# COMPLEX MULTIPLICATION FORMULAE FOR HYPERELLIPTIC CURVES OF GENUS THREE

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## Introduction

Let $\wp(u)$ be a Weierstrass elliptic function satisfying $\wp'(u)^2 = 4\wp(u)^3 - 1$. Let $\zeta := e^{\frac{2\pi i}{3}}$. Then $\wp(u)$ has a property $\wp(-\zeta u) = \zeta\wp(u)$. If $b$ is an element of $\mathbf{Z}[\zeta]$, the integer ring generated by $\zeta$, we have a $b$-multiplication formula of $\wp(u)$. If $b$ is a prime element and $b \equiv 1 \mod 3$, the $b$-multiplication formula is of the form

$$\wp(bu) = \frac{\wp(u)(\wp(u)^{\mathrm{N}b-1} + \cdots + b)}{(b\wp(u)^{\frac{\mathrm{N}b-1}{2}} + \cdots \pm 1)^2}, \tag{0.1}$$

and all the coefficients belong to $\mathbf{Z}[\zeta]$. (These facts seem to be already known to Eisenstein [**6**]). Therefore the product of the roots $\{\wp(u)\}$ except for 0 of the numerator is equal to $\pm b$, and the product of reciprocals of the roots $\{\wp(u)\}$ of the denominator is equal to $b^2$. So we have factorization of $b$ or $b^2$ in an extended integer ring of $\mathbf{Z}[\zeta]$. Analogous fact is known for a function $\wp(u)$ satisfying $\wp'(u)^2 = 4\wp(u)^3 - \wp(u)$.

By using these facts essentially, the cubic and quartic Gauss sums were deeply investigated (see [**12**] and [**13**]). So it seems natural for us to expect the existence of formulae analogous to (0.1) for curves of higher genus. A remarkable formula was discovered by D.Grant for the curve of genus two defined by $y^2 = x^5 + \frac{1}{4}$ ([**9**]).

The purpose of this paper is to generalize his formula. Let $C$ be a curve of genus $g(\geq 1)$ defined by $y^2 = f(x)$, where $f(x)$ is a polynomial of degree $2g+1$. Let $J$ denote the Jacobian variety of the curve $C$, and $\iota : C \hookrightarrow J$ the canonical embedding. We identify $J$ with a complex torus $\mathbf{C}^g/\Lambda$ where $\Lambda$ is a lattice of $\mathbf{C}^g$. Let $u = (u_1, \cdots, u_g)$ be the canonical coordinate system of $\mathbf{C}^g$, and $\varphi(u)$ a meromorphic function on $\mathbf{C}^g/\Lambda$. We assume that $\varphi(u)$ satisfies $\varphi(-u) = -\varphi(u)$, because the abelian functions $\varphi(u)$ we treate in this paper are odd or even functions. In the bellow, we denote by $x(u)$ and $y(u)$ the values of $x$-coordinate and $y$-coordinate, respectively, at $u$ such that $u \in \iota(C)$. Then the restriction to $\iota(C)$ of the map

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$u \mapsto \varphi(bu)$ gives an algebraic function. Hence $\varphi \circ \iota$ has a rational expression of $x(u)$ and $y(u)$. Since $x(-u) = x(u)$ and $y(-u) = -y(u)$, we have an expression

$$\varphi(bu) = \frac{y(u)P\left(x(u)\right)}{Q\left(x(u)\right)} \tag{0.2}$$

with polynomials $P(X)$ and $Q(X)$. Here we do not mention the irreducibility of right-hand side of the expression. We regard (0.2) as a generalization of (0.1). We also call such formula a *b-multiplication formula* . However, our aim is, as mentioned about (0.1), to find a nice Abelian function $\varphi(u)$ such that every one of the roots of its numerator $P(X)$ (or its denominator) is algebraic integer and the product of the roots gives a factorization of $b$ or of a product of conjugates of $b$, in a certain integer ring.

The author found several such functions $\varphi(u)$ in the family of polynomials of hyperelliptic $\wp$-functions constructed by H.F.Baker ([**2**], [**3**] and [**4**]) as Grant did, because the author believes all the roots of $P(X)$ and $Q(X)$ or all of their reciprocals are algebraic integers. We will prove that the numerator of the complex multiplication formula of each our function has required properties. Our functions $\varphi(u)$ are Abelian functions associated to the following curves: curves of genus two defined by $y^2 = x^5 + \frac{1}{4}$ (Grant's case) and by $y^2 = x^5 - x$, and of genus three defined by $y^2 = x^7 + \frac{1}{4}$ and by $y^2 = x^7 - x$. (see Theorems 6.1.6, 6.2.5. 7.1.6. and 7.2.5, respectively). Unfortunitely it is generally unknown the existence of such nice functions. So the author do not explain how to find such functions.

In Section 1, we recall the fundamental facts about hyperelliptic functions from [**2**], [**3**] and [**4**]. We introduce a well-tuned theta series $\sigma(u)$ called the sigma function, and define abelian functions called (hyperelliptic) $\wp$-functions as second derivatives of $\log\sigma(u)$. They are nice generalization of sigma function and $\wp$-function of Weierstrass. So our function $\varphi(u)$ is a rational function of $\sigma(u)$ and its (higher) derivatives. Dividing its numerator and denominator by certain power of $\sigma(u)$ or of its derivative yields the expression just obtained by rewriting (0.2) in terms of $\sigma(u)$ and its derivatives. In this expression, the denominator is a function so-called the psi function. We can prove the psi function is a polynomial of $x(u)$ or polynomial of $x(u)$ multiplied by $y(u)$ when $u \in \iota(C)$.

Now we have a rational expression of $P(x(u))$ in terms of $\sigma(u)$, $\sigma(bu)$, and their derivatives. We investigate $P(x(u))$ by using Taylor expansions of $\sigma(u)$. Such expansions are given by using differential equations of the sigma function after investigation of singularity of the theta divisor. Let $u = P_0 \in \iota(C)$ be a point such that $x(P_0) = 0$. For each of our curves such point $P_0$ is a torsion point in $J$. For instance, in the case of $\wp(u)$ used in (0.1), a point $P_0$ such that $\wp(P_0) = 0$ is $(1-\zeta)$-torsion. Suppose $P_0$ be a $c$-torsion point for a non-trivial endomorphism $c$. Assume $b \in \mathrm{End}(J)$, the ring of endomorphisms of $J$, satisfies $b \equiv 1 \mod c^2$ in $\mathrm{End}(J)$. Then we can obtain very explicitly first several terms of the Taylor expansion of $\sigma(u)$ at the image of $\infty$ and $P_0$ of $C$ by the embedding $\iota$. This expansion at $\infty$ gives the expansion of $\varphi(bu)$ on $\iota(C)$ at $\iota(\infty)$. Hence we can determine the highest term of $P(X)$.

The most difficult part is to give the Taylor expansion of $\sigma(bu)$ at $u = P_0$. Since $\sigma(bu) = \sigma(b(u-P_0)+P_0+(b-1)P_0)$ and $(b-1)P_0 \in \Lambda$ by the assumption of $b$, we may first use the expansion of $\sigma(b(v+P_0))$ at $v=0$. However, we need an explicit relation of the leading coefficients of expansions of $\sigma(b(v+P_0))$ and $\sigma(bv+P_0)$ at $v=0$. We can express $\sigma(P_0)$ as a special value of exponential of the linear form associated with the translational formula. The final form of the expansion of $\sigma(bu)$ at $u=P_0$ is obtained in Part II by using this expression. Thus we can determine the lowest term of $P(X)$ by this expansion. Grant determines the lowest term of $P(X)$ for the curve $y^2 = x^5 + \frac{1}{4}$ by induction on $b$. Since the author can not generalize such induction to other our curves, he determines it by using the Taylor expansion at $P_0$.

In Sections 6 and 7, we prove the main results for our curves of genus two and three, respectively. As an instruction, we give proofs of original formula for elliptic curves in Section 5 by the method of ours.

We do not discuss the integrality of the coefficients of $P(X)$ in this paper. For the curve $y^2 = x^5 + \frac{1}{4}$, the integrality of the coefficients of $P(X)$ is essentially proved by Grant (see [**17**]), and for the curve $y^2 = x^5 - x$, such thing seems to be proved similarly. The author are now preparing tools to investigate the coefficients for curves of genus three.

If we use most of the results up to Section 4, we can investigate lower and higher terms of the polynomial expression in terms of $x(u)$ of the numerator of an arbitrary Abelian function which is a polynomial of Baker's $\wp$-functions. Furthermore, if we take a 2-torsion point $Q_0$ instead of $P_0$ (then $y(Q_0)=0$) and $y$ instead of $x$, we can find many Abelian functions such that their coefficients have similar properties like the above $\varphi(u)$. The reason that the author do not discuss such minor formulae is that he want to find a formula which gives a non-canonical way to give certain power-root of $b$ or of a product of conjugates of $b$ as a partial product as in [**12**] and [**13**].

**Convention.** We denote, as usual, by $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{C}$ the ring of rational integers, the field of rational numbers and the field of complex numbers, respectively. The imaginary unit is denoted by $i$. For a variety $V$, the global sections of a sheaf $\mathcal{F}$ on $V$ is denoted by $\Gamma(V,\mathcal{F})$. The sheaf associated to a divisor $D$ is denoted by $\mathcal{O}(D)$. In an expression of the Laurent expansion of a function, the symbol $(d^\circ(z_1,\cdots,z_j)\geq n)$ means the terms of total degree at least $n$ with respect to the variables $z_1,\cdots,z_j$. When the member of variables or the least total degree are clear from the context, we simply use the symbol $(d^\circ \geq n)$ or the dots "$\cdots$".

For cross references, we indicate a formula as (1.2.3), and each of Lemmas, Propositions, Theorems and Remarks as 4.5.6 for example.

## Contents

# I. Hyperelliptic Abelian Functions

In this part we recall fundamentals of the theory of hyperelliptic functions.

## §1. Generalities

### 1.1. Differential forms and period matrices.

Let $C$ be a smooth projective model of a curve of genus $g > 0$ defined over $\mathbf{C}$ whose affine equation is given by $y^2 = f(x)$, where

$$f(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_{2g+1} x^{2g+1}.$$

In this paper, we keep the agreement $\lambda_{2g+1} = 1$. We use, however, the letter $\lambda_{2g+1}$ too when this notation makes easy to read an equation of homogeneous weight (for example, 1.5.1 below). The roots of the equation $f(x) = 0$ are denoted by

$$c_1,\, a_1,\, c_2,\, a_2,\, \cdots,\, c_g,\, a_g,\, c, \tag{1.1.1}$$

according to their positions (c.f. Figure 1). We denote by $\infty$ the point of $C$ at infinity. It is known that the set of

$$\omega^{(j)} := \frac{x^{j-1}dx}{2y} \quad (j = 1, \cdots, g)$$

makes a basis of the vector space $\Gamma(C, \Omega^1)$, where $\Omega^1$ is the sheaf of differential forms of the first kind (see [**15**, p.3.77]). Let

$$\eta^{(j)} := \frac{1}{2y} \sum_{k=j}^{2g-j} (k+1-j)\lambda_{k+1+j} x^k dx \quad (j = 1, \cdots, g),$$

which are differential forms of the second kind without poles except at $\infty$ (see [**2**, p.195, Ex. i] or [**3**, p.314]). We fix generators $\alpha^{(i)}$, $\beta^{(i)}$ ($i$=1, $\cdots$, $g$) of the fundamental group of $C$ such that their intersections are $\alpha^{(i)} \cdot \alpha^{(j)} = \beta^{(i)} \cdot \beta^{(j)} = 0$, $\alpha^{(i)} \cdot \beta^{(j)} = \delta_{ij}$ for $i$, $j = 1$, $\cdots$, $g$ as illustrated in Figure 1.

Figure 1

As usual we let

$$\omega' := \begin{bmatrix} \int_{\alpha^{(1)}} \omega^{(1)} & \cdots & \int_{\alpha^{(g)}} \omega^{(1)} \\ \vdots & \ddots & \vdots \\ \int_{\alpha^{(1)}} \omega^{(g)} & \cdots & \int_{\alpha^{(g)}} \omega^{(g)} \end{bmatrix}, \quad \omega'' := \begin{bmatrix} \int_{\beta^{(1)}} \omega^{(1)} & \cdots & \int_{\beta^{(g)}} \omega^{(1)} \\ \vdots & \ddots & \vdots \\ \int_{\beta^{(1)}} \omega^{(g)} & \cdots & \int_{\beta^{(g)}} \omega^{(g)} \end{bmatrix}$$

be the period matrices. Then the modulus of $C$ is given by $Z := {\omega'}^{-1}\omega''$. The lattice of periods is denoted by $\Lambda$, that is

$$\Lambda := \omega' \, {}^t[\,\mathbf{Z} \quad \mathbf{Z} \quad \cdots \quad \mathbf{Z}\,] + \omega'' \, {}^t[\,\mathbf{Z} \quad \mathbf{Z} \quad \cdots \quad \mathbf{Z}\,] \ (\subset \mathbf{C}^g).$$

We also introduce the matrices of quasi-period:

$$\eta' := \begin{bmatrix} \int_{\alpha^{(1)}} \eta^{(1)} & \cdots & \int_{\alpha^{(g)}} \eta^{(1)} \\ \vdots & \ddots & \vdots \\ \int_{\alpha^{(1)}} \eta^{(g)} & \cdots & \int_{\alpha^{(g)}} \eta^{(g)} \end{bmatrix}, \quad \eta'' := \begin{bmatrix} \int_{\beta^{(1)}} \eta^{(1)} & \cdots & \int_{\beta^{(g)}} \eta^{(1)} \\ \vdots & \ddots & \vdots \\ \int_{\beta^{(1)}} \eta^{(g)} & \cdots & \int_{\beta^{(g)}} \eta^{(g)} \end{bmatrix}.$$

**1.2. The Jacobian variety, the theta divisor.**
Let $J$ be the Jacobian variety of the curve $C$. We identify $J$ with the Picard group $\mathrm{Pic}^\circ(C)$ of the linearly equivalence classes of divisors of degree zero of $C$. Let $\mathrm{Sym}^g(C)$ be the $g$-th symmetric product of $C$. Then we have a birational map

$$\begin{aligned} \mathrm{Sym}^g(C) &\to \mathrm{Pic}^\circ(C) = J \\ (P_1, \cdots, P_g) &\mapsto \text{the class of } P_1 + \cdots + P_g - g \cdot \infty. \end{aligned}$$

As an analytic manifold, $J$ is identified with $\mathbf{C}^g/\Lambda$. We denote by $\kappa$ the canonical map $\mathbf{C}^g \to \mathbf{C}^g/\Lambda = J$. We embed $C$ into $J$ by $\iota : Q \mapsto Q - \infty$. Let $\Theta$ be the theta divisor, that is the divisor of $J$ determined by the set of classes of the form $P_1 + \cdots + P_{g-1} - (g-1) \cdot \infty$.

**1.3. The hyperelliptic sigma Function $\sigma(u)$.**
We let

$$\delta'' := {}^t\Big[\frac{1}{2} \quad \frac{1}{2} \quad \cdots \quad \frac{1}{2}\Big], \quad \delta' := {}^t\Big[\frac{g}{2} \quad \frac{g-1}{2} \quad \cdots \quad \frac{1}{2}\Big] \text{ and } \delta := \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix}.$$

For $a$ and $b$ in $\left(\frac{1}{2}\mathbf{Z}\right)^g$, we let

$$\begin{aligned} \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z) &= \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z; Z) \\ &= \sum_{n \in \mathbf{Z}^g} \exp\left[2\pi i \left\{\frac{1}{2}\, {}^t(n+a)Z(n+a) + {}^t(n+a)(z+b)\right\}\right]. \end{aligned}$$

Then the hyperelliptic sigma function on $\mathbf{C}^g$ associated with $C$ is defined by

$$\widetilde{\sigma}(u) = \exp(-\frac{1}{2} u \eta' {\omega'}^{-1}\, {}^t u) \vartheta[\delta]\,({\omega'}^{-1}\, {}^t u;\ Z)$$

up to a multiplicative constant. We fix the constant as follows.

Lemma 1.3.1. *The function* $\widetilde{\sigma}(u)$ *has the following properties*:
(1) *The lowest terms of the Taylor expansion of* $\widetilde{\sigma}(u)$ *at* $u = 0$ *contain the term* $\gamma u_1 u_3 \cdots u_g$ *if* $g$ *is odd, or* $\gamma u_1 u_3 \cdots u_{g-1}$ *if* $g$ *is even, with a non-zero constant* $\gamma$ *independent of* $u_1, \cdots, u_g$*;*
(2) *The function* $\widetilde{\sigma}(u)$ *is an odd function if* $g \equiv 1, 2 \mod 4$*, and is an even one if* $g \equiv 3, 0 \mod 4$*;*
(3) *The divisor of* $\widetilde{\sigma}(u)$ *is the pull-back of* $\Theta$ *by the map* $\kappa : \mathbf{C}^g \to \mathbf{C}^g/\Lambda = J$*.*

*Proof.* For a proof of (1), see [**3**, p.353]. The statement (2) and (3) are given in [**15**, p.3.97, p.3.100], Proposition 6.3(c), respectively. □

In this paper, we make the following normalization: we let

$$\sigma(u) := \gamma^{-1}\widetilde{\sigma}(u).$$

The constant $\gamma$ in 1.3.1 for curves of genus two is studied in [**7**]. For more details on $\sigma(u)$, we refer the reader to [**1**] and [**3**].

**1.4. Hyperelliptic Abelian functions $\wp_{jk}(u)$.**
For $j, k, \cdots, r \in \{1, \cdots, g\}$, let

$$\begin{aligned} \sigma_j(u) &= \frac{\partial}{\partial u_j}\sigma(u), \quad \sigma_{jk\cdots r}(u) = \frac{\partial}{\partial u_j}\sigma_{k\cdots r}(u), \\ \wp_{jk}(u) &= -\frac{\partial^2}{\partial u_j \partial u_k}\log\sigma(u), \quad \wp_{jk\cdots r}(u) = \frac{\partial}{\partial u_j}\wp_{k\cdots r}(u). \end{aligned} \tag{1.4.1}$$

Then the functions $\wp_{jk\cdots r}(u)$ are Abelian functions on the Jacobian variety $J$ of $C$. We call each of these functions, simply, a $\wp$-function when we talk about their uniform properties. In the genus one case, the function $\wp_{11}(u)$ is essentially the Weierstrass elliptic function.

Let $(u_1, \cdots, u_g)$ be the system of variables of $\sigma(u)$. Then we can find a set of $g$ points $(x_1, y_1), \cdots, (x_g, y_g)$ on $C$ such that

$$u_j = \int_\infty^{(x_1,y_1)} \omega^{(j)} + \cdots + \int_\infty^{(x_g,y_g)} \omega^{(j)} \quad (j = 1, \cdots, g) \tag{1.4.2}$$

with certain paths of integrals. In this situation, the $\wp$-functions are characterized as follows ([**3**, p.377]).

Lemma 1.4.1. *Assume that the variables* $u_1, \cdots, u_g$ *of* $\sigma(u)$ *depend on* $g$ *variable points* $(x_1, y_1), \cdots, (x_g, y_g)$ *of* $C$ *by the equation* (1.4.2)*. Let*

$$F(X_1, X_2) = \sum_{j=0}^{g} X_1^j X_2^j \left(\lambda_{2j+1}(X_1 + X_2) + 2\lambda_{2j}\right).$$

*Then the functions* $\wp_{jk}(u)$ *are characterized by the equations*

$$\begin{aligned} &\sum_{j=1}^{g}\sum_{k=1}^{g} \wp_{jk}(u) x_r^{j-1} x_s^{k-1} = \frac{F(x_r, x_s) - 2y_r y_s}{(x_r - x_s)^2}, \\ &x_r^g - \sum_{j=1}^{g} \wp_{jg}(u) x_r^{j-1} = 0 \end{aligned}$$

*for $r$, $s = 1, \cdots, g$ with $r \neq s$. Especially, the functions $\wp_{ij}(u)$ are defined over the field $\mathbf{Q}(\lambda_0, \cdots, \lambda_{2g+1})$, and $(-1)^{g-j}\wp_{gj}(u)$ is the elementary symmetric function of degree $g-j+1$ of $x_1, \cdots, x_g$.*

For more details on $\wp$-functions, we refer the reader to [**2**] and [**3**].

By Lemma 1.3.1(3), we know that

$$\wp_{ij}(u) \in \Gamma(J, \mathcal{O}(2\Theta)), \quad \wp_{ijk}(u) \in \Gamma(J, \mathcal{O}(3\Theta)), \quad \wp_{ijk\ell}(u) \in \Gamma(J, \mathcal{O}(4\Theta)), \tag{1.4.3}$$

where $\Gamma(J, \mathcal{O}(n\Theta))$ denotes the functions on $J$ having poles, only along $\Theta$, with at most $n$-th order.

**1.5. Algebraic relations for $\wp$-functions.**
Here we recall relations of the functions $\wp_{ij}(u)$ and $\wp_{ijk\ell}(u)$.

PROPOSITION 1.5.1. *Let $\wp_{ijk\ell} := \wp_{ijk\ell}(u)$ and $\wp_{ij} := \wp_{ij}(u)$ for simplicity. The following equations hold for $g = 1$, $2$ and $3$:*

(1) $\wp_{3333} - 6\wp_{33}^2 = 2\lambda_5\lambda_7 + 4\lambda_6\wp_{33} + 4\lambda_7\wp_{32}$,

(2) $\wp_{3332} - 6\wp_{33}\wp_{32} = 4\lambda_6\wp_{32} + 2\lambda_7(3\wp_{31} - \wp_{22})$,

(3) $\wp_{3331} - 6\wp_{31}\wp_{33} = 4\lambda_6\wp_{31} - 2\lambda_7\wp_{21}$,

(4) $\wp_{3322} - 4\wp_{32}^2 - 2\wp_{33}\wp_{22} = 2\lambda_5\wp_{32} + 4\lambda_6\wp_{31} - 2\lambda_7\wp_{21}$,

(5) $\wp_{3321} - 2\wp_{33}\wp_{21} - 4\wp_{32}\wp_{31} = 2\lambda_5\wp_{31}$,

(6) $\wp_{3311} - 4\wp_{31}^2 - 2\wp_{33}\wp_{11} = 2\Delta$,

(7) $\wp_{3222} - 6\wp_{32}\wp_{22} = -4\lambda_2\lambda_7 - 2\lambda_3\wp_{33} + 4\lambda_4\wp_{32} + 4\lambda_5\wp_{31} - 6\lambda_7\wp_{11}$,

(8) $\wp_{3221} - 4\wp_{32}\wp_{21} - 2\wp_{31}\wp_{22} = -2\lambda_1\lambda_7 + 4\lambda_4\wp_{31} - 2\Delta$,

(9) $\wp_{3211} - 4\wp_{31}\wp_{21} - 2\wp_{32}\wp_{11} = -4\lambda_0\lambda_7 + 2\lambda_3\wp_{31}$,

(10) $\wp_{3111} - 6\wp_{31}\wp_{11} = 4\lambda_0\wp_{33} - 2\lambda_1\wp_{32} + 4\lambda_2\wp_{31}$,

(11) $\wp_{2222} - 6\wp_{22}^2$
$= -8\lambda_2\lambda_6 + 2\lambda_3\lambda_5 - 6\lambda_1\lambda_7 - 12\lambda_2\wp_{33} + 4\lambda_3\wp_{32} + 4\lambda_4\wp_{22} + 4\lambda_5\wp_{21} - 12\lambda_6\wp_{11} + 12\Delta$,

(12) $\wp_{2221} - 6\wp_{22}\wp_{21} = -4\lambda_1\lambda_6 - 8\lambda_0\lambda_7 - 6\lambda_1\wp_{33} + 4\lambda_3\wp_{31} + 4\lambda_4\wp_{21} - 2\lambda_5\wp_{11}$,

(13) $\wp_{2211} - 4\wp_{21}^2 - 2\wp_{22}\wp_{11} = -8\lambda_0\lambda_6 - 8\lambda_0\wp_{33} - 2\lambda_1\wp_{32} + 4\lambda_2\wp_{31} + 2\lambda_3\wp_{21}$,

(14) $\wp_{2111} - 6\wp_{21}\wp_{11} = -2\lambda_0\lambda_5 - 8\lambda_0\wp_{32} + 2\lambda_1(3\wp_{31} - \wp_{22}) + 4\lambda_2\wp_{21}$,

(15) $\wp_{1111} - 6\wp_{11}^2 = -4\lambda_0\lambda_4 + 2\lambda_1\lambda_3 + 4\lambda_0(4\wp_{31} - 3\wp_{22}) + 4\lambda_1\wp_{21} + 4\lambda_2\wp_{11}$,

*where*

$$\Delta = \wp_{32}\wp_{21} - \wp_{31}\wp_{22} + \wp_{31}^2 - \wp_{33}\wp_{11}.$$

*These equations are presented under the convention that if $g = 1$ or $2$ then $\lambda_i$ with $i > 2g+1$ and $\wp$-functions whose suffix contain $j$ bigger than $g$ are all zero.*

Note that when $g = 1$ the equation (15) above is a well-known equation derived from $\wp'(u)^2 = 4f(\wp(u))$. We refer the reader to [**4**] for the proof of this Proposition.

**1.6. The algebraic addition formulae.**

Here we present algebraic addition formulae which express each function $\wp_{k\ell}(u+v)$ as a rational function of $\{\wp_{ij}(u)\}$, $\{\wp_{ij}(v)\}$, $\{\wp_{hij}(u)\}$ and $\{\wp_{hij}(v)\}$ with $1 \leqq h \leqq g$, $1 \leqq i \leqq g$ and $1 \leqq j \leqq g$.

Proposition 1.6.1. $\dfrac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2}$ *can be expressed as a polynomial in the* $g(g+1)$ *functions* $\{\wp_{ij}(u)\}$ *and* $\{\wp_{ij}(v)\}$ *with coefficients in* $\mathbf{Q}$.

For a proof of this Proposition we refer the reader to [**3**].

Corollary 1.6.2. *Each function* $\wp_{ij\cdots r}(u+v)$ *has a rational expression in terms of the functions* $\{\wp_{ij}(u)\}$, $\{\wp_{ij}(v)\}$, $\{\wp_{hij}(u)\}$ *and* $\{\wp_{hij}(v)\}$ *with coefficients in* $\mathbf{Q}(\lambda_0, \cdots, \lambda_{2g+1})$.

*Proof.* After logarithmically differentiating the expression of 1.6.1 by $u_i$ and $v_i$, respectively, by adding the obtained two equations, we have a rational expression of $2\dfrac{\partial}{\partial u_i}\log\sigma(u+v) - 4\dfrac{\partial}{\partial u_i}\log\sigma(u) - 4\dfrac{\partial}{\partial v_i}\log\sigma(v)$ in the functions $\{\wp_{ij}(u)\}$, $\{\wp_{ij}(v)\}$, $\{\wp_{hij}(u)\}$ and $\{\wp_{hij}(v)\}$. We operate $\dfrac{\partial}{\partial u_j}$ to this expression. Then we have a rational expression of $\wp_{ij}(u+v)$ in the functions $\{\wp_{ij}(u)\}$, $\{\wp_{ij}(v)\}$, $\{\wp_{hij}(u)\}$, $\{\wp_{hij}(v)\}$, $\{\wp_{ijk\ell}(u)\}$, $\{\wp_{ijk\ell}(v)\}$, $\{\wp_{ijk\ell m}(u)\}$ and $\{\wp_{ijk\ell m}(v)\}$. We can obatin the desired expression by using the equations in 1.5.1. $\square$

**1.7. Geometry of the theta divisor.**

We fix the local parameter of every point of $C$. To make clear the following argument we define the local parameter $t$ at each point $P$ by

$$t = \begin{cases} x - x(P) & \text{if } P \text{ is an ordinary point,} \\ y & \text{if } P \text{ is a branch point different from } \infty, \\ \dfrac{1}{\sqrt{x}} & \text{if } P = \infty. \end{cases} \tag{1.7.1}$$

Here we call $P$ a branch point if $y(P) = 0$ or $\infty$, and an ordinary point otherwise.

We determine the singular locus of the theta divisor $\Theta$ by using certain matrix attached to a positive divisor of $C$. Here our argument is based on [**5**, pp. 85-86]. For a point $P$ of $C$, let $t$ be the local parameter defined above. We denote by $P_t$ the point of $C$ such that the value of $t$ at $P_t$ is $t$. Then we define for $\mu \in \Gamma(C, \Omega^1)$

$$D^\ell\mu(P) = \frac{d^\ell}{dt^\ell}\int_\infty^{P_t}\mu\bigg|_{t=0}.$$

Since $\mu$ is a holomorphic form, $D^\ell\mu(P)$ takes finite value at every point $P$. Let $D := \sum_{j=1}^{k} n_j P_j$ be a positive divisor. We define by $B(D)$ the matrix with $\deg D := \sum n_j$ columns and $g$ rows whose $(n_1 + \cdots + n_{j-1} + \ell, i)$-entry is $D^\ell\omega^{(i)}(P_j)$, where $1 \leqq \ell \leqq n_j - 1$. This matrix $B(D)$ informs us singularity of $\Theta$ in $J$ at the point determined by the divisor $D - (\deg D)\infty$. For $\mu \in \Gamma(C, \Omega^1)$, we can find uniquely $c_1, \cdots, c_g \in \mathbf{C}$ such that $\mu = c_1\omega^{(1)} + \cdots + c_g\omega^{(g)}$. In this situation, the three statements

(1) $\mu \in \Gamma(C, \Omega^1(-D))$,

(2) $D^\ell\mu(P_j) = 0$ for all $j$ and $\ell$ with $1 \leqq j \leq k$ and $1 \leqq \ell \leqq n_j - 1$, and

(3) $B(D)\begin{bmatrix} c_1 \\ \vdots \\ c_g \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

are equivalent. So $\dim \Gamma(C, \Omega^1(-D)) = g - \mathrm{rank} B(D)$. The Riemann-Roch theorem states

$$\dim \Gamma(C, \mathcal{O}(D)) = \deg D - g + 1 + \dim \Gamma(C, \Omega^1(-D)).$$

Hence

$$\dim \Gamma(C, \mathcal{O}(D)) = \deg D + 1 - \mathrm{rank} B(D). \tag{1.7.2}$$

However, by [**1**, p.190, (4.5)], singularity of $\Theta$ is known as follows.

LEMMA 1.7.1. *The singular locus of* $\Theta$ *is the points determined by the elements of* $\{P_1 + \cdots + P_{g-1} - (g-1)\infty \mid \dim \Gamma(C, \mathcal{O}(P_1 + \cdots + P_{g-1})) > 1\}$.

By (1.7.2), $\dim \Gamma(C, \mathcal{O}(D)) = 1$ if and only if $\mathrm{rank} B(D) = \deg D$. So we can determine the singular locus of $\Theta$ by calculating $\mathrm{rank} B(D)$. The result is

LEMMA 1.7.2. (1) *If* $g = 2$, $\Theta$ *is non-singular.*
(2) *If* $g = 3$, $\Theta$ *has only one singular point at the origin* $O = (0, 0, 0)$.

*Proof.* We firstly show (2). For two points $P_1$ and $P_2$, we can calculate $B(D)$ and its rank in each case that $P_1 = P_2$ or $P_1 \neq P_2$, and that each $P_i$ is $\infty$, a branch point different from $\infty$, or an ordinary point. Then we see the rank of $B(D)$ is 1 only when $P_1 = P_2 = \infty$ and is $2 (= \deg(P_1 + P_2))$ otherwise. According to 1.7.1 and the statement above this Lemma, we conclude the assertion (2). The assertion (1) is shown by a similar explicit calculation of the matrix $B(P)$ for each point $P$. □

## §2. Taylor and Laurent Expansions

In this section, we give lower terms of the Taylor expansion of $\sigma(u)$ at each point on the curve $C$.

**2.1. Taylor expansion of $\sigma(u)$ at $O$.**
Let $O = (0, \cdots, 0) \in \mathbf{C}^g$.

Proposition 2.1.1.
(1) *If $g = 1$, then the Taylor expansion of $\sigma(u)$ is of the following form*:

$$\sigma(u) = u + (d^\circ \geq 1).$$

(2) *If $g = 2$, then the Taylor expansion of $\sigma(u)$ is of the following form*:

$$\sigma(u) = u_1 + \frac{1}{6}\lambda_2 u_1^3 - \frac{1}{3}\lambda_5 u_2^3 + (d^\circ \geq 5), \quad (\lambda_5 = 1).$$

(3) *If $g = 3$, then Taylor expansion of $\sigma(u)$ is of the form*

$$\begin{aligned}\sigma(u) =& u_1 u_3 - u_2^2 - \frac{\lambda_0}{3}{u_1}^4 - \frac{\lambda_1}{3}{u_1}^3 u_2 - \frac{\lambda_2}{2}{u_1}^2{u_2}^2 - \frac{\lambda_3}{3}u_1{u_2}^3 - \frac{\lambda_4}{3}{u_2}^4 \\ &+\frac{\lambda_2}{6}{u_1}^3 u_3 - \frac{\lambda_5}{3}{u_2}^3 u_3 - \frac{\lambda_6}{2}{u_2}^2 u_3 2 + \frac{\lambda_6}{6}u_1{u_3}^3 - \frac{\lambda_7}{3}u_2{u_3}^3 + (d^\circ \geq 6), \quad (\lambda_7 = 1),\end{aligned}$$

*and the coefficient of the term $u_3^6$ is $\frac{\lambda_7}{45}$.*

Proposition 2.1.1 will be used in Sections 5, 6 and 7. The last statement about a term of degree six is only used in 3.2.3.

*Proof of* 2.1.1. We omit the proof of the statement (1) because it is well-known fact. The proof of the statement (2) was given by Baker, and is reproduced in [**7**, pp. 129-130]. Let us prove (3). Since $\sigma(-u) = \sigma(u)$, the terms of odd total degree are vanish. From [**3**, p.353], we know that the constant term is vanish, and the form of terms of second order is $u_1 u_3 - u_2^2$. Hence $\sigma_{22}(O) \neq 0$, $\sigma_{31}(O) \neq 0$ and the other partial derivatives of second order are vanish. The method to compute terms of higher degree is essentially as same as in the proof of (2) in [**7**]. We set $u = O$, after operating $\frac{\partial^2}{\partial u_1 \partial u_3}$ or $\frac{\partial^2}{\partial u_2^2}$ to the equations, of $\sigma(u)$ and its partial derivatives, obtained from (6), (8) and (11) of 1.5.1 by multiplying $\sigma(u)^2$, then we have the following six equations:

$$\begin{aligned}
(\sigma^2\Delta)_{31}(O) &= -\frac{1}{2}\sigma_{3311}(O), \\
(\sigma^2\Delta)_{31}(O) &= \sigma_{3311}(O), \\
(\sigma^2\Delta)_{31}(O) &= -\frac{2}{3}\lambda_4 + (-\frac{1}{12}\sigma_{2222} + \sigma_{3221})(O), \\
(\sigma^2\Delta)_{22}(O) &= (\sigma_{3311} - 2\sigma_{3221})(O), \\
(\sigma^2\Delta)_{22}(O) &= 4\lambda_4 + (\frac{1}{2}\sigma_{2222} + 2\sigma_{3221})(O), \\
(\sigma^2\Delta)_{22}(O) &= -\frac{4}{3}\lambda_4 - \frac{1}{6}\sigma_{2222}(O).
\end{aligned}$$

These equations yield $\sigma_{2222}(O) = -8\lambda_4$, $\sigma_{3221}(O) = \sigma_{3311}(O) = 0$. Furthermore, we rewrite the leftover eleven equations in 1.5.1 by $\sigma(u)$ and its partial derivatives by the definition of $\wp$-functions. Multiplying $\sigma(u)^2$ to, for instance, the equation obtained from 1.5.1(1) yields

$$\begin{aligned}&\sigma_{3333}(u)\sigma(u) + 4\sigma_{333}(u)\sigma_3(u) - \sigma_{33}(u)^2\\ =&2\lambda_5\lambda_7\sigma(u)^2 + 4\lambda_6(\sigma_3(u)^2 - \sigma_{33}(u)\sigma(u)) + 4\lambda_7(\sigma_3(u)\sigma_2(u) - \sigma_{32}(u)\sigma(u)).\end{aligned}$$

After operating $\frac{\partial^2}{\partial u_2^2}$ on this, by plugging $u = O$, we have

$$-\sigma_{3333}(O)\sigma_{22}(O) = 0. \tag{2.1.1}$$

Since $\sigma_{22}(O) \neq 0$, we obtain that $\sigma_{3333}(O) = 0$. This shows that the term of $u_3^4$ vanishes. The proofs of the other statements are done by repeating the same operation as which gave rise to (2.1.1) from 1.5.1(1). The leftover equations (2), (3), (4), (5), (7), (9), (10), (12), (13), (14) and (15) of 1.5.1 give rise to the coefficients of the terms of $u_2u_3^3$, $u_1u_3^3$, $u_2^2u_3^2$, $u_1u_2u_3^2$, $u_2^3u_3$, $u_1^2u_2u_3$, $u_1^3u_3$, $u_1u_2^3$, $u_1^2u_2^2$, $u_1^3u_2$ and $u_1^4$, respectively. Finaly we can show $\sigma_{333333}(O) = 16\lambda_7$ by setting $u = O$ after operating $\frac{\partial^4}{\partial u_3^4}$ on 1.5.1(1) with multiplied by $\sigma(u)^2$. $\square$

**2.2. Taylor expansion of $\sigma(u)$ at each point of $C$ other than $O$.** Here we give the Taylor expansion of $\sigma(u)$ at each point on the curve $\iota(C)$ other than $O = \iota(\infty)$.

PROPOSITION 2.2.1. *Let $P$ be an arbitrary point of $\kappa^{-1}\iota(C)$ different from points in $\Lambda$. Then the following statements hold.*
*(1) If $g = 1$ then $\sigma(P) \neq 0$.*
*(2) If $g = 2$ then $\sigma_2(P) \neq 0$ and $\sigma_1(P) = -x(P)\sigma_2(P)$. Furthermore the partial derivatives at $P$ of third degree are written by ones of first and second degree as in the following :*

$$\begin{aligned}\sigma_{111}(P) =&(3\frac{\sigma_{21}\sigma_{11}}{\sigma_2} - \frac{3}{2}\frac{\sigma_{22}\sigma_{11}\sigma_1}{\sigma_2^2} - 3\frac{\sigma_{21}^2\sigma_1}{\sigma_2^2} + 3\frac{\sigma_{21}\sigma_{22}\sigma_1^2}{\sigma_2^3}\\ &- \frac{3}{4}\frac{\sigma_{22}^2\sigma_1^3}{\sigma_2^4} - 2\lambda_1\sigma_2 + 4\lambda_2\sigma_1 - 3\lambda_3\frac{\sigma_1^2}{\sigma_2} - 3\lambda_4\frac{\sigma_1^3}{\sigma_2^2} - 3\lambda_5\frac{\sigma_1^4}{\sigma_2^3})(P),\\ \sigma_{112}(P) =&(\frac{1}{2}\frac{\sigma_{22}\sigma_{11}}{\sigma_2} + \frac{\sigma_{21}^2}{\sigma_2} - \frac{\sigma_{22}\sigma_{21}\sigma_1}{\sigma_2^2} + \frac{1}{4}\frac{\sigma_{22}^2\sigma_1^2}{\sigma_2^3} + \lambda_3\sigma_1 + \lambda_4\frac{\sigma_1^2}{\sigma_2} + \lambda_5\frac{\sigma_1^3}{\sigma_2^2})(P),\\ \sigma_{122}(P) =&(\frac{\sigma_{22}\sigma_{21}}{\sigma_2} - \frac{1}{4}\frac{\sigma_{22}^2\sigma_1}{\sigma_2} - \lambda_4\sigma_1 - \lambda_5\frac{\sigma_1^2}{\sigma_2})(P),\\ \sigma_{222}(P) =&(\frac{3}{4}\frac{\sigma_{22}^2}{\sigma_2} + \lambda_4\sigma_2 + \lambda_5\sigma_1)(P).\end{aligned}$$

*(3) If $g = 3$ then $\sigma_3(P) = 0$, $\sigma_2(P) \neq 0$, $\sigma_1(P) = -x(P)\sigma_2(P)$ and $(\sigma^2\Delta)(P) = (\lambda_7\frac{\sigma_1^3}{\sigma_2})(P)$. Furthermore, the partial derivatives at $P$ of third degree are written by*

*ones of first and second degree as in the following form:*

$$\begin{aligned}
\sigma_{111}(P) =&(-3\frac{\sigma_{21}\sigma_{11}}{\sigma_2} - \frac{3}{2}\frac{\sigma_{22}\sigma_{11}\sigma_1}{\sigma_2^2} - 3\frac{\sigma_{21}^2\sigma_1}{\sigma_2^2} + 3\frac{\sigma_{21}\sigma_{22}\sigma_1^2}{\sigma_2^3} - \frac{3}{4}\frac{\sigma_{22}^2\sigma_1^3}{\sigma_2^4} - 2\lambda_1\sigma_2 + 4\lambda_2\sigma_1 \\
&- 3\lambda_3\frac{\sigma_1^2}{\sigma_2} - 3\lambda_4\frac{\sigma_1^3}{\sigma_2^2} - 3\lambda_5\frac{\sigma_1^4}{\sigma_2^3} + 3\lambda_6\frac{\sigma_1^5}{\sigma_2^4} + \frac{3}{4}\lambda_7\frac{\sigma_1^6}{\sigma_2^5})(P), \\
\sigma_{112}(P) =&(\frac{1}{2}\frac{\sigma_{22}\sigma_{11}}{\sigma_2} + \frac{\sigma_{21}^2}{\sigma_2} - \frac{\sigma_{22}\sigma_{21}\sigma_1}{\sigma_2^2} + \frac{1}{4}\frac{\sigma_{22}^2\sigma_1^2}{\sigma_2^3} + \lambda_3\sigma_1 \\
&+ \lambda_4\frac{\sigma_1^2}{\sigma_2} + \lambda_5\frac{\sigma_1^3}{\sigma_2^2} - \lambda_6\frac{\sigma_1^4}{\sigma_2^3} - \lambda_7\frac{\sigma_1^5}{\sigma_2^4})(P), \\
\sigma_{122}(P) =&(\frac{\sigma_{22}\sigma_{21}}{\sigma_2} - \frac{1}{4}\frac{\sigma_{22}^2\sigma_1}{\sigma_2} - \lambda_4\sigma_1 - \lambda_5\frac{\sigma_1^2}{\sigma_2} + \lambda_6\frac{\sigma_1^3}{\sigma_2^2} + \lambda_7\frac{\sigma_1^4}{\sigma_2^3})(P), \\
\sigma_{222}(P) =&(\frac{3}{4}\frac{\sigma_{22}^2}{\sigma_2} + \lambda_4\sigma_2 + \lambda_5\sigma_1 - 3\lambda_6\frac{\sigma_1^2}{\sigma_2} + 3\lambda_7\frac{\sigma_1^3}{\sigma_2^2})(P), \\
\sigma_{113}(P) =&(\frac{\sigma_{32}\sigma_{11}}{\sigma_2})(P), \\
\sigma_{123}(P) =&(\frac{\sigma_{32}\sigma_{21}}{\sigma_2})(P), \\
\sigma_{133}(P) =&(\frac{\sigma_{32}\sigma_{31}}{\sigma_2} + \lambda_7\frac{\sigma_1^2}{\sigma_2})(P), \\
\sigma_{223}(P) =&(\frac{\sigma_{32}\sigma_{22}}{\sigma_2} - 2\lambda_7\frac{\sigma_1^2}{\sigma_2})(P), \\
\sigma_{233}(P) =&(\frac{\sigma_{32}^2}{\sigma_2} - \lambda_7\sigma_1)(P), \\
\sigma_{333}(P) =&- 2\lambda_7\sigma_2(P).
\end{aligned}$$

*Proof.* The assertion (1) is well-known. We show (3). Since

$$\frac{\sigma_1}{\sigma_2}(P) = \frac{\sigma_1\sigma_3 - \sigma_{13}\sigma}{\sigma_2\sigma_3 - \sigma_{23}\sigma}(P) = \frac{\wp_{13}}{\wp_{23}}(P) = \frac{x_1x_2x_3}{-x_1x_2 - x_2x_3 - x_3x_1}\Bigg|_{\substack{x_1=\infty \\ x_2=\infty \\ x_3=x(P)}} = -x(P),$$

$$\frac{\sigma_3}{\sigma_2}(P) = \frac{{\sigma_3}^2 - \sigma_{33}\sigma}{\sigma_2\sigma_3 - \sigma_{23}\sigma}(P) = \frac{\wp_{33}}{\wp_{23}}(P) = \frac{x_1 + x_2 + x_3}{-x_1x_2 - x_2x_3 - x_3x_1}\Bigg|_{\substack{x_1=\infty \\ x_2=\infty \\ x_3=x(P)}} = 0,$$

and $P \neq O$ it must be $\sigma_2(P) \neq 0$ and $\sigma_3(P) = 0$ by virtue of 1.3.1(3) and 1.7.2(2). We get $\sigma_{33}^2(P) = 0$ by setting $u = P$ to the equation which is obtained from 1.5.1(1) by writing it in terms of $\sigma(u)$ and its partial derivatives with multiplied by $\sigma(u)^2$. Hence

$$\sigma_{33}(P) = 0. \tag{2.2.1}$$

We note that $\Delta(u) \in \Gamma(J, \mathcal{O}(2\Theta))$ by the equations (6), (8) or (11) of 1.5.1. So we get

$$(\sigma_1\sigma_{32})(P) = (\sigma_2\sigma_{31})(P) \tag{2.2.2}$$

by plugging $u = P$ in the equation

$$\begin{aligned}(\sigma^3\Delta)(u) =&(\sigma_3\sigma_2\sigma_{21} - \sigma_2\sigma_1\sigma_{32} + \sigma_3\sigma_1\sigma_{22} + \sigma_2^2\sigma_{31} - 2\sigma_3\sigma_1\sigma_{31} + \sigma_3^2\sigma_{11} - \sigma_1^2\sigma_{33}\\ &+ \sigma_{32}\sigma_{21}\sigma - \sigma_{31}\sigma_{22}\sigma + \sigma_{31}^2\sigma - \sigma_{11}\sigma_{33}\sigma)(u).\end{aligned}$$

Here we have used (2.2.1), $\sigma_3(P) = 0$ and $\sigma(P) = 0$. The rest of our proof are also done by repeating the same operation as above. Though the facts (2.2.1), (2.2.2) and $\sigma_2(P) \neq 0$ are used often in the following, we do not mention in the proof when they used. The equation 1.5.1(6) gives rise to

$$(\sigma^2\Delta)(P) = \lambda_7\left(\frac{{\sigma_1}^3}{\sigma_2}\right)(P). \tag{2.2.3}$$

Then the equations (8) and (11) of 1.5.1 give rise to the formulae for $\sigma_{321}(P)$ and $\sigma_{222}(P)$ by (2.2.3). The equations (3) and (15) of 1.5.1 are not nessesary here. The leftover equations (2), (4), (5), (7), (9), (10), (12), (13) and (14) of 1.5.1 give rise to the formulae for $\sigma_{333}(P)$, $\sigma_{332}(P)$, $\sigma_{331}(P)$, $\sigma_{322}(P)$, $\sigma_{311}(P)$, $\sigma_{221}(P)$, $\sigma_{211}(P)$ and $\sigma_{111}(P)$, respectively. The assertion (2) is obtained by a similar calculation. $\square$

**2.3. The Laurent expansions of analytic coordinates on $C$.**

There are two different coordinates which identify a point of $\kappa^{-1}\iota(C)$ or $\iota(C)$, the analytic coordinate $u = (u_1, \cdots, u_g)$ and a pair of solution $(x, y)$ of the algebraic affine equation defining $C$. This subsection is used to make relate these coordinates. If $u \in \kappa^{-1}\iota(C)$ and $\kappa(u) = \iota(x, y)$, then, by (1.4.2),

$$u_j = \int_{\infty}^{(x,y)} \omega^{(j)} \quad (j = 1, \cdots, g) \tag{2.3.1}$$

with certain paths of integrals.

Lemma 2.3.1. *The Laurent expansion of $x(u)$ and $y(u)$ at $u = O$ on the pull-back $\kappa^{-1}\iota(C)$ of $C$ to $\mathbf{C}^g$ are*

$$x(u) = \frac{1}{u_g^2} + (d^\circ(u_g) \geq -1), \quad y(u) = -\frac{1}{u_g^{2g+1}} + (d^\circ(u_g) \geq -2g).$$

*Proof.* We take $t = \dfrac{1}{\sqrt{x}}$ as a local parameter at $O$ along $\kappa^{-1}\iota(C)$. If $u$ is in $\kappa^{-1}\iota(C)$ and sufficiently near $O$. We are agree to that $t$, $u = (u_1, \cdots u_g)$ and $(x, y)$ are coordinates of the same point on $C$. Then

$$\begin{aligned}u_g &= \int_{\infty}^{(x,y)} \frac{x^{g-1}dx}{2y}\\ &= \int_{\infty}^{(x,y)} \frac{x^{-3/2}dx}{2\sqrt{1 + \lambda_{2g}\frac{1}{x} + \cdots + \lambda_0\frac{1}{x^{2g+1}}}}\\ &= \int_0^t \frac{t^3\cdot\left(-\frac{2}{t^3}\right)dt}{2 + (d^\circ \geq 1)}\\ &= -t + (d^\circ(t) \geq 2).\end{aligned}$$

Hence $x(u) = \frac{1}{u_g^2} + (d^\circ(u_g) \geq -1)$ and our assertion is proved. $\square$

LEMMA 2.3.2. *If $u \in \kappa^{-1}\iota(C)$, then the following statements hold.*
(1) *If $g = 2$ then*

$$u_1 = \frac{1}{3}u_2^3 + (d^\circ(u_2) \geq 4).$$

(2) *If $g = 3$ then*

$$u_1 = \frac{1}{5}u_3^5 + (d^\circ(u_3) \geq 6), \quad u_2 = \frac{1}{3}u_3^3 + (d^\circ(u_3) \geq 4).$$

*Proof.* Similar argument as we have $u_g = -t + (d^\circ(t) \geq 2)$ in 2.3.1 gives

$$u_{g-1} = -\frac{1}{3}t^3 + (d^\circ(t) \geq 4), \quad u_{g-2} = -\frac{1}{5}t^5 + (d^\circ(t) \geq 6).$$

Hence we have the desired formulae. □

The following lemma gives an expression of the Taylor expansion of analytic coordinates with respect to the local parameter $y$ at branch points different from $\infty$ along $\kappa^{-1}\iota(C)$.

LEMMA 2.3.3. *Let $(a, 0)$ be a branch point of $C$ different from $\infty$, that is $f(a) = 0$, and let $P$ denote a point of $\mathbf{C}^g$ such that $\kappa(P) = \iota(a, 0)$. Choose $v = (v_1, \cdots, v_g)$ such that $\kappa(v + P) = \iota(x, y)$. Then the Taylor expansion of $v_i$ as a function of $y$ is of the following form:*
(1) *If $g \geq 1$ then* $v_1 = \dfrac{1}{f'(a)}y + \dfrac{f''(a)}{3f'(a)^2}y^3 + (d^\circ(y) \geq 5)$. (2) *If $g \geq 2$ then* $v_2 = \dfrac{a}{f'(a)}y + \dfrac{1 + af''(a)}{3f'(a)^2}y^3 + (d^\circ(y) \geq 5)$. (3) *If $g \geq 3$ then* $v_3 = \frac{a^2}{f'(a)}y + \frac{a(2+af''(a))}{3f'(a)^2}y^3 + (d^\circ(y) \geq 5)$.

*Proof.* Let $g = 3$. Since $f'(a) \neq 0$ and $y^2 = f(x) = f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\cdots$,

$$x = a + \frac{1}{f'(a)}y^2 + \frac{f''(a)}{2f'(a)^2}y^4 + (d^\circ \geq 6).$$

Therefore we have

$$\begin{aligned}
v_3 &= \int_{(0,0)}^{(x,y)} \frac{x^2 dx}{2y} \\
&= \int_0^y (a + \frac{1}{f'(a)}y^2 + \frac{f''(a)}{2f'(a)^2}y^4 + (d^\circ \geq 6))^2 (\frac{1}{f'(a)} + \frac{f''(a)}{f'(a)^2}y^2 + (d^\circ \geq 4))dy \\
&= \int_0^t (\frac{a^2}{f'(a)} + (\frac{a(2 + a^2 f''(a))}{f'(a)^2})y^2 + (d^\circ \geq 4))dy \\
&= \frac{a^2}{f'(a)}y + \frac{a(2 + f''(a))}{3f'(a)^2}y^3 + (d^\circ \geq 5).
\end{aligned}$$

The formulae for $v_1$ and $v_2$ are obtained by the same way. For $g = 1$ or $g = 2$, the formulae are also shown similarly. □

## §3. The translational formula of $\sigma(u)$

In this section, we discuss the translational formula and the Riemann form of $\sigma(u)$. We also give a generalization of Weber's psi function ([**20**, p.150] or [**19**, p.146]) to higher genus case. Our generalization of the psi function is based on Grant [**9**].

### 3.1. The translational formula of $\sigma(u)$.

For $u \in \mathbf{C}^g$ we conventionally denote by $u'$ and $u''$ such elements of $\mathbf{R}^g$ that $u = \omega' u' + \omega'' u''$, where $\omega'$ and $\omega''$ are those defined in Section 1. We define a $\mathbf{C}$-valued $\mathbf{R}$-bilinear form $L(\ \ ,\ \ )$ by $L(u,v) = {}^t u(\eta' v' + \eta'' v'')$ for $u$, $v \in \mathbf{C}^g$. For $\ell$ in $\Lambda$, the lattice of periods as defined in Section 1, let

$$\chi(\ell) = \exp[2\pi i({}^t\ell'\delta'' - {}^t\ell''\delta') - \pi i\, {}^t\ell'\ell''],$$

where $\delta'$ and $\delta''$ are those defined in Section 1.

Lemma 3.1.1. (the translational formula) *The function $\sigma(u)$ satisfies*

$$\sigma(u+\ell) = \chi(\ell)\sigma(u)\exp L(u + \frac{1}{2}\ell, \ell)$$

*for all $u \in \mathbf{C}^g$ and $\ell \in \Lambda$.*

For a proof of this formula we refer to the reader to [**2**, p.286].

Let

$$E(u,v) = L(u,v) - L(v,u), \quad (u, v \in \mathbf{C}^g). \tag{3.1.1}$$

Then, obviously, $E(\ \ ,\ \ )$ is a $\mathbf{C}$-valued $\mathbf{R}$-bilinear form and satisfies $E(u,v) = -E(v,u)$.

Lemma 3.1.2. *The linear form $E(\ ,\ )$ has the following properties*:
(1) $E(iu,v) = E(iv,u)$,
(2) $E(u,v) = 2\pi i({}^t u' v'' - {}^t u'' v')$.
*Especially, $E(\ \ ,\ \ )$ is an $i\mathbf{R}$-valued form and $2\pi i\mathbf{Z}$-valued on $\Lambda \times \Lambda$.*

*Proof.* Statement (1) is proved in [**10**, p.85, Theorem 1.2]. Let us prove (2). In the theory of curves, it is a basic fact that ${}^t\omega'\eta'$ and ${}^t\omega''\eta''$ are symmetric. So

$$\begin{aligned}
E(u,v) =& L(u,v) - L(v,u) \\
&= {}^t u(\eta' v' + \eta'' v'') - {}^t v(\eta' u' + \eta'' u'') \\
&= {}^t v'\, {}^t\omega'\eta'\omega'^{-1}u + {}^t v'\, {}^t\omega''\eta''\omega''^{-1}u - {}^t u'\, {}^t\omega'\eta'\omega'^{-1}v - {}^t u''\, {}^t\omega''\eta''\omega''^{-1}v \\
&= {}^t v'\, {}^t\omega'\eta'(u' + Zu'') + {}^t v'\, {}^t\omega''\eta''(Z^{-1}u' + u'') \\
&\quad - {}^t u'\, {}^t\omega'\eta'(v' + Zv'') - {}^t u''\, {}^t\omega''\eta''(Z^{-1}v' + v'').
\end{aligned}$$

Since ${}^t\omega'\eta'$ and $Z$ are symmetric, it follows that

$$\begin{aligned}
{}^t\omega'\eta' Z = {}^t Z\, {}^t\omega'\eta' = {}^t\omega''\, {}^t\omega'^{-1}\, {}^t\omega'\eta' = {}^t\omega''\eta', \\
{}^t\omega''\eta'' Z = {}^t Z\, {}^t\omega''\eta'' = {}^t\omega'\, {}^t\omega''^{-1}\, {}^t\omega''\eta'' = {}^t\omega'\eta''.
\end{aligned}$$

Therefore, by using the symmetricity of ${}^t\omega'\eta'$ and ${}^t\omega''\eta''$ once more, we have

$$\begin{aligned}
E(u,v) =& {}^t u'\, {}^t\omega'\eta' v' + {}^t u''\, {}^t\omega''\eta' v' + {}^t u'\, {}^t\omega'\eta'' v'' + {}^t u''\, {}^t\omega''\eta'' v'' \\
&- {}^t v'\, {}^t\omega'\eta' u' - {}^t v''\, {}^t\omega''\eta' u' - {}^t v'\, {}^t\omega'\eta'' u'' - {}^t v''\, {}^t\omega''\eta'' u''.
\end{aligned}$$

The generalized Legendre relation ${}^t\omega'\eta'' - {}^t w''\eta' = 2\pi i 1_g$ shows our assertion. $\square$

**3.2. Functions $\psi_n(u)$.**
In this subsection, we review the original and generalized Weber's psi functions defined for the (hyper)elliptic curve $C$. For the case that $J$ has complex multiplication, we treat them more extensively in 4.4.

DEFINITION 3.2.1. *Let $n \in \mathbf{Z}$.*
(1) *When $g = 1$, we let*

$$\psi_n(u) = \frac{\sigma(nu)}{\sigma(u)^{n^2}}.$$

(2) *When $g = 2$ or $3$, we let*

$$\psi_n(u) = \frac{\sigma(nu)}{\sigma_2(u)^{n^2}}.$$

PROPOSITION 3.2.2. *The function $\psi_n(u)$ is a function on $C$ if $g = 1$ and on $\Theta$ if $g \geq 2$. In other words, as a function on $\mathbf{C} = \kappa^{-1}(C)$ if $g = 1$ and on $\kappa^{-1}(\Theta)$ if $g \geq 2$, it is periodic with respect to the lattice $\Lambda$. Furthermore $\psi_n(u)$ restricted to $u \in \kappa^{-1}\iota(C)$ is a polynomial of $x(u)$ if $g = 1$, $2$ with $n$ odd or $g = 3$ with $n$ even, and is a polymomial of $x(u)$ multiplied by $y(u)$ if $g = 1$, $2$ with $n$ even or $g = 3$ with $n$ odd.*

*Proof.* We follow [**9**, p.126, Lemma 1]. We have $(-1)^*\Theta = \Theta$, because our theta divisor is comming from a hyperelliptic curve. So $n^*\Theta = n^2\Theta$ ([**16**, p.59]). Hence the function

$$\phi_n(u) := \frac{\sigma(nu)}{\sigma(u)^{n^2}}$$

is a trivial theta function. On the other hand, by 3.1.1, we have $\sigma(n(u+\ell)) = \chi(n\ell)\sigma(nu)\exp[n^2L(u+\frac{1}{2}\ell,\ell)]$. By the definition of $\chi(\quad)$, $\chi(n\ell)$ is equal to $\chi(\ell)$ or 1 if $n$ is odd or even, respectively. So we have

$$\phi_n(u+\ell) = \phi_n(u)$$

for all $u \in \mathbf{C}^g$ and $\ell \in \Lambda$. Hence the proof of first statement for $g = 1$ is completed. Now assume $g = 2$ or $3$. Because of

$$\wp_{22}(u) = \Big(\frac{\sigma_2^2 - \sigma_{22}\sigma}{\sigma^2}\Big)(u),\ \wp_{222}(u) = \Big(\frac{-2\sigma_2^3 + 3\sigma_2\sigma_{22}\sigma + \sigma_{222}\sigma^2}{\sigma^3}\Big)(u)$$

and of $\sigma(u) = 0$ for all $u \in \kappa^{-1}\Theta$, we have

$$\frac{\phi_n(u)}{\wp_{22}(u)^{\frac{n^2}{2}}} = \frac{\sigma(nu)}{\sigma_2(u)^{n^2}} \text{ or } \frac{\phi_n(u)}{-\frac{1}{2}\wp_{222}(u)\wp_{22}(u)^{\frac{n^2-3}{2}}} = \frac{\sigma(nu)}{\sigma_2(u)^{n^2}}$$

for all $u \in \kappa^{-1}(\Theta)$ if $n$ is even or odd, respectively. Thus $\psi_n(u)$ is a function on $\Theta$. Hence the first statement. For $u \in \kappa^{-1}\iota(C)$, $u = O$ if and only if $\sigma_2(u) = 0$ by 2.2.1. Therefore $\psi_n(u)$ has, as a function on $C$, only pole at $u = 0$. So it must be a polynomial of $x(u)$ and $y(u)$. The last statement is shown by $x(-u) = x(u)$, $y(-u) = -y(u)$ and 1.3.1(3). $\square$

We compute $\psi_n$ for $n = 2$, $3$ and $4$ in 3.2.4 bellow. To do so we give the following

Lemma 3.2.3. *Let $C$ be the hyperelliptic curve of genus $g(\geq 2)$ defined in* 1.1. *Let $P$ be a point of $C$ different from $\infty$. If $nP \in \Theta$ with $n = g$ or $g+1$, then $P$ is a branch point, that is $y(P) = 0$.*

*Proof.* Since $nP \in \Theta$, we have $g-1$ points $Q_1, \cdots, Q_{g-1}$ such that, as divisors, $nP$ is linearly equivalent to $Q_1 + \cdots + Q_{g-1} + (n-g+1)\infty$ ([**15**, pp.3.28-29]). For a point $Q$ of $C$, we here denote by $\overline{Q}$ the point $(x(Q), -y(Q))$. We first assume $n = g$. In this case, there exists a function $G$ on $C$ whose divisor is $(Q_1 + \cdots + Q_{g-1} + \infty) - nP$. Since $P \neq \infty$, $G$ may not be a constant function. However, there is no non-constant function whose poles are bounded by a divisor $\sum_{j=1}^{g} P_j$ such that $P_j \neq \infty$ and $P_j \neq \overline{P_i}$ for every $i$ and $j$ with $i \neq j$ ([**15**, p.3.30]). Since $P \neq \infty$, it must be $P = \overline{P}$, and hence $y(P) = 0$.
Secondly, we assume $n = g+1$. Then there exists a function $G$ on $C$ whose divisor is $(Q_1 + \cdots + Q_{g-1} + 2\infty) - nP$. The divisor of the function $(x - x(P)) \cdot G$ is $Q_1 + \cdots + Q_{g-1} + \overline{P} - (n-1)P$. This function may not be a constant. So, by the same argument as in the case $n = g$, we have $P = \overline{P}$, and hence $y(P) = 0$. Now we have shown the assertion. □

Lemma 3.2.4.
(1) *If $g = 1$ then $\psi_2(u) = -2y(u)$ and if $g = 2$ then $\psi_2(u) = 2y(u)$.*
(2) *If $g = 2$ or $g = 3$ then $\psi_3(u) = -8y(u)^3$.*
(3) *If $g = 3$ then $\psi_4(u) = 64y(u)^4$.*

*Proof.* (1) When $g = 1$, 2.1.1(1) implies

$$\psi_2(u) = \frac{\sigma(2u)}{\sigma(u)^4} = \frac{2u + (d^\circ \geq 2)}{(u + (d^\circ \geq 2))^4} = \frac{2}{u^3} + \cdots .$$

Thus 2.3.1 and 3.2.2 imply $\psi_2(u) = -2y(u)$ for $u \in C$. When $g = 2$, 2.1.1(2) and 2.3.2(1) imply

$$\begin{aligned}
\psi_2(u)|_{u \in \kappa^{-1}\iota(C)} =& \frac{\sigma(2u)}{\sigma_2(u)^4} \\
=& \frac{2u_1 + \frac{1}{6}\lambda_2 8u_1^3 - \frac{1}{3}\lambda_5 8u_2^3 + (d^\circ \geq 5)}{(-u_2^2 + (d^\circ \geq 4))^4} \\
=& \frac{-2u_2^3 + (d^\circ \geq 5)}{(-u_2^2 + (d^\circ \geq 4))^4} \\
=& -\frac{2}{{u_2}^5} + \cdots .
\end{aligned}$$

Thus 2.3.1 and 3.2.2 imply $\psi_2(u) = 2y(u)$ for $u \in C$.
(2) When $g = 2$, we have

$$\psi_3(u)|_{u \in \kappa^{-1}\iota(C)} = \frac{\sigma(3u)}{\sigma_2(u)^9} = \frac{3u_1 + \frac{1}{6}\lambda_2 27u_1^3 - \frac{1}{3}27u_2^3 + (d^\circ \geq 5)}{(-u_2^2 + (d^\circ \geq 4))^9} = 8\frac{1}{u_2^{15}} + \cdots$$

by 2.1.1(2) and 2.3.2(1). Let $P = (x(u), y(u))$ and assume $\psi_3(u) = 0$. Then we have $\sigma(3u) = 0$ because $\sigma_2(u) = 0$ if and only if $u = O$ as seen in 2.2.1(2). So $3P \in \Theta$. By 3.2.3, it must be $P = \infty$ or $P = [-1]P$. This means $y(u) = \infty$ or $y(u) = -y(u)$. Hence we have known, for $u \in \kappa^{-1}\iota(C)$, that $\psi_3(u) = 0$ is equivalent to $y(u) = 0$. So $\psi_3(u)$ must be of the form

$$\psi_3(u)|_{u\in\kappa^{-1}\iota(C)} = -8y(u) \prod_{y(P)=0} (x(u) - x(P)) \tag{3.2.1}$$

by 3.2.2. To determine the product for points $P$, we look at the vanishing order at each $P$ such as $y(P) = 0$. Let $P = (a, 0)$. Assume $u = v + P \in \kappa^{-1}\iota(C)$. Then $y = y(v + P)$ is a local parameter at $P$. Since

$$\begin{aligned}
&\psi_3(v + P)|_{v+P\in\kappa^{-1}\iota(C)} \\
=&\frac{\sigma(3(v + P))}{\sigma_2(v + P)^9} \\
=&\frac{\sigma(3v + P)\chi(2P) \exp L(3v + P + P, 2P)}{\sigma_2(v + P)^9} \\
=&\frac{(3\sigma_1(P)v_1 + 3\sigma_2(P)v_2 + (d^\circ \geq 3)) \exp 4L(P, P)(1 + (d^\circ(v_1, v_2) \geq 1))}{(\sigma_2(P) + (d^\circ(v_1, v_2) \geq 1))^9},
\end{aligned}$$

it follows from the first statement of 2.2.1(2) and 2.3.3 that

$$\psi_3(v + P)|_{v+P\in\kappa^{-1}\iota(C)} = (d^\circ(y) \geq 3).$$

This argument is independent of the choice of $a$. So the factors of the product in (3.2.1) contain $x(v + P) - a$ for all $a$ with $f(a) = 0$. Thus the product must be equal to $y(u)^2$. Hence

$$\psi_3(u)|_{u\in\kappa^{-1}\iota(C)} = -8y(u)^3.$$

When $g = 3$, we have

$$\begin{aligned}
\psi_3(u)|_{u\in\kappa^{-1}\iota(C)} =&\frac{\sigma(3u)}{\sigma_2(u)^9} \\
=&\frac{9u_1u_3 - 9u_2^2 - 81\frac{\lambda_7}{3}u_2u_3^3 + 3^6\frac{\lambda_7}{45}u_3^6 + \cdots}{(-2u_2 - \frac{\lambda_7}{3}u_3^3 + \cdots)^9} \\
=&8\frac{u_3^6 + (d^\circ(u_3) \geq 8)}{(-u_3^3 + (d^\circ(u_3) \geq 5))^9} \\
=&-\frac{8}{u_3^{21}} + \cdots
\end{aligned}$$

for $u \in \kappa^{-1}\iota(C)$ by 2.1.1(3) and 2.3.2(2). Let $P = (x(u), y(u))$ and assume $\psi_3(u) = 0$. Then we have $\sigma(3u) = 0$ because $\sigma_2(u) = 0$ if and only if $u = O$ as seen in 2.2.1(3). Therefore $3P \in \Theta$. By 3.2.3, it must be $P = \infty$ or $P = [-1]P$. This means $y(u)=\infty$ or $y(u)=-y(u)$. Hence we have known, for $u\in C$, that $\psi_3(u)=0$ is equivalent to $y(u)=0$. So $\psi_3(u)$ must be of the form

$$\psi_3(u)|_{u\in\kappa^{-1}\iota(C)} = -8y(u)\prod_{y(P)=0}(x(u)-x(P)) \tag{3.2.2}$$

by 3.2.2. As in the case $g=2$, we look at the vanishing order at a point $P=(a,0)\in C$. By using the Taylor expansion 2.2.1(3) we have

$$\begin{aligned}
\psi_3(v+P)|_{v+P\in\kappa^{-1}\iota(C)} &= \frac{\sigma(3(v+P))}{\sigma_2(v+P)^9}\\
=&\frac{\sigma(3v+P)\chi(2P)\exp L(3v+P+P,2P)}{\sigma_2(v+P)^9}\\
=&\frac{(3(\sigma_1(P)v_1+\sigma_2(P)v_2+\sigma_3(P)v_3)+(d^\circ\geq 3))\exp 4L(P,P)(1+(d^\circ(v_1,v_2,v_3)\geq 1))}{(\sigma_2(P)+(d^\circ(v_1,v_2,v_3)\geq 1))^9}.
\end{aligned}$$

So 2.3.3 and the first statement of 2.2.1(3) give

$$\psi_3(v+P)|_{v+P\in\kappa^{-1}\iota(C)} = (d^\circ(y)\geq 3).$$

This argument is independent of the choice of $a$ with $f(a)=0$. So the factors of the product in (3.2.2) contain $x(v+P)-a$ for all $a$ with $f(a)=0$. Thus the product must be equal to $y(u)^2$. Hence

$$\psi_3(u)|_{u\in\kappa^{-1}\iota(C)} = -8y(u)^3.$$

(3) We have

$$\begin{aligned}
\psi_4(u)|_{u\in\kappa^{-1}\iota(C)} =&\frac{\sigma(4u)}{\sigma_2(u)^{16}}\\
=&\frac{16u_1u_3-16u_2^2-4^4\frac{\lambda_7}{3}u_2u_3^3+4^6\frac{\lambda_7}{45}u_3^6+\cdots}{(-2u_2-\frac{\lambda_7}{3}u_3^3+\cdots)^{16}}\\
=&\frac{64u_3^6+(d^\circ(u_3)\geq 8)}{(-u_3^3+(d^\circ(u_3)\geq 5))^{16}}\\
=&\frac{64}{u_3^{42}}+\cdots
\end{aligned}$$

for $u\in\kappa^{-1}\iota(C)$ by 2.1.1(3). Let $P=(x(u),y(u))$ and assume $\psi_4(u)=0$. Then we have $\sigma(4u)=0$ because of that $\sigma_2(u)=0$ if and only if $u=O$ as seen in 2.2.1(3). Hence $4P\in\Lambda$. By 3.2.3, it must be $P=\infty$ or $P=[-1]P$. This means $y(u)=\infty$ or $y(u)=-y(u)$. Hence we have shown, for $u\in\kappa^{-1}\iota(C)$, that $\psi_4(u)=0$ is equivalent to $y(u)=0$. So $\psi_4(u)$ must be of the form

$$\psi_4(u)|_{u\in\kappa^{-1}\iota(C)} = 64\prod_{y(P)=0}(x(u)-x(P)) \tag{3.2.3}$$

by 3.2.2. As in the proof of (2), we look at the vanishing order of $\psi_4(u)$ at a point $P=(a,0)\in C$. We take $y=y(u)$ as a local parameter at $P$ along $\kappa^{-1}\iota(C)$. Let $u=v+P$ on $\kappa^{-1}\iota(C)$. We first show that $\sigma(4v)=(d^\circ(y(u))\geq 6)$. By 2.3.3, we have

$$\begin{aligned}
v_1v_3-v_2^2 =&\big(\frac{1}{f'}y+\frac{f''}{3f'^2}y^2+(d^\circ\geq 5)\big)\big(\frac{a^2}{f'}y+\frac{a(2+af'')}{3f'^2}y^3+(d^\circ\geq 5)\big)\\
&-\big(\frac{a}{f'}y+\frac{1+af''}{f'^2}y^3+(d^\circ\geq 5)\big)^2\\
=&(d^\circ(y)\geq 6),
\end{aligned}$$

$$\begin{aligned}
-\frac{\lambda_0}{3}v_1^4-\frac{\lambda_1}{3}v_1^3v_2-\lambda_2v_1^2v_2^2-\frac{\lambda_3}{3}v_1v_2^3&\\
-\frac{\lambda_4}{3}v_2^4+\frac{2\lambda_2}{3}v_1^3v_3-\frac{\lambda_5}{3}v_2^3v_3-\frac{\lambda_6}{2}v_2^2v_3^2+\frac{\lambda_6}{6}v_1v_3^3-\frac{\lambda_7}{3}v_2v_3^3&\\
=\frac{1}{f'^4}\big(-\frac{\lambda_0}{3}-\frac{\lambda_1a}{3}-\lambda_2a^2+\frac{2\lambda_2a^2}{3}-\frac{\lambda_3a^3}{3}-\frac{\lambda_4a^4}{3}&\\
-\frac{\lambda_5a^5}{3}-\frac{\lambda_6a^6}{2}+\frac{\lambda_6a^6}{6}-\frac{\lambda_7a^7}{3}\big)y^4+(d^\circ\geq 6)&\\
=(d^\circ(y)\geq 6),&
\end{aligned}$$

where we simply write $f'$ and $f''$ instead of $f'(a)$ and $f''(a)$, respectively. By 3.1.1, we have

$$\psi_4(v+P)|_{v+P\in\kappa^{-1}\iota(C)}=\frac{\sigma(4(v+P))}{\sigma_2(v+P)^{16}}=\frac{\sigma(4v)\chi(4P)\exp L(4v+2P,2P)}{\sigma_2(v+P)^{16}}$$

Therefore

$$\psi_4(v+P)|_{v+P\in\kappa^{-1}\iota(C)}=(d^\circ(y)\geq 6).$$

This argument is independent of the choice of $a$ with $f(a)=0$. So the factors of the product in (3.2.3) contain $x(v+P)-a$ for all $a$, $f(a)=0$, with multiplicity at least three. Hence the product must be equal to $y(u)^6$. Therfore we have shown

$$\psi_4(u)|_{v+P\in\kappa^{-1}\iota(C)}=64y(u)^6,$$

and we have established the proof. $\square$

## §4. Curves of cyclotomic type

### 4.1. Automorphisms of $C$ and endomorphisms of $J$.

In this subsection, we treat the case when the affine equation of the curve $C$ is given by $y^2 = x^m + \frac{1}{4}$ or $y^2 = x^n - x$ with $m$ and $n$ odd. In this paper we say such a curve to be *of cyclotomic type.* In the latter case, if $n-1$ be a power of 2, then we call such a curve to be of 2-*primary* (*cyclotomic*) *type.*

In the first case, we let $\zeta = \exp(\frac{2\pi i}{m})$. Then there are automorphisms

$$\lceil \pm\zeta^j \rceil : C \to C, \quad (x, y) \mapsto (\zeta^j x, \pm y)$$

for $j = 0, \cdots, m-1$. Especially, $\lceil \pm\zeta^j \rceil \infty = \infty$, $\lceil \zeta^j \rceil (0, \frac{1}{2}) = (0, \frac{1}{2})$ and $\lceil -1 \rceil (-4^{-1/m}, 0) = (-4^{-1/m}, 0)$.

In the second case, we let $\zeta = \exp(\dfrac{\pi i}{n-1})$. Then there are automorphisms

$$\lceil \zeta^j \rceil : C \to C, \quad (x, y) \mapsto (\zeta^{2j} x, \zeta^j y)$$

for $j = 0, \cdots, n-1$. We have $\lceil \zeta^j \rceil \infty = \infty$ and $\lceil \zeta^j \rceil (0,0) = (0,0)$.

In each of the cases, each automorphism extends to an endomorphism

$$\lceil \pm\zeta^j \rceil : P_1 + \cdots + P_g - g\infty \mapsto \lceil \pm\zeta^j \rceil P_1 + \cdots + \lceil \pm\zeta^j \rceil P_g - g\infty$$

of $\mathrm{Pic}^\circ(C)$, hence, of $J$, where $P_1, \cdots, P_g$ are points of $C$. We denote by $\mathbf{Z}[\lceil \zeta \rceil]$ the subring of $\mathrm{End}(J)$ generated by $\{\lceil \zeta^j \rceil\}$. The ring $\mathbf{Z}[\lceil \zeta \rceil]$ also acts on $\mathbf{C}^g$ with $\Lambda$ being stable, that is equivalent to say $\alpha\Lambda \subset \Lambda$ for all $\alpha \in \mathbf{Z}[\lceil \zeta \rceil]$. We have obvious relations $\lceil 1 \rceil = 1$, $\lceil \zeta^j \rceil \lceil \zeta^k \rceil = \lceil \zeta^{j+k} \rceil$ and $\lceil -\zeta^j \rceil = -\lceil \zeta^j \rceil$. In each case, since $\lceil \pm\zeta^j \rceil \iota(C) = \iota(C)$, it is obvious that $\lceil \pm\zeta^j \rceil \Theta = \Theta$.

Lemma 4.1.1.
(1) *If $C$ is defined by $y^2 = x^{2g+1} + \frac{1}{4}$ then $\mathbf{Z}[\lceil \zeta \rceil] \cong \mathbf{Z}[X]/(X^{2g} + \cdots + X + 1)$ by $\lceil \zeta \rceil \mapsto X$.*
(2) *If $C$ is defined by $y^2 = x^{2g+1} - x$ then $\mathbf{Z}[\lceil \zeta \rceil] \cong \mathbf{Z}[X]/(X^{2g} + 1)$ by $\lceil \zeta \rceil \mapsto X$.*

*Proof.* The isomorphisms of (1) and (2) are easily obtained from the action

$$\lceil \zeta \rceil (u_1, u_2, \cdots, u_g) = (\zeta u_1, \zeta^2 u_2, \cdots, \zeta^g u_g)$$

and

$$\lceil \zeta \rceil (u_1, u_2, \cdots, u_g) = (\zeta u_1, \zeta^3 u_2, \cdots, \zeta^{2g-1} u_g),$$

respectively. □

Let $b$ be an element of $\mathbf{Z}[\lceil \zeta \rceil]$. In the following, we will investigate the $b$-multiplication for $\sigma(u)$, that is $\sigma(bu)$, and pull-back of $b$-multiplication for $\Theta$, that is $b^*\Theta$. If $b \in \mathbf{Z}$ then most results of this section are quite simple. However, for our main results, one of the most important cases would be when $b$ is an imaginary number in $\mathbf{Z}[\lceil \zeta \rceil]$.

### 4.2. The Riemann form for a curve of cyclotomic type.

Definition 4.2.1. *The function $\sigma(u) = \sigma(u; Z)$ is said to be a* normalized *theta function ( in the sence of* [**10**, p.87] *or* [**18**, p.20] *) if the form $L(u,v)$ defined in* 3.1 *is hermitian, that is equivalent to say $L(v,u) = \overline{L(u,v)}$, where the bar means the complex conjugate. If that is so,*

$$L(u,v) = \frac{1}{2i}[E(iu,v) + iE(u,v)]$$

*for all $u$, $v \in \mathbf{C}^g$.*

Lemma 4.2.2. *Let $\eta'$ and $\eta''$ be the period matrix of differential forms of second kind as is defined in* 1.1. *If $\eta'^{-1}\eta'' = \overline{Z}$ then $\sigma(u)$ is a normalized theta function.*

*Proof.* By the definition of $L(\ ,\ )$, $L(iu,v) = iL(u,v)$. We will show that $L(u,iv) = -iL(u,v)$. Let us define $w'$ and $w'' \in \mathbf{R}^g$ by $i\omega'^{-1}v = w' + Zw''$. Then $-i\overline{\omega'^{-1}}v = w' + \overline{Z}w''$. Since $\omega'^{-1}v = v' + Zv''$ and $\overline{\omega'^{-1}v} = v' + \overline{Z}v''$, we have

$$\begin{aligned} L(u,iv) &= u(\eta'w' + \eta''w'') \\ &= u\eta'(w' + \overline{Z}w'') \\ &= u\eta'(\overline{i\omega'^{-1}v}) \\ &= u\eta'(-i)(v' + \overline{Z}v'') \\ &= -iu(\eta'v' + \eta''v'') \\ &= -iL(u,v). \end{aligned}$$

Since $E(\ ,\ )$ is $\mathbf{R}$-valued, we have $L(u,v) = \overline{L(v,u)}$ by 3.1.2(1) and the relation of $L(\ ,\ )$ and $E(\ ,\ )$ in 4.2.1. Therefore we have the assertion. □

Proposition 4.2.3. *If $C$ is of cyclotomic type, then $\eta'^{-1}\eta'' = \overline{Z}$. Hence $\sigma(u;Z)$ is normalized because of* 4.2.2.

*Proof.* In our case, the differential forms $\eta^{(1)}, \cdots, \eta^{(g)}$ defined in 1.1 are

$$\eta^{(1)} = (2g-1)\frac{x^{2g-1}}{2y}dx,\ \eta^{(2)} = (2g-3)\frac{x^{2g-2}}{2y}dx,\ \cdots,\ \eta^{(g)} = \frac{x^g}{2y}dx.$$

Let $C$ be the curve defined by $y^2 = x^{2g+1} + \frac{1}{4}$ (resp. $y^2 = x^{2g+1} - x$) and let

$$K_i = \int_{(0,\frac{1}{2})}^{(-4^{\frac{-1}{2g+1}},0)} \omega^{(i)},\ H_i = \int_{(0,\frac{1}{2})}^{(-4^{\frac{-1}{2g+1}},0)} \eta^{(i)}$$

$$(\text{resp. } K_i = \int_{(0,0)}^{(1,0)} \omega^{(i)},\ H_i = \int_{(0,0)}^{(1,0)} \eta^{(i)})$$

be integrals along the real axis. Then we have

$$\int_{(0,\frac{1}{2})}^{(-4^{-1/(2g+1)}\zeta^k,0)} \omega^{(i)} = \int_{(0,\frac{1}{2})}^{(-4^{-1/(2g+1)},0)} \lceil \zeta^k \rceil \omega^{(i)} = \zeta^{ki} K_i,$$

$$\int_{(0,\frac{1}{2})}^{(-4^{-1/(2g+1)}\zeta^k,0)} \eta^{(i)} = \int_{(0,\frac{1}{2})}^{(-4^{-1/(2g+1)},0)} \lceil \zeta^k \rceil \eta^{(i)} = \zeta^{(2g-i+1)k} H_i = \zeta^{-ki} H_i$$

$$\text{(resp. } \int_{(0,0)}^{(\zeta^k,0)} \omega^{(i)} = \int_{(0,0)}^{(1,0)} \lceil \zeta^k \rceil \omega^{(i)} = \zeta^{(2i-1)k} K_i,$$

$$\int_{(0,\frac{1}{2})}^{(\zeta^k,0)} \eta^{(i)} = \int_{(0,\frac{1}{2})}^{(1,0)} \lceil \zeta^k \rceil \eta^{(i)} = \zeta^{(2(2g-i)+1)k} H_i = \zeta^{(-2i+1)k} H_i),$$

where each of integrals is along the segment with a constant argument. Let us compute the periods matrices $\eta'$ and $\eta''$ by choosing paths $\alpha^{(j)}$ and $\beta^{(j)}$ as a join of segments of line in $x$-plane with constant $\arg x$ as in Figure 2.

Figure 2

Then we are led to the following relations:

$$\int_{\alpha^{(j)}} \eta^{(i)} = \frac{H_i}{K_i} \overline{\int_{\alpha^{(j)}} \omega^{(i)}}, \quad \int_{\beta^{(j)}} \eta^{(i)} = \frac{H_i}{K_i} \overline{\int_{\beta^{(j)}} \omega^{(i)}}$$

for all $i$ and $j$. Hence

$$\eta' = \begin{bmatrix} \frac{H_1}{K_1} & & \\ & \ddots & \\ & & \frac{H_g}{K_g} \end{bmatrix} \overline{\omega'}, \quad \eta'' = \begin{bmatrix} \frac{H_1}{K_1} & & \\ & \ddots & \\ & & \frac{H_g}{K_g} \end{bmatrix} \overline{\omega''},$$

So we have $\eta'^{-1}\eta'' = \overline{\omega'^{-1}}\overline{\omega''} = \overline{Z}$. $\square$

For each $b \in \mathbf{Z}[\lceil \zeta \rceil]$ we denote by $\overline{b}$ the involution in $\mathbf{Z}[\lceil \zeta \rceil]$ induced by $\overline{\lceil \zeta^j \rceil} = \lceil \zeta^{-j} \rceil$.

Proposition 4.2.4. *If $C$ is of cyclotomic type, then*

$$E(bu,v)=E(u,\overline{b}v),\ \ L(bu,v)=L(u,\overline{b}v),$$

*for all $u$, $v\in\mathbf{C}^g$ and $b\in\mathbf{Z}[\lceil\zeta\rceil]$.*

*Proof.* Since $\lceil\zeta\rceil$ is an automorphism of $\Lambda$, there exists a matrix $M(\zeta^j)$ with entries in $\mathbf{Z}$ such that

$$\lceil\zeta^j\rceil u=[\omega'\,\omega'']M(\zeta^j)\begin{bmatrix}u'\\ u''\end{bmatrix}.$$

Since $\lceil\zeta^j\rceil$ is an automorphism of $C$ over $\mathbf{Q}$, it induces an automorphism of the fundamental group of $C$. Hence

$${}^tM(\zeta^j)IM(\zeta^j)=I$$

with $I=\begin{bmatrix}0 & 1_g\\ -1_g & 0\end{bmatrix}$ and $M(\zeta^j)M(\zeta^{-j})=1_{2g}$. Thus we have ${}^tM(\zeta^j)I=IM(\zeta^j)^{-1}=IM(\zeta^{-j})$. We define $U'$ and $U''$ by $\lceil\zeta^j\rceil u=\omega'U'+\omega''U''$ or equivalently by $\begin{bmatrix}U'\\ U''\end{bmatrix}=M(\zeta^j)\begin{bmatrix}u'\\ u''\end{bmatrix}$, and let $\begin{bmatrix}V'\\ V''\end{bmatrix}=M(\zeta^{-j})\begin{bmatrix}v'\\ v''\end{bmatrix}$, where the letters $u'$, $u''$, $v'$ and $v''$ are used under the convention of 3.1. Then 3.1.2(2) and the above equation give

$$\begin{aligned}
E(\lceil\zeta^j\rceil u,v)&=2\pi i({}^tU'v''-{}^tU''v')\\
&=2\pi i[{}^tU'\ {}^tU'']I\begin{bmatrix}v'\\ v''\end{bmatrix}\\
&=2\pi i[{}^tu'\ {}^tu'']\,{}^tM(\zeta^j)I\begin{bmatrix}v'\\ v''\end{bmatrix}\\
&=2\pi i[{}^tu'\ {}^tu'']I\,{}^tM(\zeta^{-j})\begin{bmatrix}v'\\ v''\end{bmatrix}\\
&=2\pi i[{}^tu'\ {}^tu'']I\begin{bmatrix}V'\\ V''\end{bmatrix}\\
&=2\pi i({}^tu'V''-{}^tu''V')\\
&=E(u,\overline{\lceil\zeta^j\rceil}v).
\end{aligned}$$

By linearlity the proof of the first equation is completed. The second is obtained by the relation in 4.2.1. □

Lemma 4.2.5. *If $C$ is of cyclotomic type, then there is $j\in\mathbf{Z}$ such that*

$$\sigma(\lceil\zeta\rceil u)=\zeta^j\sigma(u).$$

*In particular,*
(1) *If the genus of $C$ is 1 or 2, that is $C$ is defined by $y^2=y^3+\frac{1}{4}$, $y^2=y^3-x$, $y^2=y^5+\frac{1}{4}$ or $y^2=y^5-x$, then $\sigma(\lceil\zeta\rceil u)=\zeta\sigma(u)$ ;*
(2) *If $C$ is defined by $y^2=x^7+\frac{1}{4}$, then $\sigma(\lceil\zeta\rceil u)=\zeta^4\sigma(u)$ ;*

(3) *If $C$ is defined by $y^2 = x^7 - x$, then $\sigma([\zeta]u) = \zeta^6\sigma(u)$.*

*Proof.* Since $[\zeta]^*\Theta = \Theta$, two functions $\sigma([\zeta]u)$ and $\sigma(u)$ have the same divisor of zeros. So $\dfrac{\sigma([\zeta]u)}{\sigma(u)}$ is an entire function, i.e. a trivial theta function. On the other hand, by 3.1.1, we have

$$\frac{\sigma([\zeta](u+\ell))}{\sigma(u+\ell)} = \frac{\chi([\zeta]\ell)}{\chi(\ell)}\frac{\sigma([\zeta]u)}{\sigma(u)}\frac{\exp L([\zeta](u+\frac{1}{2}\ell),[\zeta]\ell)}{\exp L(u+\frac{1}{2}\ell,\ell)}.$$

Since $\chi(\ )$ is 1 or $-1$, the above quotient is equal to $\pm\dfrac{\sigma([\zeta]u)}{\sigma(u)}$ by virtue of 4.2.4. Therefore the function $\dfrac{\sigma([\zeta]u)}{\sigma(u)}$ is bounded. In fact, if $M$ be the maximum of absolute values of this function on the domain

$$\left\{ u = \omega'\begin{bmatrix} u_1' \\ \vdots \\ u_g' \end{bmatrix} + \omega''\begin{bmatrix} u_1'' \\ \vdots \\ u_g'' \end{bmatrix}; 0 \le u_j' \le 1, 0 \le u_j'' \le 1 \text{ for } j = 1, \cdots g \right\},$$

then $\dfrac{\sigma([\zeta]u)}{\sigma(u)} \le M$ for all $u \in \mathbf{C}^g$. Liouville's theorem says such function is a constant function, say $\dfrac{\sigma([\zeta]u)}{\sigma(u)} = c$. Consequently, if $\zeta^k = 1$, then

$$c^k = \frac{\sigma([\zeta]u)}{\sigma(u)}\frac{\sigma([\zeta^2]u)}{\sigma([\zeta]u)}\cdots\frac{\sigma([\zeta^{k-1}]u)}{\sigma([\zeta^{k-2}]u)}\frac{\sigma(u)}{\sigma([\zeta^{k-1}]u)} = 1$$

So $c = \zeta^j$ for some $j \in \mathbf{Z}$. If $g$ is 1, 2 or 3, by looking at the Taylor expansion 2.1.1 at $O$, we get the desired formulae. $\square$

The following Lemma is used in 4.2.8 bellow.

LEMMA 4.2.6. *Let $C$ be of cyclotomic type. Let $c$ and $b$ be elements of $\mathbf{Z}[[\zeta]]$ such that $\overline{c} = c$ and such that $\overline{b} \equiv b \mod c^2$. Let $P$ be a point in $\mathbf{C}^g$ such that $cP \in \Lambda$. Then*

$$L(bP, P) \equiv L(P, bP) \mod 2\pi i\mathbf{Z}.$$

*Proof.* Since $\overline{b} - b \equiv 0 \mod c^2$, we can write $\overline{b} - b = ac^2$ with $a \in \mathbf{Z}[[\zeta]]$. Then $E(P, (\overline{b} - b)P) = E(P, ac^2P) = E(\overline{c}P, acP) = E(cP, acP) \in 2\pi i\mathbf{Z}$ by 4.2.4, because $cP \in \Lambda$ and 3.1.2(2). Therefore[1],

(4.2.1)

$$\begin{aligned} E(bP, P) =& E(P, \overline{b}P) \text{ by 4.2.4} \\ =& E(P, (\overline{b} - b)P + bP) \\ =& E(P, (\overline{b} - b)P) + E(P, bP) \equiv E(P, bP) \mod 2\pi i\mathbf{Z}. \end{aligned}$$

[1] Incidentally, since $-E(P, bP) = E(bP, P)$, we have $2E(P, bP) \equiv 0 \mod 2\pi i\mathbf{Z}$.

Furthermore, since

$$E(i \cdot bP, P) = E(iP, bP) \tag{4.2.2}$$

by 3.1.2(1), we obtain that

$$L(bP, P) = \frac{1}{2i}(E(i \cdot bP, P) + iE(bP, P)) \equiv L(P, bP) \mod 2\pi i\mathbf{Z}$$

by 4.2.1, (4.2.1) and (4.2.2). $\square$

DEFINITION 4.2.7. *Let $\rho : \zeta \mapsto \zeta^{-1}$ be the complex conjugate. Let $T$ be an element of $\mathbf{Z}[\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})]$. If $T + \rho T$ is the norm from $\mathbf{Q}(\zeta)$ to $\mathbf{Q}$, then $T$ is called a* type norm ([**11**, p.22]).

LEMMA 4.2.8. *Let $C$ be of cyclotomic type.*
*(1) Let $c$ and $b$ be elements of $\mathbf{Z}[\lceil\zeta\rceil]$ such that, as ideals, $(c^\gamma) = (c)$ for all $\gamma \in \mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$ and such that $b \equiv 1 \mod c^2$. Let $P$ be a point of $\mathbf{C}^g$ such that $cP \in \Lambda$. If $T$ is a type norm, then for all $v \in \mathbf{C}^g$.*

$$\sigma(b^T(v+P)) = \sigma(b^Tv+P)\exp[\frac{1}{2}(\mathrm{N}b-1)L(P,P)+\frac{1}{2}L(b^Tv,(b^T-1)P)]\chi((b^T-1)P).$$

*(2) Let $P_0$ be a point on $C$ such that $x(P_0) = 0$. For all $v \in \mathbf{C}^g$,*

$$\sigma(v+\lceil\zeta\rceil P_0) = \sigma(v+P_0)\exp[L(v,(\lceil\zeta\rceil-1)P_0)+\frac{1}{2}L((\overline{\lceil\zeta\rceil}-\lceil\zeta\rceil)P_0,P_0)]\chi((\lceil\zeta\rceil-1)P_0).$$

*Proof.* The assumption on $b$ and $c$ implies $b^T \equiv 1 \mod c^2$. So $(b^T - 1)P \in \Lambda$ and 3.1.1 gives

$$\begin{aligned}\sigma(b^T(v+P)) =&\sigma(b^Tv + P + (b^T-1)P)\\ =&\sigma(b^Tv+P)\exp(L(b^Tv+P+\frac{1}{2}(b^T-1)P,(b^T-1)P)\chi((b^T-1)P).\end{aligned}$$

Here

$$\begin{aligned}&L(b^Tv + P + \frac{1}{2}(b^T-1)P,(b^T-1)P)\\ =&L(\frac{1}{2}(b^T+1)P,(b^T-1)P) + L(b^Tv,(b^T-1)P)\\ =&\frac{1}{2}L((b^T+1)P,(b^T-1)P) + L(b^Tv,(b^T-1)P)\\ \equiv&\frac{1}{2}(L(b^TP,b^TP) - L(P,P)) + L(b^Tv,(b^T-1)P) \mod 2\pi i\mathbf{Z} \text{ by 4.2.6}\\ =&\frac{1}{2}(L(b^T\overline{b^T}P,P) - L(P,P)) + L(b^Tv,(b^T-1)P) \text{ by 4.2.4}\\ =&\frac{1}{2}(\mathrm{N}b-1)L(P,P) + L(b^Tv,(b^T-1)P).\end{aligned}$$

Hence we have (1). The formula (2) is obtained by calculation like (1). $\square$

### 4.3. Action for the theta divisor.

In this subsection, the curve $C$ is still assumed to be of cyclotomic type. For $b \in \mathbf{Z}[\lceil\zeta\rceil]$ we denote by $b^*\Theta$ the pull-back of $\Theta$ with respect to the endomorphism $b$. Therefore $\kappa^{-1}(b^*\Theta)$ is just the divisor of zeros of $\sigma(bu)$, and $E(bu, bv)$ is the Riemann form associated to this divisor.

The following proposition seems to be true for all $C$ of cyclotomic type. But the author has no proof of it for 2-primary type (see 4.1) except for the curve defined by $y^2 = x^5 - x$. We denote by $\approx$ the algebraic equivalence and by $\sim$ the linear equivalence.

Proposition 4.3.1. *Assume that $g \geq 2$ and that $C$ is not of 2-primary type. Let $\varepsilon_1, \cdots, \varepsilon_n$ and $b$ be elements of $\mathbf{Z}[\lceil\zeta\rceil]$, and let $\ell_0, \ell_1, \cdots, \ell_n$ be rational integers. Let $\rho : \zeta \mapsto \zeta^{-1}$ be the complex conjugate. If $b^{1+\rho} = \ell_0 + \ell_1\varepsilon_1^2 + \cdots + \ell_n\varepsilon_n^2$, then*

$$b^*\Theta \sim \ell_0 \cdot \Theta + \ell_1 \cdot \varepsilon_1^*\Theta + \cdots + \ell_n \cdot \varepsilon_n^*\Theta.$$

If $C$ is the curve defined by $y^2 = x^5 + \frac{1}{4}$, 4.3.1 is proved in [**9**, p.126, Lemma 1], We firstly prove the following lemma as in [**9**].

Lemma 4.3.2. *Assume that $g \geq 2$ and that $C$ is not of 2-primary type. Let $D$ be a divisor of $J$. If $D \approx 0$ and $\lceil\pm\zeta\rceil^*D \sim D$, then $D \sim 0$.*

*Proof.* We prove by using the dual Abelian variety of $J$. Since $\Theta$ gives a principal polarization of $J$ and $D \approx 0$, $D \sim \Theta_u - \Theta$ for some $u \in J$, where $\Theta_u$ denotes the translation of $\Theta$ by $u$ ([**16**, p.77, Theorem 1]). Since $\lceil\zeta^j\rceil(\Theta) = \Theta$, we have $\lceil\pm\zeta^j\rceil(\Theta_u) = \Theta_{\lceil\pm\zeta^j\rceil u} \sim \Theta_u$. Hence $\lceil\pm\zeta^j\rceil u = u$ by [**14**, p.186, 6.6]. Because $n-1$ is not a power of 2, there is an integer $\nu$ such that $1 - \lceil\zeta^\nu\rceil$ and 2 are coprime in $\mathrm{End}(J)$. The above linear equivalences imply that $u$ is 2-torsion and $1 - \lceil\zeta^\nu\rceil$-torsion. Hence $u = O$ and so $D \sim 0$. □

*Proof of* 4.3.1. For a divisor $D$ in $J$, we denote by $E_D(\ ,\ )$ the Riemann form associated to $D$ which takes values in $2\pi i\mathbf{Z}$ on $\Lambda \times \Lambda$([**11**, p.68]). Then

$$\begin{aligned}
E_{b^*\Theta}(u,v) =& E(bu, bv) \\
=& E(b\bar{b}u, v) \text{ (by 4.2.4)} \\
=& E(b^{1+\rho}u, v) \\
=& E((\ell_0 + \ell_1\varepsilon_1^2 + \cdots + \ell_n\varepsilon_n^2)u, v) \\
=& \ell_0 E(u,v) + \ell_1 E(\varepsilon_1^2 u, v) + \cdots + \ell_n E(\varepsilon_n^2 u, v) \\
=& \ell_0 E(u,v) + \ell_1 E(\varepsilon_1 u, \varepsilon_1 v) + \cdots + \ell_n E(\varepsilon_n u, \varepsilon_n v) \\
=& E_{\ell_0\cdot\Theta + \ell_1\cdot\varepsilon_1^*\Theta + \cdots + \ell_n\cdot\varepsilon_n^*\Theta}(u,v).
\end{aligned}$$

Thus $b^*\Theta \approx \ell_0 \cdot \Theta + \ell_1 \cdot \varepsilon_1^*\Theta + \cdots + \ell_n \cdot \varepsilon_n^*\Theta$. Since the both divisors are invariant by the action $\lceil\pm\zeta\rceil^*$, 4.3.2 implies they are linearly equivalent. □

For a curve of 2-primary type, the proof above can not be applied. Here we give a proof only for the curve defined by $y^2 = x^5 - x$, for the case that $\varepsilon_1$ of 4.3.1 is certain special element. Note that, for this curve, the map $\mathbf{Z}[\lceil\zeta\rceil] \to \mathrm{End}(J)$ is known to be injective and the image is isomorphic to $\mathbf{Z}[\zeta]$ by $\lceil\zeta^j\rceil \mapsto \zeta$ (see also 6.2).

Proposition 4.3.3. *Assume that $C$ is defined by $y^2 = x^5 - x$. Let $\varepsilon_1 = 1 + \sqrt{2} = 1 + \zeta - \zeta^3$ and let $b$ be an element of $\mathbf{Z}[\zeta]$. Let $\rho : \zeta \mapsto \zeta^{-1}$ be the complex conjugate. If $b^{1+\rho} = \ell_0 + \ell_1 \varepsilon_1^2$ with rational integers $\ell_0$ and $\ell_1$, then $b^*\Theta \sim \ell_0 \cdot \Theta + \ell_1 \cdot \varepsilon_1^*\Theta$.*

*Proof.* We prove the stament in somewhat extended form. First of all, we note the following. Let $b \in \mathbf{Z}[\zeta]$ and let $b = p + q\zeta + r\zeta^2 + s\zeta^3$ with integers $p$, $q$, $r$ and $s$. Then we have

$$b^{1+\rho} = \left(p^2 + q^2 + r^2 + s^2 + \frac{3}{2}(-pq + ps - rs + qr)\right) + \left(\frac{1}{2}(pq - ps + rs - qr)\right)\varepsilon_1^2.$$

So, in the expression $b^{1+\rho} = \ell_0 + \ell_1\varepsilon_1^2$ for arbitrary $b \in \mathbf{Z}[\zeta]$ with $\ell_0$ and $\ell_1 \in \mathbf{Q}$, it is actually $2\ell_0$ and $2\ell_1 \in \mathbf{Z}$. Now let us prove that, for every $b \in \mathbf{Z}[\zeta]$, if $2b^{1+\rho} = 2\ell_0 + 2\ell_1\varepsilon_1^2$ then $2(b^*\Theta) \sim 2\ell_0 \cdot \Theta + 2\ell_1 \cdot \varepsilon_1^*\Theta$, and if moreover $\ell_0$, $\ell_1 \in \mathbf{Z}$ then $(b^*\Theta) \sim \ell_0 \cdot \Theta + \ell_1 \cdot \varepsilon_1^*\Theta$ by induction with respect to $p$, $q$, $r$ and $s$. In the following we note that $[\zeta^j]^*\Theta = \Theta$. If four or three of $p$, $q$, $r$ and $s$ are 0, the statement is trivial. We frequently apply [**16**, p.58, Corollary 2]. We get that

$$\begin{aligned}
\Theta =&(1 + i - i)^*\Theta \\
\sim&(1 + i)^*\Theta + (1 - i)^*\Theta + 0^*\Theta - 3\Theta \\
=&(1 + i)^*\Theta + ((i + 1)(-i))^*\Theta - 3\Theta \\
=&2 \cdot (1 + i)^*\Theta - 3\Theta.
\end{aligned}$$

Hence $(1+i)^*\Theta \sim 2\cdot\Theta$ and $(\zeta-\zeta^3)^*\Theta = ((1+i)(-\zeta^3))^*\Theta \sim 2\cdot\Theta$. For the pull-back of $1 + \zeta$, from

$$\begin{aligned}
\varepsilon_1^*\Theta =&(1 + \zeta - \zeta^3)^*\Theta \\
\sim&(1 + \zeta)^*\Theta + (1 - \zeta^3)^*\Theta + (\zeta - \zeta^3)^*\Theta - 3\Theta \\
\sim&(1 + \zeta)^*\Theta + ((\zeta + 1)(-\zeta^3))^*\Theta + 2\Theta - 3\Theta \\
=&2 \cdot (1 + \zeta)^*\Theta - \Theta,
\end{aligned}$$

we have $2\cdot(1+\zeta)^*\Theta \sim -\Theta+\varepsilon_1^*\Theta$ and $(1-\zeta)^*\Theta = (1+\zeta-\zeta-\zeta)^*\Theta \sim 4\Theta-(1+\zeta)^*\Theta$. These are a part of the disired results since $(1+\zeta)^{1+\rho} = \frac{1}{2}(-1+\varepsilon_1^2)$. Therefore the statement is shown for $1+\zeta^3 = \zeta^3(1-\zeta)$, $\zeta+\zeta^2 = \zeta(1+i)$ and $\zeta^2+\zeta^3 = \zeta(1+\zeta)$. By using these results, we can check easily the statement for $b$ with three or four of $p$, $q$, $r$ and $s$ being 1. The rest of the proof is completed by induction as follows. If the statement is true for $b$ and $b-\zeta^j$ then it is true for $b+\zeta^j$. In fact, let $b^{1+\rho} = \ell_0+\ell_1\varepsilon_1^2$ and $(b - \zeta^j)^{1+\rho} = b^{1+\rho} - (\zeta^{-j}b + \zeta^j b^\rho) + 1 = m_0 + m_1\varepsilon_1^2$. Then $(\zeta^{-j}b + \zeta^j b^\rho) = (\ell_0 - m_0 + 1) + (\ell_1 - m_1)\varepsilon_1^2$. Thus $(b + \zeta^j)^{1+\rho} = (2\ell_0 - m_0 + 2) + (2\ell_1 - m_1)\varepsilon_1^2$. Note that the coefficients $2\ell_0 - m_0 + 2$ and $m_0$, $2\ell_1 - m_1$ and $m_1$ are of the same parity. On the other hand,

$$b^*\Theta \sim (b + \zeta^j)^*\Theta + (b - \zeta^j)^*\Theta + 0^*\Theta - b^*\Theta - 2 \cdot \Theta$$

yields

$$(b + \zeta^j)^*\Theta \sim 2b^*\Theta - (b - \zeta^j)^*\Theta + 2 \cdot \Theta.$$

So we have

$$2 \cdot (b+\zeta^j)^*\Theta \sim 2(2\ell_0 - m_0 + 2) \cdot \Theta + 2(2\ell_1 - m_1) \cdot \varepsilon_1^*\Theta.$$

Furthermore if $2\ell_0 - m_0 + 2$ and $2\ell_1 - m_1 \in \mathbf{Z}$, then we have

$$(b+\zeta^j)^*\Theta \sim (2\ell_0 - m_0 + 2) \cdot \Theta + (2\ell_1 - m_1) \cdot \varepsilon_1^*\Theta.$$

Hence the statement is also true for $b+\zeta^j$. Similarly, if the statement is true for $b$ and $b+\zeta^j$ then it is true for $b-\zeta^j$. Therefore we have shown the assertion for all $b$. □

**4.4. Further generalization of psi functions.**
Here we construct a generarized Weber's psi function.

LEMMA 4.4.1. *Let $b$ be an element of $\mathbf{Z}[\lceil\zeta\rceil]$. Under the notation of* 4.3.1 *or* 4.3.3, *the function*

$$\phi_b(u) = \frac{\sigma(bu)}{\sigma(u)^{\ell_0}\sigma(\varepsilon_1 u)^{\ell_1}\cdots\sigma(\varepsilon_n u)^{\ell_n}}$$

*on $\mathbf{C}^g$ satisfies*

$$\phi_b(u+\ell) = \pm\phi_b(u)$$

*for all $u \in \mathbf{C}^g$ and $\ell \in \Lambda$. Here the signature $\pm$ is independent of $u$. Moreover if $C$ is not of 2-primary type or is defined by $y^2 = x^5 - x$, then*

$$\phi_b(u+\ell) = \phi_b(u)$$

*for all $u \in \mathbf{C}^g$ and $\ell \in \Lambda$.*

The author can not follow the proof of [**9**, Section 3] for $C$ defined by $y^2 = x^5 + \frac{1}{4}$. Here we give another proof.

*Proof.* As is shown in 4.3.1 or 4.3.3,

$$E(bu, bv) = \ell_0 E(u,v) + \ell_1 E(\varepsilon_1 u, \varepsilon_1 v) + \cdots + \ell_n E(\varepsilon_n u, \varepsilon_n v).$$

Because of this and $i(bu) = b(iu)$ for all $u \in \mathbf{C}^g$, we have

$$L(bu, bv) = \ell_0 L(u,v) + \ell_1 L(\varepsilon_1 u, \varepsilon_1 v) + \cdots + \ell_n L(\varepsilon_n u, \varepsilon_n v).$$

Hence

$$\begin{aligned}\sigma(b(u+\ell)) =& \sigma(bu+b\ell)\\ =& \chi(b\ell)\sigma(bu)\exp[L(b(u+\frac{1}{2}\ell), b\ell)]\\ =& \chi(b\ell)\sigma(bu)\exp[\ell_0 L(u+\frac{1}{2}\ell, \ell)\exp[\ell_1 L(\varepsilon_1(u+\frac{1}{2}\ell), \varepsilon_1\ell)]\\ & \cdots\exp[\ell_n L(\varepsilon_n(u+\frac{1}{2}\ell), \varepsilon_n\ell)]\end{aligned}$$

by 3.1.1. On the other hand, we have

$$\sigma(\varepsilon_j(u+\ell)) = \chi(\varepsilon\ell)\sigma(\varepsilon_j u)\exp[L(\varepsilon_j(u+\frac{1}{2}\ell), \varepsilon_j\ell)]$$

for $j = 1, \cdots, n$ by 3.1.1. Since $\chi(\lambda) = \pm 1$ for $\lambda \in \Lambda$, we get $\phi_b(u+\ell) = \pm\phi_b(u)$ for all $u \in \mathbf{C}^g$ and $\ell \in \Lambda$. As $\phi_b(u+\ell)/\phi_b(u)$ is a meromorphic function, the signature $\pm$ must be determined by $\ell$. Now we assume that $C$ is not of 2-primary type or the curve defined by $y^2 = x^5 - x$. Then 4.3.1 and 4.3.3 imply that the divisor of $\phi_b(u)$ is the pull-back of a divisor of a function with respect to the map $\kappa : \mathbf{C}^g \to \mathbf{C}^g/\Lambda$. Thus we can write $\phi_b(u) = f(u)e(u)$, where $f(u)$ is periodic with the periods $\Lambda$ and $e(u)$ is a trivial theta function with respect to the lattice $\Lambda$ (see [**10**, p.82]). Then we have

$$e(u+\ell) = \pm e(u).$$

As in the proof of 4.2.5, if $M$ is the maximum of $e(u)$ on the domain

$$\left\{ u = \omega' \begin{bmatrix} u_1' \\ \vdots \\ u_g' \end{bmatrix} + \omega'' \begin{bmatrix} u_1'' \\ \vdots \\ u_g'' \end{bmatrix} ; 0 \le u_j' \le 1, 0 \le u_j'' \le 1 \text{ for } j = 1, \cdots g \right\},$$

then $e(u) \le M$ for all $u \in \mathbf{C}^g$. Thus Liouville's theorem says that $e(u)$ is a constant function. Hence the signature must be $+$. So we have completed the proof. □

Since the function $\phi_b(u)$ has poles along the pull-back of $\Theta$, we modify it as in [**9**].

Definition-Proposition 4.4.2. *Let $b \in \mathbf{Z}[[\zeta]]$. Let*

$$\psi_b(u) = \frac{\sigma(bu)}{\sigma(u)^{\ell_0}} \text{ if } g = 1 \text{ and}$$
$$\psi_b(u) = \frac{\sigma(bu)}{\sigma_2(u)^{\ell_0}\sigma(\varepsilon_1 u)^{\ell_1}\cdots\sigma(\varepsilon_n u)^{\ell_n}} \text{ if } g = 2 \text{ or } 3,$$

*under the same situation of 4.3.1 or 4.3.2. Then $\psi_b(u+\ell) = \pm\psi_b(u)$ for all $u \in \kappa^{-1}\iota(C)$ and $\ell \in \Lambda$. Here the signature $\pm$ is independent of $u$. Moreover, if $C$ is of genus 1 or not of 2-primary type except the curve defined by $y^2 = x^5 - x$, then*

$$\psi_b(u+\ell) = \psi_b(u)$$

*for all $u \in \kappa^{-1}\iota(C)$ and $\ell \in \Lambda$.*

*Proof.* The proof can be given by a similar fashion as in 3.2.1 by looking at the parity of $\ell_0$. □

Remark 4.4.3. *In the rest of this paper we treat only the case $b^{1+\rho} = \ell_0 \in \mathbf{Z}$. So we need not choose $\{\varepsilon_j\}$ explicitly. We see that, in this case, $\psi_b(u)$ is a polynomial of $x(u)$ or a such polynomial multiplied by $y(u)$ by 3.2.2.*

## II. Complex Multiplication Formulae

We mention here conventions for the following three Sections. We freely use the notation of the part I. Let $\varphi(u)$ be an element of the ring

$$\mathbf{Q}[\wp_{ij}(u), \wp_{ijk}(u), \wp_{ijk\ell}(u) \big| i, j, k, \ell, \cdots = 1, \cdots, g].$$

Let $b \in \mathbf{Z}[\lceil\zeta\rceil]$. Then 1.4.1 and 1.6.2 shows that $\varphi(u)|_{u\in\iota(C)}$ can be expressed in the form

$$\varphi(bu)|_{u\in\iota(C)} = \frac{P(x(u), y(u))}{Q(x(u), y(u))},$$

where $P(X,Y)$ and $Q(X,Y) \in \mathbf{Q}(\zeta)[X,Y]$. Especially we have shown the statements about the coefficients in Theorems 5.1.3, 5.2.3, 6.1.6, and 7.1.6 below.

From now on we assume $C$ is a curve of cyclotomic type. We fix a special point $P_0$ such that $x(P_0) = 0$: if $C$ is defined by the affine equation $y^2 = x^{2g+1} + \frac{1}{4}$ then $P_0$ is the point $(0, \frac{1}{2})$, if $C$ is defined by $y^2 = x^{2g+1} - x$ then without saying $P_0$ is the point $(0,0)$.

Suppose we have labeled the roots of $f(x)$ as in (1.1.1). Such labels are described in the begining of each Subsection below. By applying the argument of our proof of 4.2.3, with the same notation, for the integrals along the paths $\alpha^{(1)}$ and $\beta^{(1)}$, we can write the entries of $\omega'$, $\omega''$, $\eta'$, and $\eta''$ by $K_j$'s and $H_j$'s.

We choose and fix a point in $\mathbf{C}^g$ whose image of the map $\kappa : \mathbf{C} \to \mathbf{C}^g/\Lambda = J$ is $P_0$. We denote such a point also by $P_0$. Throughout Sections 5, 6, and 7 such a point is assumed to be given by taking the integral (2.3.1) along the line on which the $x$-coordinate is real negative (resp. positive) and the $y$-coordinate has negative imaginary part or is real positive if the curve $C$ is defined by $y^2 = x^{2g+1} + \frac{1}{4}$ (resp. $y^2 = x^{2g+1} - x$). Then the coordinates of $\lceil\zeta^j\rceil P_0$'s can be written explicitely, as we describe in each of following Subections, in the form $\lceil\zeta^j\rceil P_0 = \omega' u' + \omega'' u''$ by taking care that the integral from $\infty$ to $(-4^{1/2g+1}, 0)$ (resp. to $(1,0)$) along negative (resp. positive) part of the real axis of $x$ is half of the one along $\alpha^{(1)}$.

In the Sections, we give explicitly the highest and lowest term of $P(X,Y)$ for each of special functions $\varphi(u)$.

In Section 5 we write $u_1$ as $u$, $K_1$ as $K$, and $H_1$ as $H$.

## §5. Elliptic curves of cyclotomic type

**5.1. The curve defined by $y^2 = x^3 + \frac{1}{4}$.**
We here give a version of the product formula of Eisenstein (see Section 8) for the curve $C$ defined by $y^2 = x^3 + \frac{1}{4}$. According to 4.1.1(1) the ring $\mathbf{Z}[\lceil\zeta\rceil]$ is isomorphic to the ring $\mathbf{Z}[\zeta]$ by $\lceil\zeta\rceil \mapsto \zeta$. So we may identify $\mathbf{Z}[\lceil\zeta\rceil]$ and $\mathbf{Z}[\zeta]$.

We let $c = -4^{-\frac{1}{3}}$, $a_1 = -4^{-\frac{1}{3}}\zeta$, $c_1 = -4^{-\frac{1}{3}}\zeta^2$ in (1.1.1). Then we have

$$\omega' = 2K(\zeta - \zeta^2),\ \omega'' = 2K(\zeta - 1),\ \eta' = 2H(\zeta^2 - \zeta),\ \eta'' = 2H(\zeta^2 - 1),$$

and

$$P_0 = K(-\zeta^2 + \zeta) - K = \frac{1}{3}\omega' + \frac{1}{3}\omega''. \tag{5.1.1}$$

Proposition 5.1.1. *$\sigma(P_0)^3 = -\exp\frac{3}{2}L(P_0, P_0)$.*

*Proof.* Because of $y(P_0) = \frac{1}{2}$, it is obtained from 3.2.4(1) and (2) that $\sigma(2P_0) = -\sigma(P_0)^4$, On the other hand, from 3.1.1, we get

$$\sigma(2P_0) = \sigma(-P_0 + 3P_0) = -\exp[\frac{3}{2}L(P_0, P_0)]\sigma(-P_0) = \exp[\frac{3}{2}L(P_0, P_0)]\sigma(P_0).$$

Here we used that $\sigma(-u) = -\sigma(u)$ and that $\chi(3P_0) = -1$ which is calculated by (5.1.1). Hence the statement. $\square$

Proposition 5.1.2. *Let $b$ be an element of $\mathbf{Z}[\zeta]$. If $b \equiv 1 \mod (1-\zeta)^2$, then*

$$\sigma(b(v + P_0)) = (-1)^{(\mathrm{N}b-1)/3}\chi((b-1)P_0)\sigma(P_0)^{\mathrm{N}b-1}\sigma(bv + P_0)(1 + (d^\circ \geq 1)).$$

*Proof.* Since $\mathrm{N}b - 1 \equiv 0 \mod 3$, the statement follows from 4.2.8(1) and

$$\exp[\frac{1}{2}(\mathrm{N}b - 1)L(P_0, P_0)] = \sigma(P_0)^{\mathrm{N}b-1}$$

which is a result of 5.1.1. $\square$

Theorem 5.1.3. (Eisenstein) *Let $b \in \mathbf{Z}[\zeta]$ and assume $b \equiv 1 \mod (1-\zeta)^2$. Then $\psi_b(u)^2\wp(bu)$ is of the form*

$$\psi_b(u)^2\wp(bu) = x(u)\sum_{\substack{0\leq j\leq \mathrm{N}b-1 \\ j\equiv 0 \mod 3}} \gamma_j x(u)^j$$

*with $\gamma_j \in \mathbf{Q}(\zeta)$. Moreover $\gamma_0 = b$ and $\gamma_{\mathrm{N}b-1} = 1$.*

*Proof.* At First, we look at the Laurent expansion at $u = O$. By 2.1.1(1), we have

$$\begin{aligned}\wp(bu)(\psi_b(u))^2 =& \frac{\sigma'(bu)^2 - \sigma''(bu)\sigma(bu)}{\sigma(u)^{2\mathrm{N}b}} \\ =& \frac{(1+\cdots)^2 - (d^\circ \geq 0)(bu + \cdots)}{(u+\cdots)^{2\mathrm{N}b}} \\ =& \frac{1}{u^{2\mathrm{N}b}} + \cdots .\end{aligned}$$

Since $\sigma(u)$ is an odd function and has only zeros at $u \in \Lambda$ by 2.2.1(1), we know that $\wp(bu)\psi_b(u)^2$ is a polynomial of $x(u)$. Thus we have $\wp(bu)\psi_b(u)^2 = x(u)^{\mathrm{N}b} + \cdots$ by 2.3.1. Secondly, we look at the Laurent expansion at $u = P_0$. Since $b - 1 \equiv 0 \mod (1-\zeta)^2$,

$$
\begin{aligned}
&\wp(b(v+P_0))(\psi_b(v+P_0))^2/\wp(v+P_0) \\
=&\frac{\sigma(b(v+P_0))^2}{\sigma(v+P_0)^{2\mathrm{N}b}} \cdot \frac{\wp(b(v+P_0))}{\wp(v+P_0)} \\
=&\frac{(-1)^{2(\mathrm{N}b-1)/3}\chi((b-1)P_0)^2\sigma(P_0)^{2\mathrm{N}b-2}\sigma(bv+P_0)^2}{\sigma(v+P_0)^{2\mathrm{N}b-2}\sigma(v+P_0)^2} \cdot \frac{\wp(bv+P_0)}{\wp(v+P_0)} + (d^\circ \geq 1) \\
&\text{(by using 5.1.2)} \\
=&\frac{\sigma(bv+P_0)^2}{\sigma(v+P_0)^2} \cdot \frac{b\wp'(bv+P_0)}{\wp'(v+P_0)} + (d^\circ \geq 1) \quad (\text{ since } \sigma(P_0) \neq 0 \text{ }) \\
=&b + (d^\circ \geq 1) \quad (\text{ since } \wp'(bP_0) = \wp'(P_0) \neq 0).
\end{aligned}
$$

Because 2.2.1(1) states the function $\psi_b(u)^2\wp(bu)$ has only pole at $u = O$ the coefficient of the lowest term must be $b$. Since $\psi_b(-\zeta u)^2\wp(-\zeta bu) = \zeta^{\mathrm{N}b}\psi_b(u)^2\wp(bu)$ because of 4.2.5(1), the function must be a polynomial of $x(u)^3$ multiplied by $x(u)$. $\square$

**5.2. The curve defined by $y^2 = x^3 - x$.**

Here we assume that the curve $C$ is defined by $y^2 = x^3 - x$. For this curve the ring $\mathbf{Z}[\lceil i \rceil]$ is also isomorphic to the ring $\mathbf{Z}[i]$ by $\lceil i \rceil \mapsto i$. So we identify $\mathbf{Z}[\lceil i \rceil]$ and $\mathbf{Z}[i]$. In this subsection, we write $u_1$ as $u$. We let $c = 1$, $a_1 = 0$, and $c_1 = -1$, in the notation of 1.1.1. The argument in the proof of 4.2.3 applied for the integrals along the paths $\alpha^{(1)}$ and $\beta^{(1)}$ gives

$$\omega' = 2K,\ \omega'' = 2Ki,\ \eta' = 2H,\ \eta'' = -2Hi.$$

As in the previous subsection we take a point in $\mathbf{C}$ whose image of the map $\kappa : \mathbf{C} \to \mathbf{C}/\Lambda = J = C$ is $P_0$ and denote it by $P_0$.

Similar path as in (5.1.1) gives

$$P_0 = iK - K = -\frac{1}{2}\omega' + \frac{1}{2}\omega''. \tag{5.2.1}$$

Proposition 5.2.1. $\sigma(P_0)^4 = \exp[2L(P_0, P_0)]$.

*Proof.* After differentiating the formula of 3.2.4(1), by setting $u = P_0$, we have $-2\sigma(P_0)^4 = 2\sigma'(2P_0)$. On the other hand, we get $\sigma(u+2P_0) = \chi(2P_0)\sigma(u)\exp[L(u+P_0, 2P_0)]$ from 3.1.1. After differentiating this, by setting $u = 0$, we have $\sigma'(2P_0) = -\exp(2L(P_0, P_0))$ because of $\sigma'(0) = 1$ and $\sigma(0) = 0$. Here we have used the fact $\chi(2P_0) = -1$ which is obtained by (5.2.1). Hence $\sigma(P_0)^4 = \exp[2L(P_0, P_0)]$. $\square$

Proposition 5.2.2. *Let $b$ be an element of $\mathbf{Z}[i]$. If $b \equiv 1 \mod 4$, then*

$$\sigma(b(v + P_0)) = \chi((b-1)P_0)\sigma(P_0)^{\mathrm{N}b-1}\sigma(bv + P_0)(1 + (d^\circ \geq 1)).$$

*Proof.* Since $b \equiv 1 \mod 4$, we have $\mathrm{N}b \equiv 1 \mod 4$. The statement follows from 4.2.8(1) and

$$\exp[\frac{1}{2}(\mathrm{N}b-1)L(P_0, P_0)] = \sigma(P_0)^{\mathrm{N}b-1}$$

which is given by 5.2.1. $\square$

Theorem 5.2.3. (Eisenstein) *Let $b \in \mathbf{Z}[i]$ and assume $b \equiv 1 \mod 4$. Then $\psi_b(u)^2\wp(bu)$ is of the form*

$$\psi_b(u)^2\wp(bu) = x(u)\sum_{\substack{0 \leq j \leq \mathrm{N}b-1 \\ j \equiv 0 \mod 2}} \gamma_j x(u)^j$$

*with $\gamma_j \in \mathbf{Q}(i)$. Moreover $\gamma_0 = b^2$ and $\gamma_{\mathrm{N}b-1} = 1$.*

*Proof.* As in the proof of 5.1.3 we have that $\wp(bu)\psi_b(u)^2 = \dfrac{1}{u^{2\mathrm{N}b}} + \cdots$, that $\wp(bu)\psi_b(u)^2$ is a polynomial of $x(u)$ with coefficients in $\mathbf{Q}(i)$, and that $\wp(bu)\psi_b(u)^2 =$

$x(u)^{\mathrm{N}b} + \cdots$. For the Laurent expansion at $u = P_0$, since $b - 1 \equiv 0 \mod 4$ and $\wp(u)$ has a double order zero at $P_0$,

$$
\begin{aligned}
&\wp(b(v + P_0))(\psi_b(v + P_0))^2/\wp(v + P_0)\\
=&\frac{\sigma(b(v + P_0))^2}{\sigma(v + P_0)^{2\mathrm{N}b}} \cdot \frac{\wp(b(v + P_0))}{\wp(v + P_0)}\\
=&\frac{\chi((b-1)P_0)^2\sigma(P_0)^{2\mathrm{N}b-2}\sigma(bv + P_0)^2}{\sigma(v + P_0)^{2\mathrm{N}b-2}\sigma(v + P_0)^2} \cdot \frac{\wp(bv + P_0)}{\wp(v + P_0)} + (d^\circ \geq 1) \quad \text{(by 5.2.2)}\\
=&\frac{\sigma(bv + P_0)^2}{\sigma(v + P_0)^2} \cdot \frac{b^2\wp''(bv + P_0)}{\wp''(v + P_0)} + (d^\circ \geq 1) \quad (\text{ since } \sigma(P_0) \neq 0\ )\\
=&b^2 + (d^\circ \geq 1) \quad (\text{ since } \wp'(bP_0) = \wp'(P_0) \neq 0).
\end{aligned}
$$

Since $\psi_b(iu)^2\wp(ibu) = (-1)^{\mathrm{N}b}\psi_b(u)^2\wp(bu)$ because of 4.2.5(1), the function must be a polynomial of $x(u)^2$ multiplied by $x(u)$. $\square$

## §6. Genus two curves of cyclotomic type

### 6.1. The curve defined by $y^2 = x^5 + \frac{1}{4}$.

Now let us consider Grant's original case. So the curve $C$ is defined by $y^2 = x^5 + \frac{1}{4}$. According to the isomorphism of 4.1.1(1) the ring $\mathbf{Z}[\lceil\zeta\rceil]$ can be identified with $\mathbf{Z}[\zeta]$ by $\lceil\zeta\rceil \mapsto \zeta$. The endomorphism $\lceil -\zeta^j \rceil$ on $\mathbf{C}^2$ is described as

$$\lceil -\zeta^j \rceil(u_1, u_2) = (-\zeta^j u_1, -\zeta^{2j} u_2). \tag{6.1.1}$$

We let $c = -4^{-\frac{1}{5}}$, $a_1 = -4^{-\frac{1}{5}}\zeta$, $c_1 = -4^{-\frac{1}{5}}\zeta^2$, $a_2 = -4^{-\frac{1}{5}}\zeta^3$, $c_2 = -4^{-\frac{1}{5}}\zeta^4$, in (1.1.1). Then we have

$$\begin{aligned}
\omega' &= \begin{bmatrix} 2K_1(\zeta^3 - \zeta^4) & 2K_1(\zeta - \zeta^2) \\ 2K_2(\zeta - \zeta^3) & 2K_2(\zeta^2 - \zeta^4) \end{bmatrix}, \\
\omega'' &= \begin{bmatrix} 2K_1(-1 + \zeta - \zeta^2 + \zeta^3) & 2K_1(\zeta - 1) \\ 2K_2(-1 + \zeta^2 - \zeta^4 + \zeta) & 2K_2(\zeta^2 - 1) \end{bmatrix}, \\
\eta' &= \begin{bmatrix} 2H_1(\zeta^2 - \zeta) & 2H_1(\zeta^4 - \zeta^3) \\ 2H_2(\zeta^4 - \zeta^2) & 2H_2(\zeta^3 - \zeta) \end{bmatrix}, \\
\eta'' &= \begin{bmatrix} 2H_1(-1 + \zeta^4 - \zeta^3 + \zeta^2) & 2H_1(\zeta^4 - 1) \\ 2H_2(-1 + \zeta^3 - \zeta + \zeta^4) & 2H_2(\zeta^3 - 1) \end{bmatrix}.
\end{aligned}$$

The point $P_0$ is

$$P_0 = \begin{bmatrix} K_1(\zeta - \zeta^2 + \zeta^3 - \zeta^4) - K_1 \\ K_2(\zeta^2 - \zeta^4 + \zeta - \zeta^3) - K_2 \end{bmatrix} = \omega' \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \end{bmatrix} + \omega'' \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}. \tag{6.1.2}$$

Then $\lceil\zeta^j\rceil P_0$ are given by (6.1.1) as follows:

$$\begin{aligned}
\lceil\zeta\rceil P_0 &= \begin{bmatrix} \zeta K_1(\zeta - \zeta^2 + \zeta^3 - \zeta^4 - 1) \\ \zeta^2 K_2(\zeta^2 - \zeta^4 + \zeta - \zeta^3 - 1) \end{bmatrix} = \omega' \begin{bmatrix} -\frac{3}{5} \\ -\frac{4}{5} \end{bmatrix} + \omega'' \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}, \\
\lceil\zeta^2\rceil P_0 &= \begin{bmatrix} \zeta^2 K_1(\zeta - \zeta^2 + \zeta^3 - \zeta^4 - 1) \\ \zeta^4 K_2(\zeta^2 - \zeta^4 + \zeta - \zeta^3 - 1) \end{bmatrix} = \omega' \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \end{bmatrix} + \omega'' \begin{bmatrix} \frac{1}{5} \\ -\frac{4}{5} \end{bmatrix}, \\
\lceil\zeta^3\rceil P_0 &= \begin{bmatrix} \zeta^3 K_1(\zeta - \zeta^2 + \zeta^3 - \zeta^4 - 1) \\ \zeta K_2(\zeta^2 - \zeta^4 + \zeta - \zeta^3 - 1) \end{bmatrix} = \omega' \begin{bmatrix} -\frac{3}{5} \\ \frac{1}{5} \end{bmatrix} + \omega'' \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}, \\
\lceil\zeta^4\rceil P_0 &= \begin{bmatrix} \zeta^4 K_1(\zeta - \zeta^2 + \zeta^3 - \zeta^4 - 1) \\ \zeta^3 K_2(\zeta^2 - \zeta^4 + \zeta - \zeta^3 - 1) \end{bmatrix} = \omega' \begin{bmatrix} \frac{2}{5} \\ -\frac{4}{5} \end{bmatrix} + \omega'' \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}.
\end{aligned} \tag{6.1.3}$$

Now let us compute the Taylor expansion at $u = P_0$ explicitly. Since

$$(\lceil\zeta\rceil - 1)P_0 = \omega' \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \omega'' \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.1.4}$$

by (6.1.2) and (6.1.3), we have $\chi((\lceil\zeta\rceil - 1)P_0) = 1$. After substituting this to 4.2.8(2) and differentiating it by $v_2$, by setting $v = O$, we have

$$\sigma_2(\lceil\zeta\rceil P_0) = \sigma_2(P_0) \exp \frac{1}{2} L((\overline{\lceil\zeta\rceil} - \lceil\zeta\rceil)P_0, P_0),$$

where we have used that $\sigma(P_0) = 0$. Because of 4.2.5(1) and $\sigma_2(P_0) \neq 0$ (see 2.2.1(2)), it must be

$$\exp \frac{1}{2} L((\overline{\lceil\zeta\rceil} - \lceil\zeta\rceil)P_0, P_0) = \zeta^4.$$

Therefore 4.2.8(2) gives rise to

$$\sigma(v + \lceil\zeta\rceil P_0) = \zeta^4 \sigma(v + P_0) \exp[L(v, (\lceil\zeta\rceil - 1)P_0)]. \tag{6.1.5}$$

After operating $\frac{\partial^2}{\partial u_i \partial u_j}$ to (6.1.5), by setting $v = O$, we have

$$\sigma_{ij}(\lceil\zeta\rceil P_0) = \zeta^4 \sigma_{ij}(P_0) + \sigma_i(P_0)(-\eta'_{1j} - \eta'_{2j})\zeta^4 + \sigma_j(P_0)(-\eta'_{1i} - \eta'_{2i})\zeta^4 \tag{6.1.6}$$

by (6.1.4). For the case $i = j = 1$, (6.1.6) is of no use because $\sigma_1(P_0) = 0$. But 2.2.1 gives $\sigma_{11}(P_0) = 2\sqrt{\lambda_0}\sigma_2(P_0) = \sigma_2(P_0)$. Set $i = 1$ and $j = 2$ in (6.1.6), then $\sigma_{12}(P_0) = 2H_1(\zeta^2 + \zeta^4)\sigma_2(P_0)$. By a similar fashion, we get $\sigma_{22}(P_0) = 4H_2(\zeta^4 + \zeta^3)\sigma_2(P_0)$ by setting $i = j = 2$ in (6.1.6). Although these explicit values are unnecessary to prove 6.1.6 below, we mention this here to make 6.1.1 below clean. The Taylor expansion at $O$ is given by 2.1.1(2). Thus we have arrived at

PROPOSITION 6.1.1. *Assume $C$ be defined by $y^2 = x^5 + \frac{1}{4}$. Let $P_0$ be the point whose coordinate is given by* (6.1.2). *Then*

$$\begin{aligned}
&(1) && \sigma(u) = u_1 - \frac{1}{3}u_2^3 + (d^\circ \geq 5),\\
&(2) && \sigma(v + P_0) = \sigma_2(P_0)\big(v_2 + \frac{1}{2}v_1^2 + \gamma_{12}v_1v_2 + \frac{\gamma_{22}}{2}v_2^2 + \frac{\gamma_{12}}{2}v_1^3\\
&&& \quad + (\frac{\gamma_{22}}{4} + \frac{\gamma_{12}^2}{2})v_1^2v_2 + \frac{\gamma_{12}\gamma_{22}}{2}v_1v_2^2 + \frac{\gamma_{22}^2}{8}v_2^3 + (d^\circ \geq 3)\big),
\end{aligned}$$

*where* $\gamma_{12} = 2H_1(\zeta^2 + \zeta^4)$ *and* $\gamma_{22} = 4H_2(\zeta^3 + \zeta^4)$.

PROPOSITION 6.1.2. $\sigma_2(P_0)^5 = \exp \frac{5}{2} L(P_0, P_0)$.

*Proof.* Because of $y(P_0) = \frac{1}{2}$, it is obtained from 3.2.4(1) and (2) that

$$\sigma(2P_0) = \sigma_2(P_0)^4, \ \sigma(3P_0) = \sigma_2(P_0)^9.$$

On the other hand, from 3.1.1, we get

$$\sigma(3P_0) = \sigma(-2P_0 + 5P_0) = -\exp[\frac{5}{2}L(P_0, P_0)]\sigma(2P_0).$$

Here we used that $\sigma(-u) = -\sigma(u)$ and that $\chi(5P_0) = 1$ which is given by (6.1.2). Therefore we obtain

$$-\sigma_2(P_0)^9 = -\exp[\frac{5}{2}L(P_0, P_0)]\sigma_2(P_0)^4$$

and the statement. □

We denote by $\tau$ the element of $\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$ such that $\zeta^\tau = \zeta^2$. Then $1 + \tau$ is a type norm (see 4.2.7) in $\mathbf{Z}[\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})]$.

LEMMA 6.1.3. *If* $b \in \mathbf{Z}[\zeta]$ *and* $b \equiv 1 \mod (1-\zeta)$, *then* $\chi((b^{1+\tau^{-1}}-1)P_0)^{\mathrm{N}b-1} = 1$.

*Proof.* If N$b$ is odd, the statement is trivial. So we assume N$b$ is even. For $\ell \in \Lambda$, it is easily verified from the definition that the value $\chi(\ell)$ is determined only by $\ell \mod 2\Lambda$. By the assumption of $b$, we may write $b^{1+\tau^{-1}} = (a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4)(1-\zeta) + 1$. Since 2 is a prime in $\mathbf{Z}[\zeta]$, we have $b \equiv 0 \mod 2$ and hence $b^{1+\tau^{-1}} \equiv 0 \mod 2$. By simple calculation, we see that $a_1 \equiv a_3 \equiv 1 \mod 2$ and $a_2 \equiv a_4 \equiv 0 \mod 2$. Therefore

$$\begin{aligned}\chi((b^{1+\tau^{-1}}-1)P_0) =&\chi((\zeta+\zeta^3)(1-\zeta)P_0)\\ =&\chi((\zeta-\zeta^2+\zeta^3-\zeta^4)P_0)\\ =&1\end{aligned}$$

because of

$$(\zeta-\zeta^2+\zeta^3-\zeta^4)P_0 = \omega' \begin{bmatrix}-2\\-1\end{bmatrix} + \omega'' \begin{bmatrix}1\\1\end{bmatrix}$$

which is obtained from (6.1.3). □

PROPOSITION 6.1.4. *Let $b$ be an element of* $\mathbf{Z}[\zeta]$. *If* $b \equiv 1 \mod (1-\zeta)^2$, *then*

$$\sigma(b^{1+\tau^{-1}}(v+P_0)) = \sigma_2(P_0)^{\mathrm{N}b-1}\chi((b^{1+\tau^{-1}}-1)P_0)\sigma(b^{1+\tau^{-1}}v+P_0)(1+(d^\circ \geq 1)).$$

*Proof.* The statement follows from 4.2.8(1) and

$$\exp[\frac{1}{2}(\mathrm{N}b-1)L(P_0,P_0)] = \sigma_2(P_0)^{\mathrm{N}b-1}$$

which is given by 6.1.2. □

LEMMA 6.1.5. *Let $\varphi(u)$ denote the function $(\wp_{12}^2 - \wp_{22}\wp_{11})(u)$. Then it has the following properties.*
(1) $\varphi([\zeta]u) = \zeta^4\varphi(u)$,
(2) $\varphi(u) \in \Gamma(J, \mathcal{O}(3\Theta))$,
(3) *the Taylor expansions of $\sigma(u)^3\varphi(u)$ at $O$ and $P_0$ are of the form*

$$\begin{aligned}\sigma(u)^3\varphi(u) &= 2u_2 + (d^\circ(u_1,u_2) \geq 2) \text{ and}\\ \sigma(v+P_0)^3\varphi(v+P_0) &= \sigma_2(P_0)^3(-1+(d^\circ(v_1,v_2)\geq 1)).\end{aligned}$$

*Proof.* The statement (1) follows from 4.2.5 and the definition of $\wp$-functions. The statement (2) follows from

$$(\sigma^3\varphi)(u) = -\sigma_2(u)^2\sigma_{11}(u) - \sigma_1(u)^2\sigma_{22}(u) - 2\sigma_1(u)\sigma_2(u)\sigma_{12}(u) + \sigma_{12}(u)^2\sigma(u).$$

The statement (3) is easily derived from a calculation by using the equation above and 6.1.1. □

THEOREM 6.1.6. (Grant[**9**]) *Let* $\varphi(u) := (\wp_{12}^2 - \wp_{22}\wp_{11})(u)$. *Let* $b \in \mathbf{Z}[\zeta]$ *and assume* $b \equiv 1 \mod (1-\zeta)^2$. *Then* $\psi_{b^{1+\tau^{-1}}}(u)^3\varphi(b^{1+\tau^{-1}}u)$ *is of the form*

$$\psi_{b^{1+\tau^{-1}}}(u)^3\varphi(b^{1+\tau^{-1}}u) = 2y(u) \sum_{\substack{0 \le j \le 3(\mathrm{N}b-1) \\ j \equiv 0 \mod 5}} \gamma_j x(u)^j$$

*with every* $\gamma_j \in \mathbf{Q}(\zeta)$. *Moreover* $\gamma_{3(\mathrm{N}b-1)} = (-1)^{\mathrm{N}b} b^{1+\tau}$ *and* $\gamma_0 = -1$.

*Proof.* At first, we look at the Laurent expansion at $u = O$. By 6.1.5(3) and 6.1.1(1), we have

$$\begin{aligned}
&\psi_{b^{1+\tau^{-1}}}(u)^3\varphi(b^{1+\tau^{-1}}u)\Big|_{u\in\kappa^{-1}\iota(C)} \\
&= \frac{\sigma(b^{1+\tau^{-1}}u)^3\varphi(b^{1+\tau^{-1}}u)}{\sigma_2(u)^{3\mathrm{N}b}} \\
&= \frac{2(b^{1+\tau^{-1}})^\tau u_2 + (d^\circ(u_2) \ge 2)}{(-u_2^2 + (d^\circ(u_2) \ge 4))^{3\mathrm{N}b}} \\
&= (-1)^{\mathrm{N}b} 2b^{\tau+1} \frac{1}{u_2^{6\mathrm{N}b-1}} + \cdots \\
&= (-1)^{\mathrm{N}b} 2b^{\tau+1} \frac{-1}{u_2^5}\left(\frac{1}{u_2^2}\right)^{3(\mathrm{N}b-1)} + \cdots \\
&= (-1)^{\mathrm{N}b} 2b^{\tau+1} y(u)(x(u)^{3(\mathrm{N}b-1)} + \text{“lower terms of power of } x(u)^5\text{”}).
\end{aligned}$$

Here we used 2.3.1 and the fact that $\psi_{b^{1+\tau^{-1}}}(u)\varphi(b^{1+\tau^{-1}}u)\Big|_{u\in\kappa^{-1}\iota(C)}$ is a polynomial of $x(u)$ multiplied by $y(u)$, which is deduced from that this function is odd and $\sigma_2$ has only zeroes at $u \in \Lambda$ by the first statement of 2.2.1(2). Secondly, we look at the Laurent expansion at $u = P_0$ ($\kappa(P_0) = \iota(0, \frac{1}{2})$). Since $b \equiv 1 \mod (1-\zeta)^2$ we have $b^{\tau+1} \equiv 1 \mod (1-\zeta)^2$. Because of $(1-\zeta)P_0 \in \Lambda$ and $\varphi(u)$ being periodic, we have $\varphi(b^{1+\tau^{-1}}(v+P_0)) = \varphi(b^{1+\tau^{-1}}v + P_0)$. Consequently, 6.1.4, 6.1.1, 6.1.5 and 6.1.3 imply

$$\begin{aligned}
&\psi_{b^{1+\tau^{-1}}}(v+P_0)^3\varphi(b^{1+\tau^{-1}}(v+P_0))|_{v+P_0\in\kappa^{-1}\iota(C)} \\
=&\frac{\sigma(b^{1+\tau^{-1}}(v+P_0))^3\varphi(b^{1+\tau^{-1}}(v+P_0))}{\sigma_2(b^{1+\tau^{-1}}(v+P_0))^{3\mathrm{N}b}}\Big|_{v+P_0\in\kappa^{-1}\iota(C)} \\
=&\frac{\sigma_2(P_0)^{3(\mathrm{N}b-1)}\sigma(b^{1+\tau^{-1}}v+P_0)^3\chi((b^{1+\tau^{-1}}-1)P_0)^3(1+(d^\circ(v_2)\ge 1))\varphi(b^{1+\tau^{-1}}v+P_0)}{[\sigma_2(b^{1+\tau^{-1}}v+P_0)\chi((b^{1+\tau^{-1}}-1)P_0)(1+(d^\circ(v_2)\ge 1))]^{3\mathrm{N}b}} \\
=&\frac{\sigma_2(P_0)^{3(\mathrm{N}b-1)}\sigma_2(P_0)^3(-1+(d^\circ(v_1)\ge 1))}{\sigma_2(b^{1+\tau^{-1}}v+P_0)^{3\mathrm{N}b}}\chi((b^{1+\tau^{-1}}-1)P_0)^{3(1-\mathrm{N}b)} \\
=&-1+(d^\circ(v_1)\ge 1) \\
=&-2y(u)(1+(d^\circ(x(u))\ge 2)).
\end{aligned}$$

Furthermore, since $\psi_{b^{1+\tau^{-1}}}([-\zeta]u)^3\varphi(b^{1+\tau^{-1}}[-\zeta]u) = -\zeta^{2\mathrm{N}b-2}\psi_{b^{1+\tau^{-1}}}(u)^3\varphi(b^{1+\tau-1}u)$ by 4.2.5(1), the function must be a polynomial of $x(u)^5$ multiplied by $y(u)$. $\square$

**6.2. The curve defined by $y^2 = x^5 - x$.**
We treat here the other genus two curve $C$ defined by $y^2 = x^5 - x$. The ring $\mathbf{Z}[\lceil\zeta\rceil]$ can also be identified with $\mathbf{Z}[\zeta]$ by 4.1.1(2). The endomorphism $\lceil -\zeta^j \rceil$ acts as

$$\lceil -\zeta^j \rceil(u_1, u_2) = (-\zeta^j u_1, -\zeta^{3j} u_2)$$

because $\lceil\zeta\rceil\omega^{(1)} = \zeta\omega^{(1)}$ and $\lceil\zeta\rceil\omega^{(2)} = \zeta^2\omega^{(2)}$. We let $c = 1$, $a_1 = i$, $c_1 = -1$, $a_2 = -i$, $c_2 = 0$, in (1.1.1). In this case

$$\begin{aligned}
\omega' &= \begin{bmatrix} -2K_1\zeta^3 & 2K_1(\zeta^2 - \zeta) \\ -2K_2\zeta & 2K_2(\zeta^6 - \zeta^3) \end{bmatrix}, \\
\omega'' &= \begin{bmatrix} 2K_1(\zeta^2 - \zeta + 1) & 2K_1(-\zeta + 1) \\ 2K_2(\zeta^6 - \zeta^3 + 1) & 2K_2(-\zeta^3 + 1) \end{bmatrix}, \\
\eta' &= \begin{bmatrix} -2H_1\zeta^5 & 2H_1(\zeta^6 - \zeta^7) \\ -2H_2\zeta^7 & 2H_2(\zeta^2 - \zeta^5) \end{bmatrix}, \\
\eta'' &= \begin{bmatrix} 2H_1(\zeta^6 - \zeta^7 + 1) & 2H_1(-\zeta^7 + 1) \\ 2H_2(\zeta^2 - \zeta^5 + 1) & 2H_2(-\zeta^5 + 1) \end{bmatrix}.
\end{aligned}$$

Our choice of $P_0$ in $\mathbf{C}^2$ gives

$$P_0 = \begin{bmatrix} K_1(\zeta - \zeta^2 + \zeta^3) - K_1 \\ K_2(\zeta^3 - \zeta^6 + \zeta) - K_2 \end{bmatrix} = \omega' \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} + \omega'' \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}, \tag{6.2.1}$$

and

$$\lceil\zeta\rceil P_0 = \begin{bmatrix} \zeta K_1(\zeta - \zeta^2 + \zeta^3 - 1) \\ \zeta^3 K_2(\zeta^3 - \zeta^6 + \zeta - 1) \end{bmatrix} = \omega' \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + \omega'' \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}. \tag{6.2.2}$$

Taking care of the fact $\chi((\lceil\zeta\rceil - 1)P_0) = 1$ deduced from

$$(\lceil\zeta\rceil - 1)P_0 = \omega' \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \omega'' \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.2.3}$$

which is given by (6.2.1) and (6.2.2), we have by similar argument to Subsection 6.1

$$\sigma(v + \lceil\zeta\rceil P_0) = \zeta^6 \sigma(v + P_0) \exp[L(v, (\lceil\zeta\rceil - 1)P_0)]. \tag{6.2.4}$$

instead of (6.1.5). After operating $\frac{\partial^2}{\partial u_i \partial u_j}$ to (6.2.4), by setting $v = O$, we have

$$\sigma_{ij}(\lceil\zeta\rceil P_0) = \zeta^6 \sigma_{ij}(P_0) + \sigma_i(P_0)(\eta'_{1j} + \eta'_{2j})\zeta^6 + \sigma_j(P_0)(\eta'_{1i} + \eta'_{2i})\zeta^6 \tag{6.2.5}$$

by (6.2.3). Instead of (6.1.6), we here use (6.2.5). Then we have $\sigma_{12}(P_0) = H_1(-1-(\sqrt{2}-1)i)\sigma_2(P_0)$ and $\sigma_{22}(P_0) = 2H_2(-1+\sqrt{2}+i)\sigma_2(P_0)$. From 2.2.1(2) we have $\sigma_{11}(P_0) = 2\sqrt{\lambda_0}\sigma_2(P_0) = 0$. Summing up with 2.1.1(2), we arrived at

Proposition 6.2.1. *Assume $C$ be defined by $y^2 = x^5 - x$. Let $P_0$ be the point whose coordinate is given by* (6.2.1). *Then*

(1) $$\sigma(u) = u_1 - \frac{1}{3}u_2^3 + (d^\circ \geq 5),$$

(2) $$\sigma(v + P_0) = \sigma_2(P_0)(v_2 + \gamma_{12}v_1v_2 + \frac{\gamma_{22}}{2}v_2^2 - \frac{1}{3}v_1^3 + \frac{\gamma_{12}^2}{2}v_1^2v_2 + \frac{\gamma_{12}\gamma_{22}}{2}v_1v_2^2 + \frac{\gamma_{22}^2}{8}v_2^3 + (d^\circ \geq 4)),$$

*where $\gamma_{12} = H_1(-1 - (\sqrt{2} - 1)i)$ and $\gamma_{22} = 2H_2(-1 + \sqrt{2} + i)$.*

Proposition 6.2.2. $\sigma_2(P_0)^4 = \exp 2L(P_0, P_0)$.

*Proof.* Take $y = y(u)$ as a local parameter at $P_0$ along $\kappa^{-1}\iota(C)$. By 3.2.2(1), we have $2y(u)\sigma_2(u)^4 = \sigma(2u)$. After differentiating this with respect to $y$, by setting $u = P_0$, we get $2\sigma_2(P_0)^4 = 2\sigma_1(2P_0)$ because of $y(P_0) = 0$ and $\sigma(2P) = 0$ which is led from the fact $2P_0 \in \Lambda$. Moreover, after differentiating with respect to $u_1$ the equation $\sigma(u + 2P_0) = \chi(2P_0)\sigma(u)\exp L(u + P_0, 2P_0)$, by settting $u = P_0$, we get $\sigma_1(2P_0) = \exp 2L(P_0, P_0)$ because $\sigma(O) = 0$, $\sigma_1(O) = 1$ and $\chi(2P_0) = 1$. Here the last is obtained from (6.2.1) and the definition of $\chi(\ )$. Thus, we have proved the statement. □

We denote by $\tau$ the element of $\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$ such that $\zeta^\tau = \zeta^3$. Then $1 + \tau$ is a type norm (see 4.2.7) in $\mathbf{Z}[\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})]$.

Proposition 6.2.3. *Let $b$ be an element of $\mathbf{Z}[\zeta]$. If $b \equiv 1 \mod 4$, then*

$$\sigma(b^{1+\tau}(v + P_0)) = \sigma_2(P_0)^{\mathrm{N}b-1}\sigma(b^{1+\tau}v + P_0)(1 + (d^\circ \geq 1)).$$

*Proof.* By the assumption, $b^{1+\tau} - 1 \equiv 0 \mod 4$ and hence $(b^{1+\tau} - 1)P_0 \in 2\Lambda$. So $\chi((b^{1+\tau} - 1)P_0) = 1$. Moreover $\mathrm{N}b - 1 \equiv 0 \mod 4$ and $2P_0 \in \Lambda$, the statement follows from 4.2.8(1) and

$$\exp[\frac{1}{2}(\mathrm{N}b - 1)L(P_0, P_0)] = \sigma_2(P_0)^{\mathrm{N}b-1}$$

which is given by 6.2.2. □

Lemma 6.2.4. *Let*

$$\varphi(u) := \left(\frac{1}{8}(\wp_{2222} - 6\wp_{22}^2)\wp_{111} + \frac{1}{4}(\wp_{1112} - 6\wp_{11}\wp_{12})\wp_{222})\right)(u).$$

*Then it has the following properties.*
(1) $\varphi([\zeta]u) = \zeta^3\varphi(u)$,
(2) $\varphi(u) \in \Gamma(J, \mathcal{O}(5\Theta))$,
(3) *the Taylor expansions of $\sigma(u)^5\varphi(u)$ at $O$ and $P_0$ are of the form*

$\sigma(u)^5\varphi(u) = u_2^2 + c_1u_1^2 + c_2u_1u_2 + (d^\circ(u_1, u_2) \geq 4)$ *for some constants $c_1$ and $c_2$,*
$\sigma(v + P_0)^5\varphi(v + P_0) = \sigma_2(P_0)^5(1 + (d^\circ(v_1, v_2) \geq 1))$.

*Proof.* The statement (1) follows from 4.2.5(1) and the definition of $\wp$-functions. Since

$$\begin{aligned}\sigma(u)^2(\wp_{2222}-6\wp_{22}^2)(u) &= (-\sigma_{2222}\sigma+4\sigma_{222}\sigma_2-3\sigma_{22}^2)(u),\\ \sigma(u)^3\wp_{111}(u) &= (-2\sigma_1^3+3\sigma_1\sigma_{11}\sigma-\sigma_{111}\sigma^2)(u),\\ \sigma(u)^2(\wp_{1112}-6\wp_{11}\wp_{12}^2)(u) &= (-\sigma_{1112}\sigma+3\sigma_{112}\sigma_1+\sigma_{111}\sigma_2-3\sigma_{11}\sigma_{12})(u),\\ \sigma(u)^3\wp_{222}(u) &= (-2\sigma_2^3+3\sigma_2\sigma_{22}\sigma-\sigma_{222}\sigma^2)(u)\end{aligned}$$

the statement (2) holds. The expansion in 6.2.1(1) gives

$$\begin{aligned}&(-\sigma_{2222}\sigma+4\sigma_{222}\sigma_2-3\sigma_{22}^2)(u)\\ =&-(d^\circ\geq 1)(u_1+(d^\circ\geq 3))+4(-2+(d^\circ\geq 1))(-u_2^2+(d^\circ\geq 4))-3(-2u_2+(d^\circ\geq 3))^2\\ =&-4u_2^2+c_1'u_1^2+c_2'u_1u_2+(d^\circ\geq 4) \text{ for some constants } c_1' \text{ and } c_2',\\ &(-2\sigma_1^3+3\sigma_1\sigma_{11}\sigma-\sigma_{111}\sigma^2)(u)\\ =&-2(1+(d^\circ\geq 1))^3+3(1+(d^\circ\geq 1))(d^\circ\geq 3)(d^\circ\geq 1)-(d^\circ\geq 3)(d^\circ\geq 1)^2\\ =&-2+(d^\circ\geq 2),\\ &(-\sigma_{1112}\sigma+3\sigma_{112}\sigma_1-3\sigma_{111}\sigma_2-3\sigma_{11}\sigma_{12})(u)\\ =&-(d^\circ\geq 1)(u_1+(d^\circ\geq 3))+3(d^\circ\geq 2)(d^\circ\geq 4)+(d^\circ\geq 2)(d^\circ\geq 2)-3(d^\circ\geq 3)(d^\circ\geq 3)\\ =&(d^\circ\geq 2),\\ &(-2\sigma_2^3+3\sigma_2\sigma_{22}\sigma-\sigma_{222}\sigma^2)(u)\\ =&-2(d^\circ\geq 2)^3+3(d^\circ\geq 2)(d^\circ\geq 1)(d^\circ\geq 1)-(d^\circ\geq 0)(d^\circ\geq 1)^2\\ =&(d^\circ\geq 2).\end{aligned}$$

Therefore

$$\sigma(u)^5\varphi(u)=u_2^2+c_1u_1^2+c_2u_1u_2+(d^\circ\geq 4)$$

for some constants $c_1$ and $c_2$. Similarly, 6.2.1(2) gives

$$\begin{aligned}&(-2\sigma_1^3+3\sigma_1\sigma_{11}\sigma-\sigma_{111}\sigma^2)(v+P_0)\\ =&\sigma_2(P_0)^3[-2(d^\circ\geq 2)^3+3(d^\circ\geq 2)(d^\circ\geq 1)(d^\circ\geq 1)-(d^\circ\geq 0)(d^\circ\geq 1)^2]\\ =&(d^\circ\geq 2),\\ &(-\sigma_{1112}\sigma+3\sigma_{112}\sigma_1+\sigma_{111}\sigma_2-3\sigma_{11}\sigma_{12})(v+P_0)\\ =&\sigma_2(P_0)^2[-(d^\circ\geq 0)(d^\circ\geq 1)+3(d^\circ\geq 0)(d^\circ\geq 2)\\ &+(-2+(d^\circ\geq 1))(1+(d^\circ\geq 1))-3(d^\circ\geq 1)(d^\circ\geq 0)]\\ =&\sigma_2(P_0)^2(-2+(d^\circ\geq 1)),\\ &(-2\sigma_2^3+3\sigma_2\sigma_{22}\sigma-\sigma_{222}\sigma^2)(v+P_0)\\ =&\sigma_2(P_0)^3[-2(1+(d^\circ\geq 1))^3+3(1+(d^\circ\geq 1))(d^\circ\geq 0)(d^\circ\geq 1)-(d^\circ\geq 0)(d^\circ\geq 1)^2\\ =&\sigma_2(P_0)^3(-2+(d^\circ\geq 1)).\end{aligned}$$

Hence

$$\sigma(u)^5\varphi(v+P_0)=\sigma_2(P_0)^5(1+(d^\circ\geq 1)).$$

This is (3). $\square$

THEOREM 6.2.5. *Let $\varphi(u)$ be as in 6.2.4. Let $b \in \mathbf{Z}[\zeta]$ and assume $b \equiv 1 \mod 4$. Then $\psi_{b^{1+\tau}}(u)^5\varphi(b^{1+\tau}u)$ is of the form*

$$\psi_{b^{1+\tau}}(u)^5\varphi(b^{1+\tau}u) = \sum_{\substack{0\le j\le 5\mathrm{N}b-1 \\ j\equiv 0 \mod 4}} \gamma_j x(u)^j$$

*with $\gamma_j \in \mathbf{Q}(\zeta)$. Moreover $\gamma_{5\mathrm{N}b-1} = b^{2(1+\tau)}$ and $\gamma_0 = 1$.*

*Proof.* We look at the Laurent expansion at $u = O$. By 6.2.4(3) and 6.2.1(1), we have

$$\begin{aligned}
\psi_{b^{1+\tau}}(u)\varphi(b^{1+\tau}u)\Big|_{u\in\kappa^{-1}\iota(C)} &= \frac{\sigma(b^{1+\tau}u)^5\varphi(b^{1+\tau}u)}{\sigma_2(u)^{5\mathrm{N}b}} \\
&= \frac{-(b^{1+\tau})^{2\tau}u_2^2 + (d^\circ(u_2)\ge 4)}{(-u_2^2 + (d^\circ(u_2)\ge 4))^{5\mathrm{N}b}} \\
&= b^{2(\tau+1)}\frac{1}{u_2^{10\mathrm{N}b-2}} + \cdots \\
&= b^{2(\tau+1)}\left(\frac{1}{u_2^2}\right)^{5\mathrm{N}b-1} + \cdots \\
&= b^{2(\tau+1)}x(u)^{5\mathrm{N}b-1} + \text{“lower terms of power of } x(u)\text{”}.
\end{aligned}$$

Here we used 2.3.1 and the fact that $\psi_{b^{1+\tau}}(u)\varphi(b^{1+\tau}u)\big|_{u\in\kappa^{-1}\iota(C)}$ is a polynomial of $x(u)$, which is deduced from that this function is even and $\sigma_2$ has only zeroes at $u \in \Lambda$ by the first statement of 2.2.1(2). So we look at the Laurent expansion at $u = P_0$ ($\kappa(P_0) = \iota(0,0)$). Since $b \equiv 1 \mod 4$ we have $b^{\tau+1} \equiv 1 \mod 4$. Hence, because of $(1-\zeta)P_0 \in \Lambda$ and $\varphi(u)$ being periodic, we have $\varphi(b^{1+\tau}(v+P_0)) = \varphi(b^{1+\tau}v + P_0)$. Therefore, 6.2.3, 6.2.1 and 6.2.4 imply

(6.2.7)

$$\begin{aligned}
&\psi_{b^{1+\tau}}(v+P_0)^5\varphi(b^{1+\tau}(v+P_0))|_{v+P_0\in\kappa^{-1}\iota(C)} \\
=&\frac{\sigma(b^{1+\tau}(v+P_0))^5\varphi(b^{1+\tau}(v+P_0))}{\sigma_2(b^{1+\tau}(v+P_0))^{5\mathrm{N}b}}\Big|_{v+P_0\in\kappa^{-1}\iota(C)} \\
=&\frac{\sigma_2(P_0)^{5(\mathrm{N}b-1)}\sigma(b^{1+\tau}v+P_0)^5\chi((b^{1+\tau}-1)P_0)^5(1+(d^\circ(v_1)\ge 1))\varphi(b^{1+\tau}v+P_0)}{\sigma_2(P_0)^{5\mathrm{N}b}(1+(d^\circ\ge 1))\chi((b^{1+\tau}-1)P_0)^{5\mathrm{N}b}} \\
=&\frac{\sigma_2(P_0)^{5(\mathrm{N}b-1)}\sigma_2(P_0)^5(1+(d^\circ(v_1)\ge 1))\chi((b^{1+\tau}-1)P_0)^{5(1-\mathrm{N}b)}}{\sigma_2(P_0)^{5\mathrm{N}b}(1+(d^\circ\ge 1))} \\
=&\chi(b^{1+\tau}-1)^{5(1-\mathrm{N}b)}(1+(d^\circ(v_1)\ge 1)).
\end{aligned}$$

Since $b^{1+\tau}-1$ is divisible by 4, $\chi((b^{1+\tau}-1)P_0) = 1$. Hence continuing the last of (6.2.7) is equal to

$$1 + (d^\circ(x(u)) \ge 2).$$

According to 4.2.5(1), $\psi_{b^{1+\tau}}(\lceil\zeta\rceil u)^5\varphi(b^{1+\tau}\lceil\zeta\rceil u) = \zeta^{3(5\mathrm{N}b-1)}\psi_{b^{1+\tau}}(u)^5\varphi(b^{1+\tau}u)$, and hence the function must be a polynomial of $x(u)^4$. $\square$

## §7. Genus three curves of cyclotomic type

### 7.1. The curve defined by $y^2 = x^7 + \frac{1}{4}$.

Let us treate genus three case. First example is the curve $C$ defined by $y^2 = x^7 + \frac{1}{4}$. As in Sections 5 and 6 the ring $\mathbf{Z}[[\zeta]]$ is isomorphic to $\mathbf{Z}[\zeta]$. Then $[-\zeta^j]$ acts such as

$$[-\zeta^j](u_1, u_2) = (-\zeta^j u_1, -\zeta^{2j} u_2, -\zeta^{3j} u_3).$$

We let $c = -4^{-\frac{1}{7}}$, $a_1 = -4^{-\frac{1}{7}}\zeta$, $c_1 = -4^{-\frac{1}{7}}\zeta^2$, $a_2 = -4^{-\frac{1}{7}}\zeta^3$, $c_2 = -4^{-\frac{1}{7}}\zeta^4$, $a_3 = -4^{-\frac{1}{7}}\zeta^5$, $c_3 = -4^{-\frac{1}{7}}\zeta^6$, in (1.1.1). Then

$$\omega' = \begin{bmatrix} 2K_1(\zeta^5 - \zeta^4) & 2K_1(\zeta^3 - \zeta^4) & 2K_1(\zeta - \zeta^2) \\ 2K_2(\zeta^3 - \zeta) & 2K_2(\zeta^6 - \zeta) & 2K_2(\zeta^2 - \zeta^4) \\ 2K_3(\zeta - \zeta^5) & 2K_3(\zeta^2 - \zeta^5) & 2K_3(\zeta^3 - \zeta^6) \end{bmatrix},$$

$$\omega'' = \begin{bmatrix} 2K_1(\zeta^5 - \zeta^4 + \zeta^3 - \zeta^2 + \zeta - 1) & 2K_1(\zeta^3 - \zeta^2 + \zeta - 1) & 2K_1(\zeta - 1) \\ 2K_2(\zeta^3 - \zeta + \zeta^6 - \zeta^4 + \zeta^2 - 1) & 2K_2(\zeta^6 - \zeta^4 + \zeta^2 - 1) & 2K_2(\zeta^2 - 1) \\ 2K_3(\zeta - \zeta^5 + \zeta^2 - \zeta^6 + \zeta^3 - 1) & 2K_3(\zeta^2 - \zeta^6 + \zeta^3 - 1) & 2K_3(\zeta^3 - 1) \end{bmatrix},$$

$$\eta' = \begin{bmatrix} 2H_1(\zeta^2 - \zeta^3) & 2H_1(\zeta^4 - \zeta^3) & 2H_1(\zeta^6 - \zeta^5) \\ 2H_2(\zeta^4 - \zeta^6) & 2H_2(\zeta - \zeta^6) & 2H_2(\zeta^5 - \zeta^3) \\ 2H_3(\zeta^6 - \zeta^2) & 2H_3(\zeta^5 - \zeta^2) & 2H_3(\zeta^4 - \zeta) \end{bmatrix},$$

$$\eta'' = \begin{bmatrix} 2H_1(\zeta^2 - \zeta^3 + \zeta^4 - \zeta^5 + \zeta^6 - 1) & 2H_1(\zeta^4 - \zeta^5 + \zeta^6 - 1) & 2H_1(\zeta^6 - 1) \\ 2H_2(\zeta^4 - \zeta^6 + \zeta - \zeta^3 + \zeta^5 - 1) & 2H_2(\zeta - \zeta^3 + \zeta^5 - 1) & 2H_2(\zeta^5 - 1) \\ 2H_3(\zeta^6 - \zeta^2 + \zeta^5 - \zeta + \zeta^4 - 1) & 2H_3(\zeta^5 - \zeta + \zeta^4 - 1) & 2K_3(\zeta^4 - 1) \end{bmatrix}.$$

The point $P_0$ in $\mathbf{C}^3$ is given by

$$P_0 = \begin{bmatrix} K_1(\zeta - \zeta^2 + \zeta^3 - \zeta^4 + \zeta^5 - \zeta^6) - K_1 \\ K_2(\zeta^2 - \zeta^4 + \zeta^6 - \zeta + \zeta^3 - \zeta^5) - K_2 \\ K_3(\zeta^3 - \zeta^6 + \zeta^2 - \zeta^5 + \zeta - \zeta^4) - K_3 \end{bmatrix} = \omega' \begin{bmatrix} \frac{3}{7} \\ \frac{2}{7} \\ \frac{1}{7} \end{bmatrix} + \omega'' \begin{bmatrix} \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \end{bmatrix}. \tag{7.1.1}$$

Then

$$\begin{aligned} [\zeta]P_0 &= \omega' \begin{bmatrix} -\frac{4}{7} \\ -\frac{5}{7} \\ -\frac{6}{7} \end{bmatrix} + \omega'' \begin{bmatrix} \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \end{bmatrix}, \; [\zeta^2]P_0 = \omega' \begin{bmatrix} \frac{3}{7} \\ \frac{2}{7} \\ \frac{1}{7} \end{bmatrix} + \omega'' \begin{bmatrix} \frac{1}{7} \\ \frac{1}{7} \\ -\frac{6}{7} \end{bmatrix}, \\ [\zeta^3]P_0 &= \omega' \begin{bmatrix} -\frac{4}{7} \\ -\frac{5}{7} \\ \frac{1}{7} \end{bmatrix} + \omega'' \begin{bmatrix} \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \end{bmatrix}, \; [\zeta^4]P_0 = \omega' \begin{bmatrix} \frac{3}{7} \\ \frac{2}{7} \\ \frac{1}{7} \end{bmatrix} + \omega'' \begin{bmatrix} \frac{1}{7} \\ -\frac{6}{7} \\ \frac{1}{7} \end{bmatrix}, \\ [\zeta^5]P_0 &= \omega' \begin{bmatrix} -\frac{4}{7} \\ \frac{2}{7} \\ \frac{1}{7} \end{bmatrix} + \omega'' \begin{bmatrix} \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \end{bmatrix}, \; [\zeta^6]P_0 = \omega' \begin{bmatrix} \frac{3}{7} \\ \frac{2}{7} \\ \frac{1}{7} \end{bmatrix} + \omega'' \begin{bmatrix} -\frac{6}{7} \\ \frac{1}{7} \\ \frac{1}{7} \end{bmatrix}. \end{aligned} \tag{7.1.2}$$

The Taylor expansion at $u = P_0$ is computed as follows. Taking care of the fact $\chi(([\zeta] - 1)P_0) = 1$ deduced from

$$([\zeta] - 1)P_0 = \omega' \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} + \omega'' \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7.1.3}$$

which is given by (7.1.1) and (7.1.2), and the fact $\sigma_2(\lceil\zeta\rceil P_0)=\zeta^2\sigma_2(P_0)\neq 0$ deduced from 4.2.5(2), instead of (6.1.5) we can deduce

(7.1.4) $$\sigma(v+\lceil\zeta\rceil P_0)=\zeta^2\sigma(v+P_0)\exp[L(v,(\lceil\zeta\rceil-1)P_0)].$$

from 4.2.8(2). After operating $\frac{\partial^2}{\partial u_i\partial u_j}$ to (7.1.4), by setting $v=O$, we have

(7.1.5)
$$\sigma_{ij}(\lceil\zeta\rceil P_0)=\zeta^4\sigma_{ij}(P_0)+\sigma_i(P_0)(-\eta'_{1j}-\eta'_{2j}-\eta'_{3j})\zeta^2+\sigma_j(P_0)(-\eta'_{1i}-\eta'_{2i}-\eta'_{3i})\zeta^2$$

by using (7.1.3). Similar argument to Subsection 6.1 gives

$$\begin{aligned}
\sigma_{11}(P_0)=&2\sqrt{\lambda_0}\sigma_2(P_0)=\sigma_2(P_0)\neq 0, & \sigma_{12}(P_0)=&2H_1(-\zeta^3+\zeta^4+\zeta^6)\sigma_2(P_0),\\
\sigma_{22}(P_0)=&4H_2(\zeta^4+\zeta+\zeta^5)\sigma_2(P_0), & \sigma_{13}(P_0)=&0,\\
\sigma_{23}(P_0)=&2H_2(-\zeta^2+\zeta^5+\zeta^4)\sigma_2(P_0), & \sigma_{33}(P_0)=&0
\end{aligned}$$

provided (6.1.6) is replaced by (7.1.5). Since the Taylor expansion at $O$ is given by 2.1.1(3), we have obtained the following.

PROPOSITION 7.1.1. *Assume $C$ be defined by $y^2=x^7+\frac{1}{4}$. Let $P_0$ be the point whose coordinate is given by* (7.1.1). *Then*

(1) $$\sigma(u)=u_1u_3-u_2^2-\frac{1}{12}u_1^4-\frac{1}{3}u_2u_3^3+(d^\circ\geq 6),$$

(2)
$$\begin{aligned}
\sigma(v+P_0)=&\sigma_2(P_0)\big(v_2+\frac{1}{2}v_1^2+\gamma_{12}v_1v_2+\frac{\gamma_{22}}{2}v_2^2+\Big(\frac{\gamma_{12}}{6}-\frac{1}{3}\Big)v_1^3\\
&+\Big(\frac{\gamma_{22}}{8}+\frac{\gamma_{12}^2}{4}\Big)v_1^2v_2+\frac{\gamma_{12}\gamma_{22}}{4}v_1v_2^2+\frac{\gamma_{22}^2}{8}v_2^3+\frac{\gamma_{23}}{4}v_1^2v_3\\
&+\gamma_{22}\gamma_{12}v_1v_2v_3+\gamma_{22}\gamma_{23}v_2^2v_3+\frac{\gamma_{23}^2}{4}v_2v_3^2-\frac{1}{3}v_3^3+(d^\circ\geq 4)\big),
\end{aligned}$$

*where* $\gamma_{12}=2H_1(-\zeta^3+\zeta^4+\zeta^6)$, $\gamma_{22}=4H_2(\zeta^4+\zeta+\zeta^5)$ *and* $\gamma_{23}=2H_2(-\zeta^2+\zeta^5+\zeta^4)$.

PROPOSITION 7.1.2. $\sigma_2(P_0)^7=\exp\frac{7}{2}L(P_0,P_0)$.

*Proof.* Because of $y(P_0)=\frac{1}{2}$, it is obtained from 3.2.4(2) and (3) that

$$\sigma(3P_0)=\sigma_2(P_0)^9 \text{ and } \sigma(4P_0)=\sigma_2(P_0)^{16},$$

respectively. On the other hand, from 3.1.1 we get

$$\sigma(4P_0)=\sigma(-3P_0+7P_0)=-\exp[\frac{7}{2}L(P_0,P_0)]\sigma(3P_0).$$

Here we have used that $\sigma(-3P_0)=\sigma(3P_0)$ and (7.1.1) which implies $\chi(7P_0)=1$. Therefore we obtain

$$\sigma_2(P_0)^{16}=\exp[\frac{7}{2}L(P_0,P_0)]\sigma_2(P_0)^9$$

and the statement. □

We denote by $\tau$ the element of $\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$ such that $\zeta^\tau=\zeta^3$. Then $1+\tau$ is a type norm (see 4.2.7) in $\mathbf{Z}[\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})]$.

LEMMA 7.1.3. *If* $b \in \mathbf{Z}[\zeta]$ *and* $\equiv 1 \bmod (1-\zeta)$, *then* $\chi((b^{1+\tau^{-1}+\tau^{-2}}-1)P_0)^{\mathrm{N}b-1} = 1$.

*Proof.* If N$b$ is odd, the statemnt is trivial. So we assume N$b$ is even. For $\ell \in \Lambda$, it is easily verified from the definition that the value $\chi(\ell)$ is determined only by $\ell \bmod 2\Lambda$. By the assumption of $b$, we may write $b^{1+\tau^{-1}+\tau^{-1}} = (a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 + a_5\zeta^5 + a_6\zeta^6)(1-\zeta)+1$ with integers $a_j$. Since 2 is a prime in $\mathbf{Z}[\zeta]$, we have $b \equiv 0 \mod 2$ and hence $b^{1+\tau^{-1}+\tau^{-2}} \equiv 0 \mod 2$. By simple calculation, we see that $a_1 \equiv a_3 \equiv a_5 \equiv 1 \mod 2$ and $a_2 \equiv a_4 \equiv a_6 \equiv 0 \mod 2$. Therefore

$$\begin{aligned}\chi((b^{1+\tau^{-1}+\tau^{-2}}-1)P_0) =&\chi((\zeta+\zeta^3+\zeta^5)(1-\zeta)P_0)\\ =&\chi((\zeta-\zeta^2+\zeta^3-\zeta^4+\zeta^5-\zeta^6)P_0)\\ =&1\end{aligned}$$

because of

$$(\zeta-\zeta^2+\zeta^3-\zeta^4+\zeta^5-\zeta^6)P_0 = \omega'\begin{bmatrix}-3\\-2\\-1\end{bmatrix} + \omega''\begin{bmatrix}1\\1\\1\end{bmatrix}$$

which is obtained from (7.1.2). □

PROPOSITION 7.1.4. *Let* $b$ *be an element of* $\mathbf{Z}[\zeta]$. *If* $b \equiv 1 \mod (1-\zeta)^2$, *then*

$$\begin{aligned}\sigma(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0)) = \chi((b^{1+\tau^{-1}+\tau^{-2}}-1)P_0)\sigma_2(P_0)^{\mathrm{N}b-1}&\\ \sigma(b^{1+\tau^{-1}+\tau^{-2}}v+P_0)(1+(d^\circ \geq 1)).&\end{aligned}$$

*Proof.* The statement follows from 4.2.8 and

$$\exp[\frac{1}{2}(\mathrm{N}b-1)L(P_0,P_0)] = \sigma_2(P_0)^{\mathrm{N}b-1}$$

which is given by 7.1.2. □

LEMMA 7.1.5. *Let*

$$\varphi(u) := (\wp_{12}^2 - \wp_{22}\wp_{11})(u).$$

*Then it has the following properties.*
(1) $\varphi([\zeta]u) = \zeta^6\varphi(u)$,
(2) $\varphi(u) \in \Gamma(J, \mathcal{O}(3\Theta))$,
(3) *the Taylor expansions of* $\sigma(u)^3\varphi(u)$ *at* $O$ *and* $P_0$ *are of the form*

$$\begin{aligned}\sigma(u)^3\varphi(u) &= 2u_3^2 + (d^\circ(u_1,u_2,u_3) \geq 4) \text{ and}\\ \sigma(v+P_0)^3\varphi(v+P_0) &= -\sigma_2(P_0)^3(1+(d^\circ(v_1,v_2,v_3)\geq 1)).\end{aligned}$$

*Proof.* The statement (1) follows from 4.2.5(2) and the definition of $\wp$-functions. The statement (2) follows from

$$(\sigma^3\varphi)(u) = -\sigma_2(u)^2\sigma_{11}(u) - \sigma_1(u)^2\sigma_{22}(u) + 2\sigma_1(u)\sigma_2(u)\sigma_{12}(u) + \sigma_{12}(u)^2\sigma(u).$$

The statement (3) is easily derived from the equation above and 7.1.1. □

THEOREM 7.1.6. *Let $\varphi(u) = (\wp_{12}^2 - \wp_{22}\wp_{11})(u)$ as above. Let $b \in \mathbf{Z}[\zeta]$ and assume $b \equiv 1 \mod (1-\zeta)^2$.*
(1) *If* N$b$ *is odd then $\varphi_{b^{1+\tau^{-1}+\tau^{-2}}}(u)^3\varphi(b^{1+\tau^{-1}}u)$ is of the form*

$$\psi_{b^{1+\tau^{-1}+\tau^{-2}}}(u)^3\varphi(b^{1+\tau^{-1}+\tau^{-1}}u) = 2y(u)\sum_{\substack{0\le j\le \frac{9(\mathrm{N}b-1)}{2}\\ j\equiv 0 \mod 7}} \gamma_j x(u)^j$$

*with $\gamma_j \in \mathbf{Q}$. Moreover $\gamma_{\frac{9(\mathrm{N}b-1)}{2}} = b^{2(\tau+1+\tau^{-1})}$ and $\gamma_0 = -1$.*
(2) *If* N$b$ *is even then $\varphi_{b^{1+\tau^{-1}+\tau^{-2}}}(u)^3\varphi(b^{1+\tau^{-1}}u)$ is of the form*

$$\psi_{b^{1+\tau^{-1}+\tau^{-2}}}(u)^3\varphi(b^{1+\tau^{-1}+\tau^{-2}}u) = \sum_{\substack{0\le j\le \frac{9\mathrm{N}b-2}{2}\\ j\equiv 0 \mod 7}} \gamma_j x(u)^j$$

*with $\gamma_j \in \mathbf{Q}(\zeta)$. Moreover $\gamma_{\frac{9\mathrm{N}b-2}{2}} = 2b^{2(\tau+1+\tau^{-1})}$ and $\gamma_0 = -1$.*

*Proof.* At First, we look at the Laurent expansion at $u = O$. By 7.1.5(3) and 7.1.1(1), we have

(7.1.5)

$$\begin{aligned}
\psi_{b^{1+\tau^{-1}+\tau^{-2}}}(u)^3\varphi(b^{1+\tau^{-1}+\tau^{-2}}u)\Big|_{u\in\kappa^{-1}\iota(C)} &= \frac{\sigma(b^{1+\tau^{-1}+\tau^{-2}}u)^3\varphi(b^{1+\tau^{-1}+\tau^{-2}}u)}{\sigma_2(u)^{3\mathrm{N}b}}\\
&= \frac{2(b^{1+\tau^{-1}+\tau^{-2}})^{2\tau}u_3^2 + (d^\circ \ge 4)}{(-2u_2 - \frac{1}{3}u_3^3 + (d^\circ \ge 5))^{3\mathrm{N}b}}\\
&= \frac{2(b^{2(\tau+1+\tau^{-1})})u_3^2 + (d^\circ \ge 4)}{(-u_3^3 + (d^\circ \ge 5))^{3\mathrm{N}b}}\\
&= (-1)^{\mathrm{N}b}2b^{2(\tau+1+\tau^{-1})}\frac{1}{u_3^{9\mathrm{N}b-2}} + \cdots .
\end{aligned}$$

This function $\psi_{b^{1+\tau^{-1}+\tau^{-2}}}(u)^3\varphi(b^{1+\tau^{-1}+\tau^{-2}}u)\Big|_{u\in\kappa^{-1}\iota(C)}$ is odd or even and $\sigma_2$ has only zeroes at $u \in \Lambda$ by the first statement of 2.2.1(3), accordingly is a polynomial of $x(u)$ multiplied by $y(u)$ or a polynomial of $x(u)$. If N$b$ is odd, then, the last of (7.1.6) is

$$\begin{aligned}
&= 2b^{2(\tau+1+\tau^{-1})}\frac{-1}{u_3^7}\left(\frac{1}{u_3^2}\right)^{9(\mathrm{N}b-1)/2} + \cdots\\
&= 2y(u)(b^{2(\tau+1+\tau^{-1})}x(u)^{9(\mathrm{N}b-1)/2} + \text{“lower terms of power of } x(u)\text{”})
\end{aligned}$$

by 2.3.1 and 2.3.2. If N$b$ is even, then the last of (7.1.6) is

$$\begin{aligned}
&= 2b^{2(\tau+1+\tau^{-1})}\left(\frac{1}{u_3^2}\right)^{(9\mathrm{N}b-2)/2} + \cdots\\
&= 2b^{2(\tau+1+\tau^{-1})}x(u)^{(9\mathrm{N}b-2)/2)} + \text{“lower terms of power of } x(u)\text{”}
\end{aligned}$$

by 2.3.1 and 2.3.2. Secondly, we look at the Laurent expansion at $u = P_0$ ($\kappa(P_0) = \iota(0, \frac{1}{2})$). Since $b \equiv 1 \mod (1-\zeta)^2$ we have $b^{1+\tau^{-1}+\tau^{-2}} \equiv 1 \mod (1-\zeta)^2$. Because of $(1-\zeta)P_0 \in \Lambda$ and $\varphi(u)$ being periodic, we have $\varphi(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0)) = \varphi(b^{1+\tau^{-1}+\tau^{-2}}v + P_0)$. Therefore, 7.1.3, 7.1.1, 7.1.4 and 7.1.5 imply
(7.1.7)

$$
\begin{aligned}
&\psi_{b^{1+\tau^{-1}+\tau^{-2}}}(v+P_0)^3\varphi(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0))|_{v+P_0\in\kappa^{-1}\iota(C)} \\
=&\frac{\sigma(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0))^3\varphi(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0))}{\sigma_2(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0)^{3\mathrm{N}b}}|_{v+P_0\in\kappa^{-1}\iota(C)} \\
=&\Big\{\sigma_2(P_0)^{3(\mathrm{N}b-1)}\sigma(b^{1+\tau^{-1}+\tau^{-2}}v+P_0)^3\chi((b^{1+\tau^{-1}+\tau^{-2}}-1)P_0)^3(1+(d^\circ(v_1)\geq 1)) \\
&\varphi(b^{1+\tau^{-1}+\tau^{-2}}v+P_0)\Big\} \\
&/\Big\{\sigma_2(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0)^{3\mathrm{N}b}\chi((b^{1+\tau^{-1}+\tau^{-2}}-1)P_0)^{3\mathrm{N}b}\Big\} \\
=&\frac{\sigma_2(P_0)^{3(\mathrm{N}b-1)}\sigma_2(P_0)^3(-1+(d^\circ(v_1)\geq 1))}{\sigma_2(P_0)^{3\mathrm{N}b}(1+(d^\circ\geq 1))} \quad \text{(by 7.1.3)} \\
=&-1+(d^\circ(v_1)\geq 1) \\
=&\begin{cases} -2y(u)(1+(d^\circ(x(u))\geq 1)) & \text{if } \mathrm{N}b \text{ is odd,} \\ -1+(d^\circ(x(u))\geq 1) & \text{if } \mathrm{N}b \text{ is even.} \end{cases}
\end{aligned}
$$

Furthermore, since

$$\psi_{b^{1+\tau^{-1}+\tau^{-2}}}(\lceil -\zeta\rceil u)\varphi(b^{1+\tau^{-1}+\tau^{-2}}\lceil -\zeta\rceil u) = (-1)^{\mathrm{N}b}\psi_{b^{1+\tau^{-1}+\tau^{-2}}}(u)\varphi(b^{1+\tau-1+\tau^{-2}}u)$$

by 4.2.5(2), the function must be a polynomial of $x(u)^7$ if N$b$ even, or one multiplied by $y(u)$ if N$b$ is odd. $\square$

**7.2. The curve defined by $y^2 = x^7 - x$.**
Second example of genus three is the curve $C$ defined by $y^2 = x^7 - x$. The ring $\mathbf{Z}[\lceil\zeta\rceil]$ is isomorphic to the ring $\mathbf{Z}[i] \oplus \mathbf{Z}[\zeta]$ by $\lceil\zeta^j\rceil \mapsto i^j \oplus \zeta^j$ by 4.1.1(2). The endomorphism $\lceil\zeta^j\rceil$ acts such as

$$\lceil\zeta^j\rceil(u_1, u_2, u_3) = (\zeta^j u_1, i^j u_2, \zeta^{5j} u_3) \tag{7.2.1}$$

because $\lceil\zeta\rceil\omega^{(j)} = \zeta^j\omega^{(j)}$ for $j = 1, 2, 3$. We let $c = 1$, $a_1 = \zeta^2$, $c_1 = \zeta^4$, $a_2 = \zeta^6$, $c_2 = \zeta^8$, $a_3 = \zeta^{10}$, $c_3 = 0$, in (1.1.1).

As in the previous Subsections, we have

$$\omega' = \begin{bmatrix} -2K_1\zeta^5 & 2K_1(\zeta^4 - \zeta^3) & 2K_1(\zeta^2 - \zeta) \\ -2K_2\zeta^3 & 2K_2(\zeta + \zeta^3) & 2K_2(-1 - \zeta^3) \\ -2K_3\zeta & 2K_3(-\zeta^2 - \zeta^3) & 2K_3(-\zeta^4 - \zeta^5) \end{bmatrix},$$

$$\omega'' = \begin{bmatrix} 2K_1(\zeta^4 - \zeta^3 + \zeta^2 - \zeta + 1) & 2K_1(-\zeta^3 + \zeta^2 - \zeta + 1) & 2K_1(-\zeta + 1) \\ 2K_2 & 0 & 2K_2(-\zeta^3 + 1) \\ 2K_3(-\zeta^2 - \zeta^3 - \zeta^4 - \zeta^5 + 1) & 2K_3(-\zeta^3 - \zeta^4 - \zeta^5 + 1) & 2K_3(-\zeta^5 + 1) \end{bmatrix},$$

$$\eta' = \begin{bmatrix} 2H_1\zeta & 2H_1(-\zeta^2 + \zeta^3) & 2H_1(-\zeta^3 + \zeta^5) \\ 2H_2\zeta^3 & 2H_2(-\zeta^5 - \zeta^3) & 2H_2(-1 + \zeta^3) \\ 2H_3\zeta^5 & 2H_3(\zeta^4 + \zeta^3) & 2H_3(\zeta^2 + \zeta) \end{bmatrix},$$

$$\eta'' = \begin{bmatrix} 2H_1(-\zeta^2 + \zeta^3 - \zeta^4 + \zeta^5 + 1) & 2H_1(+\zeta^3 - \zeta^4 + \zeta^5 + 1) & 2H_1(\zeta^5 + 1) \\ 2H_2 & 0 & 2H_2(\zeta^3 + 1) \\ 2H_3(\zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1) & 2H_3(-\zeta^3 + \zeta^2 + \zeta + 1) & 2H_3(\zeta + 1) \end{bmatrix}.$$

Furthermore

$$P_0 = \begin{bmatrix} K_1(\zeta - \zeta^2 + \zeta^3 - \zeta^4 + \zeta^5) - K_1 \\ K_2(\zeta^3 + 1 - \zeta^3 - 1 + \zeta^3) - K_2 \\ K_3(\zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta) - K_3 \end{bmatrix} = \omega' \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + \omega'' \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix}. \tag{7.2.2}$$

and, by (7.2.1),

$$\begin{aligned} \lceil\zeta\rceil P_0 &= \begin{bmatrix} \zeta K_1(-1 + \zeta - \zeta^2 + \zeta^3 - \zeta^4 + \zeta^5 - 1) \\ \zeta^3 K_2(-1 + \zeta^3 + 1 - \zeta^3 - 1 + \zeta^3 - 1) \\ \zeta^5 K_3(-1 - \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta - 1) \end{bmatrix} \\ &= \omega' \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix} + \omega'' \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{7.2.3}$$

Let us compute the Taylor expansion at $u = P_0$ explicitly. Taking care of the fact $\chi((\lceil\zeta\rceil - 1)P_0) = 1$ deduced from

$$(\lceil\zeta\rceil - 1)P_0 = \omega' \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \omega'' \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7.2.4}$$

which is given by (7.2.2) and (7.2.3), and the fact $\sigma_2(\lceil\zeta\rceil P_0) = \zeta^3\sigma_2(P_0) \neq 0$ deduced from 4.2.5(3), we have from 4.2.8(2) by similar argument to Subsection 6.1

$$\sigma(v + \lceil\zeta\rceil P_0) = \zeta^3\sigma(v + P_0)\exp[L(v, (\lceil\zeta\rceil - 1)P_0)]. \tag{7.2.5}$$

instead of (6.1.5). After operating $\dfrac{\partial^2}{\partial u_i \partial u_j}$ to (7.2.5), by setting $v = O$, we have

$$\sigma_{ij}(\lceil\zeta\rceil P_0) = \zeta^3\sigma_{ij}(P_0) + \sigma_i(P_0)(\eta'_{j1} + \eta'_{j2} + \eta'_{j3})\zeta^3 + \sigma_j(P_0)(\eta'_{i1} + \eta'_{i2} + \eta'_{i3})\zeta^3. \tag{7.2.6}$$

by using (7.2.4). Similar argument to Subsection 6.1 gives

$$\begin{aligned} \sigma_{11}(P_0) &= 0, & \sigma_{12}(P_0) &= H_1(-1 - (2 - \sqrt{3})i)\sigma_2(P_0), \\ \sigma_{22}(P_0) &= H_2(-1 - i)\sigma_2(P_0), & \sigma_{13}(P_0) &= 0, \\ \sigma_{23}(P_0) &= H_1(-1 - (\sqrt{3} + 2)i)\sigma_2(P_0), & \sigma_{33}(P_0) &= 0, \end{aligned}$$

provided (6.1.6) is replaced by (7.2.6). Summing up these results and 2.2.1(3), we arrived at

PROPOSITION 7.2.1. *Assume $C$ be defind by $y^2 = x^7 - x$. Let $P_0$ be the point whose coordinate is given by* (7.2.1). *Then*

(1) $$\sigma(u) = u_1u_3 - u_2^2 - \frac{1}{12}u_1^4 - \frac{1}{3}u_2u_3^3 + (d^\circ \geq 6),$$

(2) $$\begin{aligned} \sigma(v + P_0) = \sigma_2(P_0)\Big(&v_2 + \gamma_{12}v_1v_2 + \frac{\gamma_{22}}{2}v_2^2 + \gamma_{13}v_1v_3 + \gamma_{23}v_2v_3 - \frac{1}{3}v_1^3 \\ &+ \frac{\gamma_{12}^2}{4}v_1^2v_2 + \frac{\gamma_{12}\gamma_{22}}{4}v_1v_2^2 + \gamma_{12}\gamma_{23}v_1v_2v_3 + \frac{\gamma_{22}^2}{8}v_2^3 \\ &+ \frac{\gamma_{22}\gamma_{23}}{4}v_2^2v_3 + \frac{\gamma_{23}^2}{4}v_2v_3^2 - \frac{1}{3}v_3^3 + (d^\circ \geq 4)\Big), \end{aligned}$$

*where* $\gamma_{12} = H_1(-1 - (2 - \sqrt{3})i)$, $\gamma_{22} = H_2(-1 - i)$ *and* $\gamma_{23} = H_1(-1 - (\sqrt{3} + 2)i)$.

PROPOSITION 7.2.2. $\sigma_2(P_0)^8 = \exp 4L(P_0, P_0)$.

*Proof.* Take $y = y(u)$ as a local parameter at $P_0$ along $\kappa^{-1}\iota(C)$. By 3.2.4(2), we have $8y(u)\sigma_2(u)^9 = \sigma(3u)$. After operating $\frac{d^3}{dy^3}$ to this, by setting $u = P_0$, we get

$$27\sigma_{111}(3P_0) + 6\sigma_2(3P_0) = -48\sigma_2(P_0)^9 \tag{7.2.7}$$

because of $y(P_0) = 0$ and 2.3.2(3). Moreover, we have the equation

$$\sigma(u + 3P_0) = \chi(2P_0)\sigma(u + P_0)\exp L(u + P_0 + P_0, 2P_0) \tag{7.2.8}$$

given by 3.1.1. After operating $\frac{\partial^3}{\partial u_1^3}$ to (7.2.8), by putting $u = 0$, we get $\sigma_{111}(3P_0) = -\sigma_{111}(P_0)\exp 2L(P_0, P_0)$ because of $\sigma(O) = 0$, $\sigma_1(O) = 1$ and $\chi(2P_0) = 1$. The last is obtained by (7.2.2) and definition of $\chi(\ )$. Similarly, after differentiating (7.2.8) with respect to $u_2$, by setting $u = 0$, we get

$$\sigma_2(3P_0) = \sigma_2(P_0)\exp 4L(P_0, P_0). \tag{7.2.9}$$

Summing up (7.2.7), (7.2.8) and (7.2.9), we arrive at the statement. $\square$

We denote by $\tau$ the element of $\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$ such that $\zeta^\tau = \zeta^5$. Then $1 + \tau$ is a type norm (see 4.2.7) in $\mathbf{Z}[\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})]$.

Proposition 7.2.3. *Let $b$ be an element of $\mathbf{Z}[\lceil\zeta\rceil]$. If $b \equiv 1 \mod 8$, then*

$$\sigma(b^{1+\tau}(v+P_0)) = \sigma_2(P_0)^{\mathrm{N}b-1}\sigma(b^{1+\tau}v+P_0)(1+(d^\circ \geq 1)).$$

*Proof.* Note that $2P_0 \in \Lambda$. By the assumption, $b^{1+\tau}-1$ is divisible by 8. So $\chi((b^{1+\tau}-1)P_0) = 1$. Moreover $\mathrm{N}b-1$ is divisible by 8, the statement follows from 4.2.8 and

$$\exp[\frac{1}{2}(\mathrm{N}b-1)L(P_0,P_0)] = \sigma_2(P_0)^{\mathrm{N}b-1}$$

which is given by 7.2.2. $\square$

Lemma 7.2.4. *Let*

$$\varphi(u) := [\frac{1}{24}(\wp_{2222}-6\wp_{22}^2)\wp_{111}+\frac{1}{2}(\wp_{1112}-6\wp_{11}\wp_{22})\wp_{222}](u).$$

*Then it has the following properties.*
(1) $\varphi(\lceil\zeta\rceil u) = \zeta^3\varphi(u)$,
(2) $\varphi(u) \in \Gamma(J, \mathcal{O}(5\Theta))$,
(3) *the Taylor expansions of* $\sigma(u)^5\varphi(u)$ *at* $O$ *and* $P_0$ *are of the form*

$$\begin{aligned}
&\sigma(u)^5\varphi(u) = -u_3^3 + (d^\circ(u_1,u_2,u_3) \geq 5)\\
&\sigma(v+P_0)^5\varphi(v+P_0) = \sigma_2(P_0)^5(-1+(d^\circ(v_1,v_2,v_3)\geq 1)).
\end{aligned}$$

*Proof.* The statement (1) follows from 4.2.5(3) and the definition of $\wp$-functions. Since

$$\begin{aligned}
\sigma(u)^2(\wp_{2222}-6\wp_{22}^2)(u) =&(-\sigma_{2222}\sigma+4\sigma_{222}\sigma_2-3\sigma_{22}^2)(u),\\
\sigma(u)^3\wp_{111}(u) =&(-2\sigma_1^3+3\sigma_1\sigma_{11}\sigma-\sigma_{111}\sigma^2)(u),\\
\sigma(u)^2(\wp_{1112}-6\wp_{11}\wp_{22}^2)(u) =&(-\sigma_{1112}\sigma+3\sigma_{112}\sigma_1-\sigma_{111}\sigma_2-3\sigma_{11}\sigma_{12})(u),\\
\sigma(u)^3\wp_{222}(u) =&(-2\sigma_2^3+3\sigma_2\sigma_{22}\sigma-\sigma_{222}\sigma^2)(u).
\end{aligned}$$

the statement (2) holds. The expansion in 7.2.1(1) gives

$$\begin{aligned}
&(-\sigma_{2222}\sigma+4\sigma_{222}\sigma_2-3\sigma_{22}^2)(u)\\
=&-(d^\circ\geq 2)(d^\circ\geq 2)+4(d^\circ\geq 2)(d^\circ\geq 2)-3(-2+(d^\circ\geq 2))^2\\
=&12+(d^\circ\geq 2),\\
&(-2\sigma_1^3+3\sigma_1\sigma_{11}\sigma-\sigma_{111}\sigma^2)(u)\\
=&-2(u_3+(d^\circ\geq 3))^3+3(d^\circ\geq 1)(d^\circ\geq 2)(d^\circ\geq 2)-(d^\circ\geq 1)(d^\circ\geq 2)^2\\
=&-2u_3^3+(d^\circ\geq 4),\\
&(-\sigma_{1112}\sigma+3\sigma_{112}\sigma_1-\sigma_{111}\sigma_2-3\sigma_{11}\sigma_{12})(u)\\
=&-(d^\circ\geq 0)(d^\circ\geq 2)+3(d^\circ\geq 1)(d^\circ\geq 1)+(d^\circ\geq 0)(d^\circ\geq 1)-3(d^\circ\geq 2)(d^\circ\geq 2)\\
=&(d^\circ\geq 2),
\end{aligned}$$

$$
\begin{aligned}
&(-2\sigma_2^3+3\sigma_2\sigma_{22}\sigma-\sigma_{222}\sigma^2)(u)\\
=&-2(d^\circ\geq 1)^3+3(d^\circ\geq 1)(d^\circ\geq 1)(d^\circ\geq 2)-(d^\circ\geq 3)(d^\circ\geq 2)^2\\
=&(d^\circ\geq 3).
\end{aligned}
$$

Therefore

$$\sigma(u)^5\varphi(u)=-u_3^3+(d^\circ\geq 5).$$

Similarly, 7.2.1(2) gives

$$
\begin{aligned}
&(-2\sigma_1^3+3\sigma_1\sigma_{11}\sigma-\sigma_{111}\sigma^2)(v+P_0)\\
=&-2(d^\circ\geq 1)^3+3(d^\circ\geq 1)(d^\circ\geq 1)(d^\circ\geq 1)-(d^\circ\geq 0)(d^\circ\geq 1)^2\\
=&(d^\circ\geq 2),\\
&(-\sigma_{1112}\sigma+3\sigma_{112}\sigma_1+\sigma_{111}\sigma_2-3\sigma_{11}\sigma_{12})(v+P_0)\\
=&\sigma_2(P_0)^2[-(d^\circ\geq 0)(d^\circ\geq 1)+3(d^\circ\geq 0)(d^\circ\geq 1)\\
&+(1+(d^\circ\geq 1))(1+(d^\circ\geq 1))-3(d^\circ\geq 1)(d^\circ\geq 0)]\\
=&\sigma_2(P_0)^2(1+(d^\circ\geq 1)),\\
&(-2\sigma_2^3+3\sigma_2\sigma_{22}\sigma-\sigma_{222}\sigma^2)(v+P_0)\\
=&\sigma_2(P_0)^3[-2(1+(d^\circ\geq 1))^3+3(d^\circ\geq 0)(d^\circ\geq 0)(d^\circ\geq 1)-(d^\circ\geq 0)(d^\circ\geq 1)^2]\\
=&\sigma_2(P_0)^3(-2+(d^\circ\geq 1)).
\end{aligned}
$$

Hence

$$\sigma(u)^5\varphi(v+P_0)=\sigma_2(P_0)^5(1+(d^\circ\geq 1)).$$

So (3) is proved. $\square$

THEOREM 7.2.5. *Let $\varphi(u)$ be as in 7.2.4. Let $b\in\mathbf{Z}[[\zeta]]$ and assume $b\equiv 1 \mod 8$. Then $\varphi_{b^{1+\tau}}(u)^5\varphi(b^{1+\tau}u)$ is of the form*

$$\psi_{b^{3(1+\tau)}}(u)^5\varphi(b^{1+\tau}u)=\sum_{\substack{0\leq j\leq \frac{15\mathrm{N}b-3}{2}\\ j\equiv 0 \mod 6}}\gamma_j x(u)^j$$

*with $\gamma_j\in\mathbf{Q}(\zeta)$. Moreover $\gamma_{\frac{15\mathrm{N}b-3}{2}}=b^{3(1+\tau)}$ and $\gamma_0=1$.*

*Proof.* At first, we look at the Laurent expansion at $u=O$. By 7.2.4(3) and 7.2.1(1), we have

$$
\begin{aligned}
&\psi_{b^{1+\tau}}(u)^5\varphi(b^{1+\tau}u)\Big|_{u\in\kappa^{-1}\iota(C)}\\
=&\frac{\sigma(b^{1+\tau}u)^5\varphi(b^{1+\tau}u)}{\sigma_2(u)^{5\mathrm{N}b}}\\
=&\frac{-(b^{1+\tau})^{3\tau}u_3^3+(d^\circ(u_2)\geq 4)}{(-2u_2-\frac{1}{3}u_3^3+(d^\circ(u_3)\geq 4))^{5\mathrm{N}b}}
\end{aligned}
$$

$$
\begin{aligned}
&= b^{3(\tau+1)} \frac{1}{u_3^{15\mathrm{N}b-3}} + \cdots \\
&= b^{3(\tau+1)} \left(\frac{1}{u_3^2}\right)^{(15\mathrm{N}b-3)/2} + \cdots \\
&= b^{3(\tau+1)} (x(u)^{(15\mathrm{N}b-3)/2} + \text{“lower terms of power of } x(u)\text{”}).
\end{aligned}
$$

Here we used 2.3.1, 2.3.2 and the fact that $\psi_{b^{1+\tau}}(u)^5\varphi(b^{1+\tau}u)\Big|_{u\in\kappa^{-1}\iota(C)}$ which fact is deduced from that this function is even and $\sigma_2$ has only zeroes at $u \in \Lambda$ by the first statement of 2.2.1(3). Secondly, we look at the Laurent expansion at $u = P_0$ ($\iota(P_0 \mod \Lambda) = (0,0)$). Since $b \equiv 1 \mod 8$ we have $b^{\tau+1} \equiv 1 \mod 8$. Because of $2P_0 \in \Lambda$ and $\varphi(u)$ being periodic, we have $\varphi(b^{1+\tau}(v+P_0)) = \varphi(b^{1+\tau}v + P_0)$. Consequently, 7.2.3, 7.2.1 and 7.2.4 imply

$$
\begin{aligned}
&\psi_{b^{1+\tau}}(v+P_0)^5\varphi(b^{1+\tau}(v+P_0))|_{v+P_0\in\kappa^{-1}\iota(C)} \\
=&\frac{\sigma(b^{1+\tau}(v+P_0))^5\varphi(b^{1+\tau}(v+P_0))}{\sigma_2(b^{1+\tau}(v+P_0))^{5\mathrm{N}b}}|_{v+P_0\in\kappa^{-1}\iota(C)} \\
=&\frac{\sigma_2(P_0)^{5(\mathrm{N}b-1)}\sigma(b^{1+\tau}v+P_0)^5(1+(d^\circ(v_1)\geq 1))\varphi(b^{1+\tau}v+P_0)}{\sigma_2(P_0)^{5\mathrm{N}b}(-1+(d^\circ\geq 1))} \\
=&\frac{\sigma_2(P_0)^{5(\mathrm{N}b-1)}\sigma_2(P_0)^5(1+(d^\circ(v_1)\geq 1))}{\sigma_2(P_0)^{5\mathrm{N}b}(1+(d^\circ\geq 1))} \\
=&-1+(d^\circ(v_1)\geq 1) \\
=&-1+(d^\circ(x(u))\geq 1).
\end{aligned}
$$

Furthermore, since $\psi_{b^{1+\tau}}(\lceil -\zeta \rceil u)^5\varphi(b^{1+\tau}\lceil -\zeta \rceil u) = -\zeta^{3(\mathrm{N}b-1)}\psi_{b^{1+\tau}}(u)^5\varphi(b^{1+\tau}u)$ by 4.2.5(3), the function must be a polynomial of $x(u)^6$. $\square$

## §8. Some remarks and comments

1. As is mentioned in the beginning of the part II, in each formula in 5.1.3, 5.2.3, 6.1.6, 6.2.6, 7.1.6 and 7.2.6, the coefficients of the right hand side, which side is a polynomial expression in $x(u)$ and $y(u)$, are contained in the field $\mathbf{Q}(\zeta)$. Furthermore we can prove that the coefficients of the right hand side of the each formula of 5.1.3 and 5.2.3 are contained in $\mathbf{Z}[e^{2\pi i/3}]$ and $\mathbf{Z}[i]$, respectively. The coefficients of the right hand side of the formula of 6.1.6 are also contained in $\mathbf{Z}[e^{2\pi i/5}]$ (see [**9**] or [**17**, p.46]). For each of the other three formulae, its coefficients seem also to be contained in the ground integer ring.

2. Theorem 5.1.3 implies

$$\prod_{\substack{P\in b^*(\wp)_0,\\ (1-\zeta)P_0\neq O\\ /\pm 1}} x(P) = (-1)^{\mathrm{N}b-1}b.$$

Theorem 5.2.3 implies

$$\prod_{\substack{P\in b^*(\wp)_0,\\ (1+i)P_0\neq O\\ /\pm 1}} x(P) = (-1)^{\mathrm{N}b-1}b^2.$$

These are versions of the product formula of Eisenstein.

3. Theorem 6.1.6 (Grant's formula) implies

$$\prod_{\substack{P\in\iota(C)\cdot(b^{1+\tau^{-1}})^*(\varphi)\\ 2P\neq O\\ /\pm 1}} x(P) = \frac{1}{b^{1+\tau}},$$

where $\cdot$ denotes an intersection of cycles in $J$. In fact the cycle $\iota(C)\cdot(b^{1+\tau^{-1}})^*(\varphi)_0$ contains only five 2-torsion points $(-4^{\frac{1}{5}}\zeta^j, 0)$ with $j = 0, \cdots, 4$ (See [**9**, p.131]). Theorem 6.2.6 also implies that the product of roots $x(u)$ of the right hand side of the formula of 6.2.6 is equal to $\dfrac{1}{b^{2(1+\tau)}}$. Similarly Theorem 7.1.6 states that the product of roots $x(u)$ of the right hand side of the formula in 7.1.6 is equal to $\dfrac{\pm 1}{b^{2(\tau+1+\tau^{-1})}}$ or $\dfrac{\pm 1}{2b^{2(\tau+1+\tau^{-1})}}$, and Theorem 7.2.6 states that the product of roots $x(u)$ of the right hand side of the formula above is equal to $\dfrac{1}{b^{3(1+\tau)}}$. These are generalizations of the product formula of Eisenstein.

4. The polynomial of $x(u)$ in the right hand side of each of the formula of 5.1.3 and 5.2.3 is known to be irreducible over the ground ring when $b$ is a prime element. It is unknown whether the other polynomials of 6.1.6, 6.2.6, 7.1.6 and 7,2,6 are irreducible.

5. The roots of each polynomial of $x(u)$ generate a finite algebraic extension over the ground field. For the genus one case, such extensions are known to be abelian. Contrarily, the extensions in higher genus case seem not to be abelian but to have very large Galois groups. For Grant's original formula, some numerical examples are given in [**17**].

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