Generalized Bernoulli-Hurwitz numbers for algebraic functions of cyclotomic type

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Generalized Bernoulli-Hurwitz Numbers

① genus g	② curve	③ Abelian integral $\left(u = u_g = \int_\infty^x \frac{x^{g-1}dx}{2y}\right)$	④ $x(u)$ $\left(= \text{inv. fct. of } x \mapsto u_g = \int_\infty^x \frac{x^{g-1}dx}{2y}\right)$ and its Laurent coefficients	⑤ von Staudt-Clausen theorem	⑥ 2nd theorem of von Staudt
0	$y^2 = x - 1$	$u(t) = t + \sum_{m=1}^{\infty}(-1)^m \binom{-\frac{1}{2}}{m} \frac{t^{2m+1}}{2m+1}$ $\left(= \sin^{-1}(u), \quad t = \frac{1}{x^{1/2}}\right)$	$x(u) = \frac{1}{\sin^2(u)} = \frac{1}{u^2} - \sum_{n=1}^{\infty}(-1)^n \frac{2^{2n}B_{2n}}{2n}\frac{u^{2n-2}}{(2n-2)!} = \sum_{\ell \in \pi\mathbb{Z}} \frac{1}{(u-\ell)^2}$	$B_{2n} \in -\sum_{\substack{p-1 \mid 2n \\ p:\text{prime}}} \frac{1}{p} + \mathbf{Z}$	If $p-1 \nmid 2n$, then $\frac{B_{2n}}{2n} \in \mathbf{Z}_{(p)}$
1	$y^2 = x^3 - 1$	$u(t) = t + \sum_{m=1}^{\infty}(-1)^m \binom{-\frac{1}{2}}{m} \frac{t^{6m+1}}{6m+1}$ $\left(t = \frac{1}{x^{1/2}}\right)$	$\wp(u) = \frac{1}{u^2} + \sum_{n=1}^{\infty} \frac{2^{6n}F_{6n}}{4n}\frac{u^{6n-2}}{(6n-2)!} = \frac{1}{u^2} + \sum_{\substack{\ell \in \varpi\mathbb{Z}[e^{2\pi i/3}] \\ \ell \neq 0}} \left(\frac{1}{(u-\ell)^2} - \frac{1}{\ell^2}\right)$ (with $\wp'^2 = 4\wp^3 - 4$, $\varpi = 2.42865\cdots$)	$F_{4n} \in -\sum_{\substack{6n=a(p-1) \\ p:\text{prime} \\ p\equiv 1 \bmod 6}} \frac{{A_p}^a}{p} + \mathbf{Z}$ $A_p = (-1)^{\frac{p-1}{6}}\binom{(p-1)/2}{(p-1)/6}$: Hasse invariant	If $p-1 \nmid 6n$, then $\frac{F_{6n}}{6n} \in \mathbf{Z}_{(p)}$
2	$y^2 = x^5 - 1$	$u_2(t) = t + \sum_{m=1}^{\infty}(-1)^m \binom{-\frac{1}{2}}{m} \frac{t^{10m+1}}{10m+1}$ $\left(t = \frac{1}{x^{1/2}}\right)$	$x(u_1,u_2) = \frac{1}{{u_2}^2} + \sum_{n=1}^{\infty} \frac{C_{10n}}{10n}\frac{{u_2}^{10n-2}}{(10n-2)!} \overset{?}{=} \sum_{(\ell_1,\ell_2)\in\Lambda} \left(\frac{1}{(u_2-\ell_2)^2} + \text{const.}\right)$ This is periodic w.r.t. the lattice $\Lambda = \left\{ \oint \left(\frac{dx}{2y}, \frac{xdx}{2y}\right)\right\}$	$C_{10n} \in -\sum_{\substack{10n=a(p-1) \\ p:\text{prime} \\ p\equiv 1 \bmod 10}} \frac{{A_p}^a}{p} + \mathbf{Z}$ $A_p = (-1)^{\frac{p-1}{10}}\binom{(p-1)/2}{(p-1)/10}$: (2, 2)-entry of the Hasse-Witt matrix w.r.t. $\left(\frac{dx}{2y}, \frac{xdx}{2y}\right)$	If $p-1 \nmid 10n$, then $\frac{C_{10n}}{10n} \in \mathbf{Z}_{(p)}$

① genus g	⑦ Kummer's original type congruence	⑧ Kummer-Adelberg type congruence	⑨ Links with zeta and L-functions	⑩ Differential equation of $x(u)$	⑪ Signature sequence	⑫ Congruence with Euler factors
0	If $p-1 \nmid 2n$ and $2n > a$, then $\sum_{r=0}^{a}\binom{a}{r}(-1)^r \frac{B_{2n+r(p-1)}}{2n+r(p-1)} \equiv 0 \mod p^a$	If $p-1 \nmid 2n$ and $2n > a$, then $\frac{B_{2n}}{2n} \equiv \frac{B_{2n+p^{a-1}(p-1)}}{2n+p^{a-1}(p-1)} \mod p^a$	$2\,\zeta(2n) = \frac{(-1)^{n-1}2^{2n}B_{2n}}{(2n)!}\pi^{2n}$ $(\pi = 3.141592\cdots)$	$x'(u)^2 = 4x(u)^3 - 4x(u)^2$	$+, -, +, -, \cdots$	If $p-1 \nmid m$ and $m \equiv n \bmod p^{a-1}(p-1)$, then $(1-p^{n-1})\frac{B_n}{n} \equiv (1-p^{m-1})\frac{B_m}{m} \mod p^a$
1	If $p \equiv 1 \bmod 6$, $p-1 \nmid 6n$ and $6n > a$, then $\sum_{r=0}^{a}\binom{a}{r}(-1)^r {A_p}^{a-r} \frac{F_{6n+r(p-1)}}{6n+r(p-1)} \equiv 0 \mod p^a$	If $p \equiv 1 \bmod 6$, $p-1 \nmid 6n$ and $6n > a$, then ${A_p}^{p^{a-1}}\frac{F_{6n}}{6n} \equiv \frac{F_{6n+p^{a-1}(p-1)}}{6n+p^{p-1}(p-1)} \mod p^a$	$\sum_{\substack{\lambda \in \mathbf{Z}[e^{2\pi i/3}] \\ \lambda \neq 0}} \frac{1}{\lambda^{6n}} = \frac{2^{6n}F_{6n}}{(6n)!}\varpi^{6n}$ $\left(\varpi = \int_1^\infty \frac{dx}{y} = 2.42865\cdots\right)$	$x'(u)^2 = 4x(u)^3 - 4$	$+, +, +, +, \cdots$	If $P \equiv 1 \bmod 3$ is degree 1 prime in $\mathbf{Z}[e^{2\pi i/3}]$, $p = P\overline{P}$, $p-1 \nmid m$ and $m \equiv n \bmod p^{a-1}(p-1)$, then ${A_p}^{(n-m)/(p-1)}\left(1-\frac{p^{m-1}}{\overline{P}^m}\right)\frac{F_m}{m} \equiv \left(1-\frac{p^{n-1}}{\overline{P}^n}\right)\frac{F_n}{n} \mod P^a$
2	If $p \equiv 1 \bmod 10$, $p-1 \nmid 10n$ and $10n > a$, then $\sum_{r=0}^{a}\binom{a}{r}(-1)^r {A_p}^{a-r} \frac{C_{10n+r(p-1)}}{10n+r(p-1)} \equiv 0 \mod p^a$	If $p \equiv 1 \bmod 10$, $p-1 \nmid 10n$ and $10n > a$, then ${A_p}^{p^{a-1}}\frac{C_{10n}}{10n} \equiv \frac{C_{10n+p^{a-1}(p-1)}}{10n+p^{p-1}(p-1)} \mod p^a$	$\sum^{*}_{\substack{\lambda \in \mathbf{Z}[e^{2\pi i/5}] \\ \lambda \neq 0}} \frac{1}{\lambda^{10n}} \overset{?}{=} \frac{C_{10n}}{(10n)!}\Omega^{10n}$ $\left(\Omega = \int_1^\infty \frac{xdx}{y} = 2.6461\cdots,\right.$ * means convergence of the sum is not justified yet.$\left.\right)$	$x'(u)^2x(u)^2 = 4x(u)^5 - 4$	$+, -, +, -, \cdots$	?

Universal Bernoulli Numbers

①	②	③	④	⑤	⑥
——	——	$u(t) = t + \sum_{n=1}^{\infty} f_n \frac{t^{n+1}}{n+1}$ (All f_ns are indeterminates.)	$\frac{1}{t(u)} = \frac{1}{u} + \sum_{n=1}^{\infty}\frac{\widehat{B}_n}{n}\frac{u^{n-1}}{(n-1)!}$	$\widehat{B}_n \in -\sum_{\substack{n=a(p-1) \\ p:\text{prime}}} \frac{{f_{p-1}}^a}{p} + \mathbf{Z}[f_1, f_2, \cdots]$	If $p-1 \nmid n$, then $\frac{\widehat{B}_n}{n} \in \mathbf{Z}_{(p)}[f_1, f_2, \cdots]$

①	⑦	⑧	⑨	⑩	⑪	⑫
——	If $p-1 \nmid n$ and $n > a$, then $\sum_{r=0}^{a}\binom{a}{r}(-1)^r {f_{p-1}}^{a-r}\frac{\widehat{B}_{n+r(p-1)}}{n+r(p-1)} \equiv 0 \mod p^{\lfloor a/2 \rfloor}\mathbf{Z}_{(p)}[f_1, f_2, \cdots]$ This estimate of the power is best possible as a linear form in a.	If $p-1 \nmid n$ and $n > a$, then ${A_p}^{p^{a-1}}\frac{\widehat{B}_n}{n} \equiv \frac{\widehat{B}_{n+p^{a-1}(p-1)}}{n+p^{p-1}(p-1)} \mod p^a$	——	——	——	——