$$

by Lemma 8.3. For a fixed $u \in \kappa^{-1}(\Theta^{[2]})$, $\notin \kappa^{-1}(\Theta^{[1]})$, the function $\kappa^{-1}(\Theta^{[1]}) \ni v \mapsto \sigma_1(u+v)$ has exactly 4 zeroes modulo Λ with allowing multiplicities. This is shown by the formula above with the same argument as the proof of Lemma 8.3. One has

$$\sigma(u+v) = 0 \ \text{ for } u \in \kappa^{-1}(W^{[2]}),\, v \in \kappa^{-1}(W^{[1]})$$

because $u + v \in \kappa^{-1}(\Theta^{[3]})$. Expanding this with respect to v_1, we get

$$0 = \sigma(u+v) = \sigma_1(u)\,v_1 + \text{“terms of higher order in } v_1\text{”}.$$

Therefore,

$$\sigma_1(u) = 0 \ \text{ for } u \in \kappa^{-1}(W^{[2]}). \tag{10.17}$$

Let $u^{(1)}$ and $u^{(2)}$ be points on $\kappa^{-1}(W^{[1]})$ such that $u^{(1)} + u^{(2)} \notin \kappa^{-1}(W^{[1]})$. Since $\Theta^{[3]}$ is symmetric with respect to the origin $\mathbf{0}$ by (5.2),

$$[-1]W^{[3]} = [-1]\Theta^{[3]} = \Theta^{[3]}, \tag{10.18}$$

there exist three points $v^{(1)}$, $v^{(2)}$, $v^{(3)} \in \kappa^{-1}(W^{[1]})$ such that

$$u^{(1)} + u^{(2)} = -v^{(1)} - v^{(2)} - v^{(3)}. \tag{10.19}$$

Then the function

$$\kappa^{-1}(W^{[1]}) \ni v \mapsto \sigma_1(u^{(1)} + u^{(2)} + v)$$

has zeroes of order 1 at $v = 0$, $v^{(1)}$, $v^{(2)}$, $v^{(3)}$ mod Λ. Expanding $\sigma(u+v)$ with respect to v_1, we have

$$\begin{aligned} 0 &= \sigma(u+v) \\ &= \sigma_1(u)\,v_1 + \sigma_2(u)\,v_2 + \sigma_{11}(u)\,\tfrac{1}{2!}{v_1}^2 + \cdots \\ &= \sigma_1(u)\,v_1 + \sigma_2(u)\,\tfrac{1}{2}{v_1}^2 + \sigma_{11}(u)\,\tfrac{1}{2!}{v_1}^2 + \text{“terms of higher order in } v_1\text{”} \end{aligned}$$

by using (5.3). Hence,

$$\sigma_{11}(u) = -\sigma_2(u) \quad \text{for } u \in \kappa^{-1}(W^{[2]}). \tag{10.20}$$

This implies

$$\left.\begin{aligned} \sigma_1(u+v) &= \sigma_{11}(u)v_1 + O({v_1}^2) \\ &= -\sigma_2(u)v_1 + O({v_1}^2) \end{aligned}\right\} \quad \text{for } u \in \kappa^{-1}(W^{[2]}),\ v \in \kappa^{-1}(W^{[1]}).$$

(10.8) and (10.17) show

$$\sigma_{11}(u) = \sigma_2(u) = 0 \quad \text{for } u \in \kappa^{-1}(W^{[1]}).$$

So, after differentiating (8.2) with respect to u_2, restricting u to $\kappa^{-1}(W^{[1]})$ gives

$$\sigma_2(u+\ell) = \chi(\ell)\sigma_2(u)\exp L(u + \tfrac{1}{2}\ell, \ell) \quad \text{for } u \in \kappa^{-1}(W^{[2]}).$$

Similar arguments give the following equations :

$$\left.\begin{aligned} \sigma_{11}(u) &= 0, \\ \sigma_2(u) &= 0, \\ \sigma_{111}(u) &= 0, \\ \sigma_{12}(u) &= 0 \end{aligned}\right\} \quad \text{for } u \in \kappa^{-1}(W^{[1]}),$$

$$\left.\begin{aligned} \sigma_{1111}(u) &= 3\,\sigma_4(u), \\ \sigma_{112}(u) &= -\sigma_4(u), \\ \sigma_{22}(u) &= -\sigma_4(u) \end{aligned}\right\} \quad \text{for } u \in \kappa^{-1}(W^{[1]}),$$

$$\left.\begin{aligned} \sigma_2(u+v) &= \sigma_{22}(u)v_2 + \sigma_{112}(u)\tfrac{1}{2!}{v_1}^2 + O({v_1}^3) \\ &= \sigma_{22}(u)\tfrac{1}{2}{v_1}^2 + \sigma_{112}(u)\tfrac{1}{2!}v_1 + O({v_1}^3) \\ &= -\sigma_4(u){v_1}^2 + O({v_1}^3) \end{aligned}\right\} \quad \text{for } u,\ v \in \kappa^{-1}(W^{[1]}),$$

$$\sigma_4(u+\ell) = \chi(\ell)\sigma_4(u)\exp L(u + \tfrac{1}{2}\ell, \ell) \quad \text{for } u \in \kappa^{-1}(W^{[1]}).$$

Now, if $\sigma_4(\kappa^{-1}(W^{[1]})) = 0$, then, by (10.12), for any $u \in \kappa^{-1}(W^{[1]})$, the function

$$\kappa^{-1}(W^{[1]}) \ni v \mapsto \sigma_2(u+v)$$

has a zero of order 3 or higher at $v = (0,0,0,0)$. On the other hand, since $-u = u^\star + u^{\star\star}$ and $v \mapsto \sigma_4(v)$ is an even function, the function (10.11) has zeroes at $v = u^\star$ and $u^{\star\star}$. Summing up, the function has at least 5 zeroes with allowing multiplicities. However, if (10.11) is not identically 0, it must have just 4 zeroes modulo Λ with allowing multiplicities. This is seen by (10.10) with the same argument as usual proof of Lemma 8.3. Therefore, (10.11) is identically 0 on $\kappa^{-1}(W^{[1]})$.

In the next instance, let us consider, for arbitrarily fixed $u \in \kappa^{-1}(W^{[2]})$, the function

$$\kappa^{-1}(W^{[1]}) \ni v \mapsto \sigma_1(u+v).$$

The discussion above says that this function has a zero of order 2 or higher at $v = (0,0,0,0)$. Since $u \in \kappa^{-1}(W^{[2]})$, one has $-u \in [-1]\kappa^{-1}(W^{[2]}) \subset [-1]\kappa^{-1}(\Theta^{[3]}) = \kappa^{-1}(W^{[3]})$ by (10.18). Hence we can rewrite it as $-u = u^{(1)} + u^{(2)} + u^{(3)}$ with some $u^{(j)} \in \kappa^{-1}(W^{[1]})$. As $\kappa^{-1}(\Theta^{[2]}) \ni u \mapsto \sigma_2(u)$ is also an even function, the function (10.7) has zeroes at $v = u^{(1)}$, $u^{(2)}$, $u^{(3)}$. Summing up our arguments, we see the function (10.7) vanishes identically on $\kappa^{-1}(W^{[2]})$.

A similar argument shows that, for any $u \in \kappa^{-1}(W^{[3]})$, the function

$$\kappa^{-1}(W^{(1)}) \ni v \mapsto \sigma(u+v)$$

is identically 0, namely, $\mathbb{C}^4 \ni u \mapsto \sigma(u)$ is identically 0. This is a contradiction. Thus, $\kappa^{-1}(W^{[1]}) \ni v \mapsto \sigma_4(v)$ is not identically 0.

Finally, we check the zeroes of $\kappa^{-1}(W^{[1]}) \ni u \mapsto \sigma_4(u)$. Suppose $\sigma_4(u) = 0$ for some $u \in \kappa^{-1}(W^{[1]})$, $\notin \Lambda$. Because $-u = u^\star + u^{\star\star}$, the definition of conjugate points, and the assumption $u \notin \Lambda$, we see $u^\star$ and $u^{\star\star} \notin \Lambda$. Now, a similar argument shows the function (10.11) for this u is identically 0. This is a contradiction. Hence we see (10.13), and an expansion

$$\sigma_4(u) = c\,{u_1}^4 + O({u_1}^5) \quad (c \neq 0),$$

where c is non zero constant. This c is exactly 1 by (7.7) and (5.3). Hence (10.16). Further argument as above shows (10.4) and (10.9). Now we have completed the proof. ∎

11. Frobenius-Stickelberger-type formulae

The initial formula of Frobenius-Stickelberger formulae is as follows:

Proposition 11.1. *For each v, the two points $v^\star$ and $v^{\star\star} \in \kappa^{-1}(W^{[1]})$ are defined as in* (9.3)*. Then, for any u and any $v \in \kappa^{-1}(W^{[1]})$, one has*

$$-\frac{\sigma_2(u+v)\,\sigma_2(u+v^\star)\,\sigma_2(u+v^{\star\star})}{\sigma_4(u)^3\,\sigma_4(v)\,\sigma_4(v^\star)\,\sigma_4(v^{\star\star})} = \begin{vmatrix} 1 & x(u) \\ 1 & x(v) \end{vmatrix}^2. \tag{11.2}$$

Proof. By (10.10) and (10.14), the left hand side is invariant under transforms $u \to u+\ell$ and $v \to v+\ell$ for $\ell \in \Lambda$. Indeed, for the transform $u \to u+\ell$, for example, the argument of the exponential factor is, by using Lemma 9.4,

$$
\begin{aligned}
&L(u+v+\tfrac{1}{2}\ell,\ell)+L(u+v^{\star}+\tfrac{1}{2}\ell,\ell)+L(u+v^{\star\star}+\tfrac{1}{2}\ell,\ell)-3\,L(u+\tfrac{1}{2}\ell,\ell)\\
&=L(v+v^{\star}+v^{\star\star},\ell)\\
&=L(0,\ell)\\
&=0,
\end{aligned}
$$

and for $v\to v+\ell$, it is

$$
\begin{aligned}
&L(u+v+\tfrac{1}{2}\ell,\ell)+L(u+v^{\star}+\tfrac{1}{2}\ell^{\star},\ell^{\star})+L(u+v^{\star\star}+\tfrac{1}{2}\ell^{\star\star},\ell^{\star\star})\\
&\qquad -L(v+\tfrac{1}{2}\ell,\ell)-L(v^{\star}+\tfrac{1}{2}\ell^{\star},\ell^{\star})-L(v^{\star\star}+\tfrac{1}{2}\ell^{\star\star},\ell^{\star\star})\\
&=L(u,\ell)+L(u,\ell^{\star})+L(u,\ell^{\star\star})\\
&=L(u,\ell+\ell^{\star}+\ell^{\star\star})\\
&=L(u,0)\\
&=0.
\end{aligned}
$$

By the results in the Section 10, as a function of the variable u modulo Λ, the left hand side of the claimed equation (11.2) has a zero of order 6 at $u=0$, and zeroes of order 2 at $u=v$, $u=v^{\star}$, $u=v^{\star\star}$; and it has no other zeroes and poles. As a function of v, since it has the same property, it must be a non zero constant times $\big(x(u)-x(v)\big)^2$. From (10.12) and (10.16), we can easily see such the multiplicative constant is -1. ■

Corollary 11.3. *The function*

$$
\frac{\sigma_2(u+v^{\star})\,\sigma_2(u+v^{\star\star})}{\sigma_4(u)^2\,\sigma_4(v^{\star})\,\sigma_4(v^{\star\star})}
$$

is invariant under the exchange $u\leftrightarrow v$. Namely, one has

$$
\frac{\sigma_2(u+v^{\star})\,\sigma_2(u+v^{\star\star})}{\sigma_4(u)^2\,\sigma_4(v^{\star})\,\sigma_4(v^{\star\star})}=\frac{\sigma_2(v+u^{\star})\,\sigma_2(v+u^{\star\star})}{\sigma_4(v)^2\,\sigma_4(u^{\star})\,\sigma_4(u^{\star\star})}.
$$

Theorem 11.4. (1) (Third stratum) *For the three variables $u^{(1)}$, $u^{(2)}$, $u^{(3)}\in\kappa^{-1}(\Theta^{[1]})$, one has*

$$
\begin{aligned}
&-\frac{\sigma_1(u^{(1)}+u^{(2)}+u^{(3)})\prod_{i<j}\sigma_2(u^{(i)}+u^{(j)\star})\sigma_2(u^{(i)}+u^{(j)\star\star})}{\prod_{j=1}^{3}\sigma_4(u^{(j)})^{6-2j+1}\,\sigma_4(u^{(j)\star})^{j-1}\,\sigma_4(u^{(j)\star\star})^{j-1}}\\
&=\begin{vmatrix}1 & x(u^{(1)}) & y(u^{(1)})\\ 1 & x(u^{(2)}) & y(u^{(2)})\\ 1 & x(u^{(3)}) & y(u^{(3)})\end{vmatrix}\begin{vmatrix}1 & x(u^{(1)}) & x^2(u^{(1)})\\ 1 & x(u^{(2)}) & x^2(u^{(2)})\\ 1 & x(u^{(3)}) & x^2(u^{(3)})\end{vmatrix}.
\end{aligned}
$$

(2) (General form) *Let $n\geqq 4$ be an integer. For n variables $u^{(1)}$, $\cdots$, $u^{(n)}\in$*

$\kappa^{-1}(\Theta^{[1]})$, one has

$$(-1)^n \frac{\sigma(u^{(1)}+\cdots+u^{(n)})\prod_{i<j}\sigma_2(u^{(i)}+u^{(j)\star})\sigma_2(u^{(i)}+u^{(j)\star\star})}{\prod_{j=1}^{n}\sigma_4(u^{(j)})^{2n-2j+1}\,\sigma_4(u^{(j)\star})^{j-1}\,\sigma_4(u^{(j)\star\star})^{j-1}}$$
$$= \pm \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(1)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(1)}) & \cdots \end{vmatrix} \begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & \cdots & x^{n-1}(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & \cdots & x^{n-1}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^{n-1}(u^{(n)}) \end{vmatrix}.$$

Here the rows of the first determinant of the right hand side is shown as n monomials in x and y in order of the order of their poles at $u = \mathbf{0}$.

Proof. Both left hand sides of (1) and (2) is periodic functions with respect to Λ because of the translational formulae for each stratum. Since the formula (1) is proved similarly to (2), we shall give a proof only of (2).

Writing $v = u^{(1)}$, we compare the two sides as functions of v. Both sides has zeroes of order 2 at $v = u^{(j)}$ $(j = 2, \cdots, n)$, and zeroes of order 1 at $v = u^{(j)\star}$ and $u^{(j)\star\star}$ $(j = 2, \cdots, n)$. Then we know $2(n-1) + 2(n-1) = 4n-4$ zeroes with multiplicities. Both sides have a pole of order $4n$ at $v = \mathbf{0}$, and have no other poles. Now, let α_1, α_2, α_3, α_4 be the remaining $(4n+2) - 4n = 4$ zeroes modulo Λ of the right hand side. By the theorem of Abel-Jacobi, we see

$$\begin{aligned}\sum_{j=2}^{n} u^{(j)} &+ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ &= \sum_{j=2}^{n} 2u^{(j)} + \sum_{j=2}^{n}(u^{(j)\star} + u^{(j)\star\star}) + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \equiv \mathbf{0} \mod \Lambda\end{aligned}$$

by $u^{(j)} + u^{(j)\star} + u^{(j)\star\star} = \mathbf{0}$. Since we can replace the α_j's modulo Λ as the "$\equiv$" can be taken to be an equality, we have

$$\sigma(v + u^{(2)} + u^{(3)} + \cdots + u^{(n)}) = \sigma(v - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4).$$

This has zeroes of order 1 at $v = \alpha_1$, α_2, α_3, α_4. Therefore, the the divisors of two sides coincide. Now, if we expand both sides with respect to v_1, the coefficients of the least order terms, namely, ones of $1/{v_1}^{4n+1}$ coincide because of the hypothesis of induction. Thus, the desired formula has been proved. ■

Remark 11.5. The left hand side is indeed symmetric under the exchanges $u^{(i)} \leftrightarrow u^{(j)}$ because the right hand side is symmetric. This is seen also by (11.3).

Remark 11.6. If we follow the usage of symbols in [9], we shall write

$$\sigma_1(u) = \sigma_{\natural^3}(u), \quad \sigma_2(u) = \sigma_{\natural^2} = \sigma_{\flat}(u), \quad \sigma_4(u) = \sigma_{\natural^1} = \sigma_{\sharp}(u).$$

12. Purely trigonal Case

If $\mu_2 = \mu_5 = \mu_4 = \mu_7 = \mu_{10} = 0$, the curve $\mathscr{C}$ has an automorphism

$$(x, y) \mapsto (x, \zeta y)$$

for $\zeta = \exp(\frac{2\pi i}{3})$. In this case, for $(x, y) \in \mathscr{C}$ or $v \in \mathbb{C}^4$, we have

$$y^\star = \zeta y, \quad y^{\star\star} = \zeta^2 y, \quad v^\star = [\zeta]v = (\zeta v_1, \zeta^2 v_2, \zeta v_4, \zeta v_7), \quad v^{\star\star} = [\zeta]^2 v.$$

Then, it is easy to see that

$$[\zeta]\,\Lambda = \Lambda, \quad \sigma([\zeta]u) = \zeta^2\,\sigma(u)$$

and a result similar to [10] is obtained.

13. Kiepert-type formula

To give our Kiepert-type formula, we need some lemmas.

Lemma 13.1. *For $u \in \kappa^{-1}(W^{[1]})$, one has*

$$\frac{\sigma_2(2u)}{\sigma_4(u)^4} = f_y\big(x(u), y(u)\big). \tag{13.2}$$

Proof. By the translational formulae, the left hand side is periodic with respect to Λ. For any $u \in \mathbb{C}^4$, we see

$$\sigma_2(u) = \tfrac{1}{4}u_2{u_1}^4 + u_4{u_1}^2 - {u_2}^3 + \text{“higher weight terms w. r. t. } u_j\text{s”}.$$

On the other hand, for $u \in \kappa^{-1}(W^{[1]})$, one has

$$\begin{aligned}\sigma_2(2u) &= 8u_2{u_1}^4 + 8u_4{u_1}^2 - 8{u_2}^3 + \text{“higher weight terms w. r. t. } u_j\text{s”}\\ &= 8(\tfrac{1}{2}{u_1}^2 + \cdots){u_1}^4 + 8(\tfrac{1}{4}{u_1}^4 + \cdots){u_1}^2 - 8(\tfrac{1}{2}{u_1}^2 + \cdots)^3 + \cdots\\ &= 3\,{u_1}^6 + \cdots .\end{aligned}$$

This and (10.16) imply that the left hand side of (13.2) has poles of order $4 \times 4 - 6 = 10$ at $u \in \Lambda$, and no pole elsewhere. Since, for u and $v \in \kappa^{-1}(W^{[1]})$, one has

$$\sigma_2(u+v) = 0 \iff v \equiv u^\star \ \text{ or } \ v \equiv u^{\star\star} \mod \Lambda,$$

we see, for $u \in \kappa^{-1}(W^{[1]})$ with $u \notin \Lambda$, that

$$\begin{aligned}&\frac{\sigma_2(2u)}{\sigma(u)^4} = 0 \iff \sigma_2(2u) = 0 \iff u \equiv u^\star \ \text{ or } \ u^{\star\star} \mod \Lambda\\ &\iff \left\{\begin{array}{l}\displaystyle\int_\infty^{(x(u),y(u))}\omega(x,y) = \int_\infty^{(x(u),y(u^\star))}\omega(x,y), \ \text{ or } \int_\infty^{(x(u),y(u^{\star\star}))}\omega(x,y)\\ \hfill\text{for suitable paths of integration}\end{array}\right.\\ &\iff y(u) = y(u^\star) \ \text{ or } \ y(u^{\star\star}).\end{aligned}$$

Hence the left hand side is a non-zero constant times

$$3\big(y(u) - y(u^\star)\big)\big(y(u) - y(u^{\star\star})\big) = f_y\big(x(u), y(u)\big)$$

However, (13) shows that

$$\frac{\sigma_2(2u)}{\sigma_4(u)^4} = \frac{3}{{u_1}^{10}} + \cdots .$$

Therefore we have the desired formula by (5.4). ∎

Lemma 13.3. *One has*

$$\lim_{v\to u}\frac{\sigma_2(u+v^\star)\sigma_2(u+v^{\star\star})}{\sigma_4(v^\star)\sigma_4(v^{\star\star})(u_1-v_1)^2}=-\frac{f_y(x(u),y(u))}{x^4(u)}. \tag{13.4}$$

Proof. By using (13.2), we have

$$\begin{aligned}f_y(x(u),y(u))\cdot\lim_{v\to u}&\frac{\sigma_2(u+v^\star)\sigma_2(u+v^{\star\star})}{\sigma_4(v^\star)\sigma_4(v^{\star\star})(u_1-v_1)^2}\\&=\lim_{v\to u}\frac{\sigma_2(u+v)\sigma_2(u+v^\star)\sigma_2(u+v^{\star\star})}{\sigma_4(u)^3\sigma_4(v)\sigma_4(u^\star)\sigma_4(v^{\star\star})(u_1-v_1)^2}\\&=-\lim_{v\to u}\Big(\frac{x(u)-x(v)}{u_1-v_1}\Big)^2=-\Big(\frac{dx}{du_1}\Big)^2=-\Big(\frac{f_y(x(u),y(u))}{x(u)^2}\Big)^2.\end{aligned}$$

This gives the formula (13.4). ■

Definition 13.5. (generalized division polynomials) For any integer $n>0$, we define a function $u\mapsto\psi_n(u)$ on $\kappa^{-1}\iota(\mathscr{C})$ by

$$\psi_n(u)=\begin{cases}1\quad(\text{if }\ n=1), & \dfrac{\sigma_2(2u)}{\sigma_4(u)^{2^2}}\quad(\text{if }\ n=2),\\ \dfrac{\sigma_1(3u)}{\sigma_4(u)^{3^2}}\quad(\text{if }\ n=3), & \dfrac{\sigma(nu)}{\sigma_4(u)^{n^2}}\quad(\text{if }\ n\geqq 4).\end{cases}$$

The translational relations (10.10), (10.14) show that these are periodic with respect to Λ, namely, $\psi_n(u+\ell)=\psi_n(u)$ for any $\ell\in\Lambda$.

Using Lemma 13.3 and

$$\lim_{v\to u}\frac{x(u)-x(v)}{u_1-v_1}=\frac{d}{du_1}x(u)=\frac{f_y(x(u),y(u))}{x(u)^2},$$

we get the following theorem by a similar argument of taking the limit of n variables close to the point u in Theorem 11.4 (2) as in the proof of Kiepert-type formulae in [9] and [10]:

Theorem 13.6. (Kiepert-type formula) *Let n be a positive integer and $u\in\kappa^{-1}\iota(\mathscr{C})$. One has*

$$\begin{aligned}\psi_n(u)=(-1)^{n(n+1)/2}&\Big(\prod_{k=1}^{n-1}k!\Big)^{-2}\Big(\frac{x^4}{f_y}\Big)^{n(n-1)/2}\\&\times\begin{vmatrix}x' & y' & (yx)' & (x^3)' & (y^2)' & \cdots\\ x'' & y'' & (yx)'' & (x^3)'' & (y^2)'' & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\\ x^{(n-1)} & y^{(n-1)} & (yx)^{(n-1)} & (x^3)^{(n-1)} & (y^2)^{(n-1)} & \cdots\end{vmatrix}\\&\times\begin{vmatrix}x' & (x^2)' & \cdots & (x^{n-1})'\\ x'' & (x^2)'' & \cdots & (x^{n-1})''\\ \vdots & \vdots & \ddots & \vdots\\ x^{(n-1)} & (x^2)^{(n-1)} & \cdots & (x^{n-1})^{(n-1)}\end{vmatrix}(u),\end{aligned}\tag{13.7}$$

where $'$ stands for $\frac{d}{du_1}=\frac{f_y}{x^2}\frac{d}{dx}$ and both determinants are of size $(n-1)\times(n-1)$.

Proof. First of all, we define a symbol by

$$\varDelta_h{}^k G(u) = G(u+h) - \Big(G(u) + \frac{G'(u)}{1!}h_1 + \frac{G''(u)}{2!}{h_1}^2 + \cdots + \frac{G^{(k)}(u)}{k!}{h_1}^k\Big)$$

Taylor's theorem states that

$$\varDelta_h{}^{k-1} G(u) = \frac{G^{(k)}(u)}{k!}{h_1}^k + O({h_1}^{k+1}). \tag{13.8}$$

We rewrite $u^{(1)} = u$ and set $h = (h_7, h_4, h_2, h_1) = u^{(2)} - u$. Then

$$h^\star = u^{(2)\star} - u^\star, \qquad h^{\star\star} = u^{(2)\star\star} - u^{\star\star},$$

and

$$\begin{aligned}
&\sigma(2u+h+u^{(3)}+\cdots+u^{(n)})\,\frac{\sigma_2(u+u^\star+h^\star)\,\sigma_2(u+u^{\star\star}+h^{\star\star})}{\sigma_4(u^\star+h^\star)\,\sigma_4(u^{\star\star}+h^{\star\star})}\\
&\times\prod_{j=3}^{n}\sigma_2(u+u^{(j)\star})\,\sigma_2(u+u^{(j)\star\star})\prod_{j=3}^{n}\sigma_2(u+h+u^{(j)\star})\,\sigma_2(u+h+u^{(j)\star\star})\\
&\quad\prod_{3\leqq i<j}\sigma_2(u^{(i)}+u^{(j)\star})\,\sigma_2(u^{(i)}+u^{(j)\star\star})\\
&\quad\Big/\Big(\sigma(u)^{2n-1}\,\sigma(u+h)^{2n-3}\prod_{j=3}^{n}\sigma_4(u^{(j)})^{2n-2j+1}\,\sigma_4(u^{(j)\star})^{j-1}\,\sigma_4(u^{(j)\star\star})^{j-1}\Big)\\
&=\begin{vmatrix}
1 & x(u) & y(u) & x^2(u) & \cdots\\
0 & \varDelta_h{}^0 x(u) & \varDelta_h{}^0 y(u) & \varDelta_h{}^0 x^2(u) & \cdots\\
1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & \cdots\\
\vdots & \vdots & \vdots & \vdots & \ddots\\
1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & \cdots
\end{vmatrix}\\
&\quad\times\begin{vmatrix}
1 & x(u) & x^2(u) & \cdots & x^{n-1}(u)\\
0 & \varDelta_h{}^0 x(u) & \varDelta_h{}^0 x^2(u) & \cdots & \varDelta_h{}^0 x^2(u)\\
1 & x(u^{(3)}) & x^2(u^{(3)}) & \cdots & x^{n-1}(u^{(3)})\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^{n-1}(u^{(n)})
\end{vmatrix}
\end{aligned}\tag{13.9}$$

$$\begin{aligned}
&=\begin{vmatrix}
1 & x(u) & y(u) & x^2(u) & \cdots\\
0 & x'(u)h_1+O({h_1}^2) & y'(u)h_1+O({h_1}^2) & (x^2)'(u)h_1+O({h_1}^2) & \cdots\\
1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & \cdots\\
\vdots & \vdots & \vdots & \vdots & \ddots\\
1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & \cdots
\end{vmatrix}\\
&\times\begin{vmatrix}
1 & x(u) & x^2(u) & \cdots & x^{n-1}(u)\\
0 & x'(u)h_1+O({h_1}^2) & (x^2)'(u)h_1+O({h_1}^2) & \cdots & (x^{n-1})'(u)h_1+O({h_1}^2)\\
1 & x(u^{(3)}) & x^2(u^{(3)}) & \cdots & x^{n-1}(u^{(3)})\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^{n-1}(u^{(n)})
\end{vmatrix}
\end{aligned}$$

by (13.8). After dividing this equation by ${h_1}^2 = (u_1 - u_1^{(2)})^2$, taking limit $h_1 \to 0$, we have by (13.4) that

$$
\begin{aligned}
&(-1)^{n+1}\sigma(2u + u^{(3)} + \cdots + u^{(n)})\,\frac{f_y(x(u),y(u))}{x^4(u)}\\
&\times \Big[\ \prod_{j=3}^{n} \sigma_2(u + u^{(j)\star})\,\sigma_2(u + u^{(j)\star\star})\ \Big]^2 \prod_{3\leqq i<j} \sigma_2(u^{(i)} + u^{(j)\star})\,\sigma_2(u^{(i)} + u^{(j)\star\star})\\
&\quad\Big/ \Big[\ \sigma_4(u)^{4n-4} \prod_{j=3}^{n} \sigma_4(u^{(j)})^{2n-2j+1}\,\sigma_4(u^{(j)\star})^{j-1}\,\sigma_4(u^{(j)\star\star})^{j-1}\ \Big]\\
&= \begin{vmatrix} 1 & x(u) & y(u) & x^2(u) & \cdots \\ 0 & x'(u) & y'(u) & (x^2)'(u) & \cdots \\ 1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & \cdots \end{vmatrix} \cdot \begin{vmatrix} 1 & x(u) & x^2(u) & \cdots & x^{n-1}(u) \\ 0 & x'(u) & (x^2)'(u) & \cdots & (x^{n-1})'(u) \\ 1 & x(u^{(3)}) & x^2(u^{(3)}) & \cdots & x^{n-1}(u^{(3)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^{n-1}(u^{(n)}) \end{vmatrix}.
\end{aligned}
$$

In the next place, by plugging $h = u^{(3)} - u$, we see by basic row operations that

$$
\begin{aligned}
&(-1)^{n+1}\sigma(3u + h + u^{(4)} + \cdots + u^{(n)})\,\frac{f_y(x(u),y(u))}{x^4(u)}\\
&\times \left(\frac{\sigma_2(u + u^\star + h^\star)\,\sigma_2(u + u^{\star\star} + h^{\star\star})}{\sigma_4(u^\star + h^\star)^2\,\sigma_4(u^{\star\star} + h^{\star\star})^2} \right)^2 \left(\prod_{j=4}^{n} \sigma_2(u + u^{(j)\star})\,\sigma_2(u + u^{(j)\star\star}) \right)^2\\
&\times \prod_{j=4}^{n} \sigma_2(u + h + u^{(j)\star})\,\sigma_2(u + h + u^{(j)\star\star}) \prod_{4\leqq i<j} \sigma_2(u^{(i)} + u^{(j)\star})\,\sigma_2(u^{(i)} + u^{(j)\star\star})\\
&\Big/ \Big(\sigma_4(u)^{4n-4}\,\sigma_4(u+h)^{2n-5} \prod_{j=4}^{n} \sigma_4(u^{(j)})^{2n-2j+1}\,\sigma_4(u^{(j)\star})^{j-1}\,\sigma_4(u^{(j)\star\star})^{j-1} \Big)\\
&= \begin{vmatrix} 1 & x(u) & y(u) & x^2(u) & \cdots \\ 0 & x'(u) & y'(u) & (x^2)'(u) & \cdots \\ 0 & {\Delta_h}^1 x(u) & {\Delta_h}^1 y(u) & {\Delta_h}^1 x^2(u) & \cdots \\ 1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & \cdots \end{vmatrix}\\
&\quad\times \begin{vmatrix} 1 & x(u) & x^2(u) & \cdots & x^{n-1}(u) \\ 0 & x'(u) & (x^2)'(u) & \cdots & (x^{n-1})'(u) \\ 0 & {\Delta_h}^1 x(u) & {\Delta_h}^1 x^2(u) & \cdots & {\Delta_h}^1 x^{n-1}(u) \\ 1 & x(u^{(4)}) & x^2(u^{(4)}) & \cdots & x^{n-1}(u^{(4)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^{n-1}(u^{(n)}) \end{vmatrix}.
\end{aligned}
$$

After dividing this equation by ${h_1}^4$, and taking the limit $h \to \mathbf{0}$, we see by (13.4) that

$$
\begin{aligned}
&(-1)^{n+3}\sigma(3u + u^{(4)} + \cdots + u^{(n)}) \left(\frac{f_y(x(u), y(u))}{x^4(u)}\right)^3 \\
&\times \left(\prod_{j=4}^{n} \sigma_2(u + u^{(j)\star})\, \sigma_2(u + u^{(j)\star\star}) \right)^3 \prod_{4 \leqq i < j} \sigma_2(u^{(i)} + u^{(j)\star})\, \sigma_2(u^{(i)} + u^{(j)\star\star}) \\
&\Big/ \left(\sigma_4(u)^{6n-9} \prod_{j=4}^{n} \sigma_4(u^{(j)})^{2n-2j+1}\, \sigma_4(u^{(j)\star})^{j-1}\, \sigma_4(u^{(j)\star\star})^{j-1} \right) \\
&= \begin{vmatrix}
1 & x(u) & y(u) & x^2(u) & \cdots \\
0 & x'(u) & y'(u) & (x^2)'(u) & \cdots \\
0 & x''(u) & y''(u) & (x^2)''(u) & \cdots \\
1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & \cdots
\end{vmatrix}
\begin{vmatrix}
1 & x(u) & x^2(u) & \cdots & x^{n-1}(u) \\
0 & x'(u) & (x^2)'(u) & \cdots & (x^{n-1}) \\
0 & x''(u) & (x^2)''(u) & \cdots & (x^{n-1})''(u) \\
1 & x(u^{(4)}) & x^2(u^{(4)}) & \cdots & x^{n-1}(u^{(4)}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^{n-1}(u^{(n)})
\end{vmatrix}.
\end{aligned}
$$

We proceed the same operations for $u^{(4)}$, $u^{(5)}$, and so on. The operation concerning $u^{(k)} = u + h$ is done for the equation

$$(-1)^{n+(k-1)(k-2)/2}\sigma(ku+h+u^{(k+1)}+\cdots+u^{(n)})\left(\frac{f_y(x(u),y(u))}{x^4(u)}\right)^{(k-1)(k-2)/2}$$

$$\left(\frac{\sigma_2(u+u^{\star}+h^{\star})\,\sigma_2(u+u^{\star\star}+h^{\star\star})}{\sigma_4(u^{\star}+h^{\star})\sigma_4(u^{\star\star}+h^{\star\star})}\right)^{k+1}\left(\prod_{j=k+1}^{n}\sigma_2(u+u^{(j)\star})\,\sigma_2(u+u^{(j)\star\star})\right)^{k}$$

$$\prod_{j=k+1}^{n}\sigma_2(u+h+u^{(j)\star})\,\sigma_2(u+h+u^{(j)\star\star})\prod_{k+1\leqq i<j}\sigma_2(u^{(i)}+u^{(j)\star})\,\sigma_2(u^{(i)}+u^{(j)\star\star})$$

$$\Big/\left[\sigma_4(u)^{2kn-k^2}\sigma_4(u+h)^{2n-2k-1}\prod_{j=k+1}^{n}\sigma_4(u^{(j)})^{2n-2j+1}\,\sigma_4(u^{(j)\star})^{j-1}\,\sigma_4(u^{(j)\star\star})^{j-1}\right]$$

$$=\begin{vmatrix} 1 & x(u) & y(u) & x^2(u) & \cdots \\ 0 & x'(u) & y'(u) & (x^2)'(u) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & x^{(k-2)}(u) & y^{(k-2)}(u) & (x^2)^{(k-2)}(u) & \cdots \\ 0 & {\Delta_h}^{k-2}x(u) & {\Delta_h}^{k-2}y(u) & {\Delta_h}^{k-2}x^2(u) & \cdots \\ 1 & x(u^{(k+1)}) & y(u^{(k+1)}) & x^2(u^{(k+1)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & \cdots \end{vmatrix}$$

$$\times\begin{vmatrix} 1 & x(u) & x^2(u) & \cdots & x^{n-1}(u) \\ 0 & x'(u) & (x^2)'(u) & \cdots & (x^{n-1})'(u) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x^{(k-2)}(u) & (x^2)^{(k-2)}(u) & \cdots & (x^{n-1})^{(k-2)}(u) \\ 0 & {\Delta_h}^{k-2}x(u) & {\Delta_h}^{k-2}x^2(u) & \cdots & {\Delta_h}^{k-2}x^{n-1}(u) \\ 1 & x(u^{(k+1)}) & (x^2)(u^{(k+1)}) & \cdots & x^{n-1}(u^{(k+1)}) \\ \vdots & \vdots & (x^2)\vdots & \ddots & \vdots \\ 1 & x(u^{(n)}) & (x^2)(u^{(n)}) & \cdots & x^{n-1}(u^{(n)}) \end{vmatrix}.$$

After using (13.8) and dividing this equation by ${h_1}^{2(k+1)}$, taking limit $h_1 \to 0$, we arrives the following equation by (13.4):

$$
\begin{aligned}
&(-1)^{n+k(k+1)/2}\sigma(ku+u^{(k+1)}+\cdots+u^{(n)})\left(\frac{f_y(x(u),y(u))}{x^4(u)}\right)^{k(k+1)/2}\\
&\times\left(\prod_{j=k+1}^{n}\sigma_2(u+u^{(j)\star})\,\sigma_2(u+u^{(j)\star\star})\right)^{k+1}\prod_{k+1\leqq i<j}\sigma_2(u^{(i)}+u^{(j)\star})\,\sigma_2(u^{(i)}+u^{(j)\star\star})\\
&\Big/\left(\sigma_4(u)^{2(k+1)n-(k+1)^2}\prod_{j=k+1}^{n}\sigma_4(u^{(j)})^{2n-2j+1}\,\sigma_4(u^{(j)\star})^{j-1}\,\sigma_4(u^{(j)\star\star})^{j-1}\right)\\
&=\begin{vmatrix}
1 & x(u) & y(u) & x^2(u) & \cdots\\
0 & x'(u) & y'(u) & (x^2)'(u) & \cdots\\
0 & x''(u) & y''(u) & (x^2)''(u) & \cdots\\
\vdots & \vdots & \vdots & \vdots & \ddots\\
0 & x^{(k-2)}(u) & y^{(k-2)}(u) & (x^2)^{(k-2)}(u) & \cdots\\
0 & x^{(k-1)}(u) & y^{(k-1)}(u) & (x^2)^{(k-1)}(u) & \cdots\\
1 & x(u^{(k+1)}) & y(u^{(k+1)}) & x^2(u^{(k+1)}) & \cdots\\
\vdots & \vdots & \vdots & \vdots & \ddots\\
1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & \cdots
\end{vmatrix}\\
&\quad\times\begin{vmatrix}
1 & x(u) & x^2(u) & \cdots & x^{n-1}(u)\\
0 & x'(u) & (x^2)'(u) & \cdots & (x^{n-1})'(u)\\
0 & x''(u) & (x^2)''(u) & \cdots & (x^{n-1})''(u)\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
0 & x^{(k-2)}(u) & (x^2)^{(k-2)}(u) & \cdots & (x^{n-1})^{(k-2)}(u)\\
0 & x^{(k-1)}(u) & (x^2)^{(k-1)}(u) & \cdots & (x^{n-1})^{(k-1)}(u)\\
1 & x(u^{(k+1)}) & (x^2)(u^{(k+1)}) & \cdots & x^{n-1}(u^{(k+1)})\\
\vdots & \vdots & (x^2)\vdots & \ddots & \vdots\\
1 & x(u^{(n)}) & (x^2)(u^{(n)}) & \cdots & x^{n-1}(u^{(n)})
\end{vmatrix}.
\end{aligned}
$$

Such a process up to $k = n$ yields the desired formula. ■

Example 13.10. As an example, we compute Kiepert-type formula (13.7) for $n = 3$. Since

$$x' = \frac{f_y}{x^2}, \quad x'' = \frac{f_y}{x^2}\cdot\frac{xf_{yx} - 2f_y}{x^3}, \quad y' = -\frac{f_x}{x^2}, \quad y'' = -\frac{f_y}{x^2}\cdot\frac{xf_{xx} - 2f_x}{x^3},$$

$$(x^2)' = 2\cdot\frac{f_y}{x}, \quad (x^2)'' = 2\cdot\frac{f_y}{x^2}\cdot\frac{xf_{yx} - f_y}{x^2}.$$

We have

$$\begin{aligned}
\psi_3(u) &= \tfrac{1}{2}\, f_y \cdot (f_{yx}f_x - f_{xx}f_y) \\
&= f_y \cdot \Big(\big(30x^3 + 18\mu_3x^2 + (-{\mu_2}^3 + 4\mu_4\mu_2 + 9\mu_6)x \\
&\qquad - \mu_5{\mu_2}^2 + \tfrac{3}{2}\mu_7\mu_2 + \mu_4\mu_5 + 3\mu_9\big)y^2 \\
&\qquad + \big(15\mu_2x^4 + (8\mu_3\mu_2 + 20\mu_5)x^3 \\
&\qquad + (-\mu_4{\mu_2}^2 + 3\mu_6\mu_2 + 12\mu_3\mu_5 + 4{\mu_4}^2)x^2 \\
&\qquad + (-\mu_7{\mu_2}^2 + 6\mu_6\mu_5 + 4\mu_7\mu_4)x \\
&\qquad - \mu_{10}{\mu_2}^2 - \mu_{12}\mu_2 + 2\mu_9\mu_5 + 2\mu_{10}\mu_4 + \tfrac{1}{2}{\mu_7}^2\big)y \\
&\qquad + ({\mu_2}^2 + 2\mu_4)x^5 + (\mu_3{\mu_2}^2 - \mu_3\mu_4 + \tfrac{15}{2}\mu_7)x^4 \\
&\qquad + (\mu_6{\mu_2}^2 - 3\mu_6\mu_4 + 4\mu_3\mu_7 + 10\mu_{10})x^3 \\
&\qquad + (\mu_9{\mu_2}^2 - 4\mu_9\mu_4 + \tfrac{3}{2}\mu_6\mu_7 + 6\mu_3\mu_{10})x^2 \\
&\qquad + (\mu_{12}{\mu_2}^2 - 4\mu_{12}\mu_4 + 3\mu_6\mu_{10})x \\
&\qquad + (\mu_{15}{\mu_2}^2 - 3\mu_{15}\mu_4 - \tfrac{1}{2}\mu_{12}\mu_7 + \mu_9\mu_{10})\Big)
\end{aligned} \tag{13.11}$$

Equation (13.11) is analogous to the 3-division polynomial of an elliptic curve (or function) for the curve $\mathscr{C}$, in the sense of Cantor [3]. The set of roots of this polynomial are just the points on $\iota(\mathscr{C})$ whose multiplication by 3 lie on $\Theta^{[2]}$, in the Jacobian variety J.

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