Explicit realization of Coble's hypersurfaces in terms of multivariate $\wp$-functions*

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Abstract

We give expressions of some Coble's hypersurfaces in algebraic explicit forms using generalised Weierstrass $\wp$ functions associated to curves of genus two and three.

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1 Introduction

In [5], Coble discussed hypersurfaces which are the singular loci of certain moduli spaces of algebraic curves. In this paper we seek explicit realisations of such surfaces using generalised Weierstrass $\wp$ functions. Our work was inspired by related work [2, 9] expressing the genus two Kummer surface using such functions.

We first outline some basic notation to describe algebraic curves. Let n and s are positive integers with $n < s$. Let $\mathscr{C}$ be the algebraic curve defined by

$$f(x, y) = 0,$$

where

$$f(x, y) = y^n + p_1(x)y^{n-1} + \cdots + p_{n-1}y - p_n(x),$$

and we complete this curve by adding a single point ∞ at infinity. Here, $p_j(x)$ is a polynomial of x of degree $\lfloor sj/n \rfloor$ of the form

$$\begin{aligned} p_j(x) &= \sum_{k:js-kn>0} \mu_{js-kn}x^k, \quad (1 \leqq j \leqq n-1) \\ p_n(x) &= x^s + \mu_{n(s-1)}x^{s-1} + \cdots + \mu_{ns}, \end{aligned}$$

where the μ_js are constants or parameters belonging to, for instance, the field of complex numbers $\mathbb{C}$.

Definition 1.1. *We define the weight, which is denoted by* wt*, by the conditions*

$$\mathrm{wt}(\mu_j) = -j, \quad \mathrm{wt}(x) = -n, \quad \mathrm{wt}(y) = -s.$$

Remark 1.2. Using this definition, all the equations in this paper are of homogeneous weight.

Coble's results apply only when considering functions related to curves of genus 2 or 3, and in this paper we consider reasonably general examples of such curves, in particular the three curves $(n, s) = (2, 5)$, $(2, 7)$, and $(3, 4)$. We denote these by $\mathscr{C}_{2,5}$, $\mathscr{C}_{2,7}$, $\mathscr{C}_{3,4}$. In particular we define

$$\begin{aligned} \mathscr{C}_{2,5} &: y^2 = x^5 + \mu_2x^4 + \mu_4x^3 + \mu_6x^2 + \mu_8x + \mu_{10}, \\ \mathscr{C}_{2,7} &: y^2 = x^7 + \mu_2x^6 + \mu_4x^5 + \mu_6x^4 + \mu_8x^3 + \mu_{10}x^2 + \mu_{12}x + \mu_{14}, \\ \mathscr{C}_{3,4} &: y^3 + (\mu_2x^2 + \mu_5x + \mu_8)y = x^4 + \mu_6x^2 + \mu_9x + \mu_{12}. \end{aligned}$$

When we discuss analytic issues, we assume these curves to be non-singular. In these cases, their genera are 2, 3, and 3, respectively; for any such non-singular (n, s) curve we have $g = (n-1)(s-1)/2$.

In the literature, the curve $\mathscr{C}_{2,5}$ is often reduced to the so-called Weierstrass form

$$y^2 = x^5 + \mu_4x^3 + \mu_6x^2 + \mu_8x + \mu_{10},$$

which can be achieved by a simple linear transformation in x. We stick to the form given

for $\mathscr{C}_{2,5}$ since this is the one used by Baker [2] and by Grant [9]. Our results can be reduced to those for the Weierstrass form by putting $\mu_2 = 0$. $\mathscr{C}_{2,7}$ can be reduced in a similar way.

In the case of $\mathscr{C}_{3,4}$, we are working directly with the Weierstrass form for conciseness. The more general $(3,4)$ curve treated in [8]

$$y^3 + (x\mu_1 x + \mu_4)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y = x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12},$$

can be studied in a similar way, and the full results are given at
`http://www.ma.hw.ac.uk/Weierstrass/Trig34/`.

A defining equation of the Kummer surface which comes from the Jacobian variety of $\mathscr{C}_{2,5}$ (for arbitrary fixed μ_j) is known from [2]. It is elegantly expressed by using generalizations of the Weierstrass $\wp$-function (see also f_1 in Theorem 3.5). We extend this idea to get an explicit expression for the Kummer variety coming from the curve $\mathscr{C}_{3,4}$. This is used to find the corresponding result when applying Coble's theory to the moduli space of the curves $\mathscr{C}_{3,4}$.

We first review some basic theory for the generalised σ and $\wp$ functions for these curves, and then explain Coble's work, we describing the case for $\mathscr{C}_{2,5}$ in some detail. In the other two cases, we only give the main results.

Convention : The ideal generated by some elements is denoted by using $\langle\ \rangle$.

2 The sigma functions and $\wp$-functions

In this Section we recall the σ function and $\wp$-functions of the curve $\mathscr{C}$, which are natural generalizations of Weierstrass σ and $\wp$-functions. For simplicity the coefficients of y (of 1st degree) and one of x^{s-1} in $f(x,y)$ are set to be 0.

For each of the three curves under consideration, the spaces of the differential forms of the first kind as follows:

$$\begin{aligned}
\mathscr{C}_{2,5} \ :\quad & \omega_1(x,y) = \frac{dx}{f_y(x,y)}, \quad \omega_2(x,y) = \frac{xdx}{f_y(x,y)}, \\
\mathscr{C}_{2,7} \ :\quad & \omega_1(x,y) = \frac{dx}{f_y(x,y)}, \quad \omega_2(x,y) = \frac{xdx}{f_y(x,y)}, \quad \omega_3(x,y) = \frac{x^2dx}{f_y(x,y)} \\
\mathscr{C}_{3,4} \ :\quad & \omega_1(x,y) = \frac{dx}{f_y(x,y)}, \quad \omega_2(x,y) = \frac{xdx}{f_y(x,y)}, \quad \omega_3(x,y) = \frac{ydx}{f_y(x,y)}.
\end{aligned}$$

Then we define g differential forms of the second kind η_j as follows. Regarding ω_i and η_j as elements in

$$U = \varinjlim_k H^0(\mathscr{C}, d\mathscr{O}(k\cdot\infty))\big/ dH^0(\mathscr{C}, \varinjlim_k \mathscr{O}(n\cdot\infty)) \simeq H^1(\mathscr{C}, \mathbb{C}),$$

the set $(\omega_1, \cdots, \omega_g, \eta_1, \cdots, \eta_g)$ forms a symplectic basis with respect to the inner product

$$U \times U \ni (\omega, \eta) \longmapsto \omega \star \eta = \operatorname*{Res}_{\infty} \left(\left(\int_{\infty}^{\infty} \omega \right) \eta(\infty) \right), \tag{2.1}$$

where $\operatorname*{Res}_{\infty}$ means taking the residue at ∞. Then, we have

$$\begin{aligned}
&\omega_i \star \eta_j = -\eta_j \star \omega_i = \delta_{ij} \qquad \text{(Kronecker's } \delta\text{)}, \\
&\omega_i \star \omega_j = \eta_i \star \eta_j = 0.
\end{aligned} \tag{2.2}$$

Though the choice of η_j is not unique, it is easily checked by the generalized Legendre relation (which follows from (2.1)) that the definition of the function $\sigma(u)$ below is independent of the choice. Indeed the bilinear form $L(\ ,\ )$ defined below, is independent of this choice. For the curve $\mathscr{C}_{2,5}$, we choose

$$\begin{aligned}
\eta_1(x,y) &= -\frac{3x^3 + 2\mu_2 x^2 + \mu_4 x}{f_y(x,y)}, \\
\eta_2(x,y) &= -\frac{x^2}{f_y(x,y)}.
\end{aligned}$$

For the curve $\mathscr{C}_{2,7}$, a choice of the η_js are

$$\begin{aligned}
\eta_1(x,y) &= -\big(5x^5 + 4\mu_2 x^4 + 3\mu_4 x^3 + 2\mu_6 x^2 \mu_8 x\big) \frac{dx}{f_y(x,y)}, \\
\eta_2(x,y) &= -\big(3x^4 + 2\mu_2 x^3 + \mu_4 x^2\big) \frac{dx}{f_y(x,y)}, \qquad \eta_3(x,y) = -\frac{x^3 dx}{f_y(x,y)}.
\end{aligned}$$

Similarly, for the curve $\mathscr{C}_{3,4}$, we choose

$$\eta_1(x,y) = -\big(5x^2y + \mu_6 y + \mu_2 y^2 + {\mu_2}^2 x^2 + \mu_5\mu_2 x\big)\frac{dx}{f_y(x,y)},$$
$$\eta_2(x,y) = -2xy\frac{dx}{f_y(x,y)}, \qquad \eta_3(x,y) = -\frac{x^2dx}{f_y(x,y)}.$$

We take a set of closed paths $\{\alpha_j \,|\, 1 \leqq i \leqq g\,\}$, $\{\beta_j \,|\, 1 \leqq i \leqq g\,\}$ on $\mathscr{C}$ which form a symplectic basis of the first homology group $H_1(\mathscr{C},\mathbb{Z})$:

$$\alpha_i \cdot \beta_j = \beta_j \cdot \alpha_i = \delta_{ij}, \quad \alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0.$$

Using these and the forms $\{\omega_i\}$, we define

$$\omega' = \left[\int_{\alpha_j} \omega_i\right], \qquad \omega'' = \left[\int_{\beta_j} \omega_i\right],$$

and $\varLambda = \omega'\,\mathbb{Z}^g + \omega''\,\mathbb{Z}^g$. Then $\varLambda$ is a lattice in $\mathbb{C}^g$. For the Jacobian variety

$$J = \mathbb{C}^g/\varLambda$$

of $\mathscr{C}$ and each integer $0 \leq k \leq g-1$, we have the map from the k-th symmetric product of $\mathscr{C}$ to J defined by

$$\mathrm{Sym}^k\mathscr{C} \ni (\mathrm{P}_1, \cdots, \mathrm{P}_k) \longmapsto \sum_{j=1}^{k}\left(\int_{\infty}^{\mathrm{P}_j} \omega_1, \cdots, \int_{\infty}^{\mathrm{P}_j} \omega_g\right) \bmod \varLambda \in J.$$

The image of this map is denoted by $\Theta^{[k]}$. We denote by

$$\kappa \,:\, \mathbb{C}^g \longrightarrow \mathbb{C}^g/\varLambda$$

which is the natural map given by $\bmod \varLambda$. Using the $\{\eta_i\}$, we define

$$\eta' = \left[\int_{\alpha_j} \eta_i\right], \quad \eta'' = \left[\int_{\beta_j} \eta_i\right].$$

For any $u \in \mathbb{C}^g$, we have uniquely $u' \in \mathbb{R}^g$ and $u'' \in \mathbb{R}^g$ satisfying $u = \omega' v' + \omega'' v''$. Then we define

$$L(u,v) = {}^t u\,(\eta' v' + \eta'' v''). \tag{2.3}$$

This L is $\mathbb{C}$-linear in the first variable and $\mathbb{R}$-linear in the second variable. The Riemann form of $\mathscr{C}$ is given by

$$E(u,v) = L(u,v) - L(v,u) \tag{2.4}$$

and written as

$$E(u,v) = 2\pi\boldsymbol{i}(u'v'' - u''v').$$

This takes values in $2\pi\boldsymbol{i}\,\mathbb{R}$ and takes values in $2\pi\boldsymbol{i}\mathbb{Z}$ on $\Lambda\times\Lambda$. Since, in our cases, the canonical divisor of $\mathscr{C}$ is linearly equivalent to $(2g-2)\cdot\infty$, we define the vector that corresponds to the Riemann constant of $\mathscr{C}$

$$\delta = \omega'\delta' + \omega''\delta'' \in \tfrac{1}{2}\Lambda.$$

Moreover, we define for any $\ell \in \Lambda$

$$\chi(\ell) = \exp\left(2\pi\boldsymbol{i}\,(\delta''\ell' - \delta'\ell + \tfrac{1}{2}\ell'\ell'')\right).$$

Now, we define the σ function for $\mathscr{C}$.

Proposition 2.5. (Characterization of the σ-function) *Using the above notation, there exists a non-zero entire function $\sigma(u)$ on the space $\mathbb{C}^g$ which satisfies*

$$\sigma(u+\ell) = \chi(\ell)\,\sigma(u)\,\exp L(u+\tfrac{1}{2}\ell,\ell) \quad (u\in\mathbb{C}^g,\ \ell\in\Lambda). \tag{2.6}$$

Such a function $\sigma(u)$ is unique up to a non-zero constant factor. (The solution space of (2.6) *is of dimension one over $\mathbb{C}$.) The solution space is independent of the choice of the $\{\alpha_j\}$ and the $\{\beta_j\}$. The function $u\mapsto\sigma(u)$ has zeroes along $\kappa^{-1}(\Theta^{[g-1]})$ of order 1 and has no zeros elsewhere.*

By using the corresponding weight as subscripts for the coordinates in the variable $u\in\mathbb{C}^g$, where $\mathrm{wt}(u_j) = +j$, we can write $\sigma(u)$ explicitly as follows:

$$\sigma(u) = \begin{cases} \sigma(u_3,u_1) & \text{for } \mathscr{C}_{2,5} \\ \sigma(u_5,u_3,u_1) & \text{for } \mathscr{C}_{2,7} \\ \sigma(u_5,u_2,u_1) & \text{for } \mathscr{C}_{3,4}. \end{cases}$$

Note that this is a different notation than [2, 9] and [8], in those papers the variables are ordered by the natural numbers and σ is written $\sigma(u_1,u_2,\dots)$. Note also that the subscripts i for the u_i denotes a *positive* weights, whereas the subscripts j for the μ_j denote *negative* weights.

We denote by $\mathbb{Q}[\boldsymbol{\mu}][[u_3,u_1]]$, $\mathbb{Q}[\boldsymbol{\mu}][[u_5,u_3,u_1]]$, and $\mathbb{Q}[\boldsymbol{\mu}][[u_5,u_2,u_1]]$, the ring of formal power series (with coefficients in the ring of polynomials in the coefficients μ_j's) with respect to the coordinates in u, for the curves $\mathscr{C}_{2,5}$, $\mathscr{C}_{2,7}$, and $\mathscr{C}_{3,4}$, respectively. It is known that the σ function has following property:

Lemma 2.7. *Up to a multiplicative constant, the function $\sigma(u)$ has following power series expansion around the origin.*
(i) *For the curve $\mathscr{C}_{2,5}$, we have*

$$\sigma(u_3,u_1) = u_3 - \tfrac{1}{3}{u_1}^3 + \text{“}\textit{higher terms in } \mathbb{Q}[\boldsymbol{\mu_j}][[u_3,u_1]]\text{”}.$$

(ii) *For the curve* $\mathscr{C}_{2,7}$*, we have*

$$\sigma(u_5, u_3, u_1) = u_5 u_1 - {u_3}^2 + \tfrac{1}{3} u_2 {u_1}^3 + \tfrac{1}{45} {u_1}^6 + \text{“}\textit{higher terms in } \mathbb{Q}[\boldsymbol{\mu_j}][[u_5, u_3, u_1]]\text{”}.$$

(iii) *For the curve* $\mathscr{C}_{3,4}$*, we have*

$$\sigma(u_5, u_2, u_1) = u_5 - u_1\, {u_2}^2 + \tfrac{1}{20}\, {u_1}^5 + \text{“}\textit{higher terms in } \mathbb{Q}[\boldsymbol{\mu_j}][[u_5, u_2, u_1]]\text{”}.$$

These leading terms are called the Schur-Weierstrass polynomials.

Definition 2.8. *We define multivariate* $\wp$-functions *by*

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j}\sigma(u), \quad \wp_{ijk}(u) = -\frac{\partial}{\partial u_k}\wp_{ij}(u),$$

and similarly for higher derivatives.

Remark 2.9. Taking $\mathscr{C}_{2,5}$ as an example, we mention here other expressions of $\wp_{ij}(u)$ and $\wp_{ijk}(u)$. For any $u = (u_3, u_1) \in \mathbb{C}^2 - \Lambda$, we have unique pair of points $\{(x_1, y_1), (x_2, y_2)\}$ on $\mathscr{C}_{2,5}$ satisfying

$$(u_1, u_3) = \left(\int_{\infty}^{(x_1,y_1)} + \int_{\infty}^{(x_2,y_2)} \right)(\omega_1, \omega_2)$$

for some choice of integration path. Then, using this correspondence between u and $\{(x_1, y_1), (x_2, y_2)\}$, we have

$$\begin{aligned}
\wp_{33}(u) &= \big((x_1 + x_2)(x_1x_2)^2 + 2\mu_2(x_1x_2)^2 + \mu_2(x_1 + x_2)x_1x_2 \\
&\qquad + 2\mu_6 x_1x_2 + \mu_8(x_1 + x_2) + 2\mu_{10} - 2y_1y_2\big)\frac{1}{(x_1 - x_2)^2}, \\
\wp_{13}(u) &= -x_1x_2, \\
\wp_{11}(u) &= x_1 + x_2, \\
\wp_{333}(u) &= 2\,\frac{y_2\,\psi(x_1, x_2) - y_1\,\psi(x_2, x_1)}{(x_1 - x_2)^3}, \text{ where} \\
\psi(x_1, x_2) &= 4\mu_{10} + \mu_8(3x_1 + x_2) + 2\mu_6 x_1(x_1 + x_2) + \mu_4 {x_1}^2(x_1 + 3x_2) \\
&\qquad + 4\mu_2 {x_1}^3 x_2 + {x_1}^3 x_2(3x_1 + x_2), \\
\wp_{133}(u) &= 2\,\frac{y_1 {x_2}^2 - y_2 {x_1}^2}{x_1 - x_2}, \\
\wp_{113}(u) &= -2\,\frac{y_1x_2 - y_2x_1}{x_1 - x_2}, \\
\wp_{111}(u) &= 2\,\frac{y_1 - y_2}{x_1 - x_2}.
\end{aligned}$$

See [2] for details of this.

3 The theory for the $(2,5)$-curve

3.1 Coble's cubic hypersurface for the $(2,5)$-curve

For a divisor D on J, we denote by $\mathscr{L}(D)$ the space of the functions F on J such that $(F)+D$, where (F) is the divisor of F, is an effective divisor, In this Section we denote

$$\tilde{\wp}(u) = \wp_{11}(u)\wp_{33}(u) - \wp_{13}(u)^2.$$

Then we have the following.

Proposition 3.1. *The spaces $\mathscr{L}(n\Theta^{[1]}))$ for $n = 1,\ 2,\ 3$ have bases as follows:*

$$\begin{aligned}
\mathscr{L}(\Theta^{[1]}) &= \mathbb{C}1,\\
\mathscr{L}(2\Theta^{[1]}) &= \mathscr{L}(\Theta^{[1]}) \oplus \mathbb{C}\wp_{11}(u) \oplus \mathbb{C}\wp_{13}(u) \oplus \mathbb{C}\wp_{33}(u),\\
\mathscr{L}(3\Theta^{[1]}) &= \mathscr{L}(2\Theta^{[1]}) \oplus \mathbb{C}\wp_{111}(u) \oplus \mathbb{C}\wp_{113}(u) \oplus \mathbb{C}\wp_{133}(u) \oplus \mathbb{C}\wp_{333}(u) \oplus \mathbb{C}\tilde{\wp}(u).
\end{aligned}$$

Proof. It is easily checked that the functions in the right hand sides are linearly independent by looking at first few terms of the power series expansion at the origin of $\sigma(u)^n$ times the functions for each corresponding n which are given by (2.7). Then, using the Riemann-Roch theorem for Abelian varieties ([11])

$$\dim_{\mathbb{C}} \mathscr{L}(n\Theta^{[1]}) = n^2 \quad (n = 1,\ 2,\ \cdots),$$

the formulae are proved. □

Remark 3.2. Defining a Hermitian quadratic form by

$$H(u,v) = \tfrac{1}{\pi}\left(E(\boldsymbol{i}u, v) + \boldsymbol{i}\, E(u,v)\right),$$

using

$$\chi(\ell + k) = \chi(\ell)\chi(k) \cdot \exp \tfrac{1}{2}\, E(\ell, k),$$

we consider the line bundle

$$L(H, \chi).$$

This is defined on pg. 20 of [11]. On the other hand, we consider for $\ell \in \Lambda$ the map defined by

$$\mathbb{C} \times \mathbb{C}^2 \ni (z,u) \longmapsto \left(z\,\chi(\ell) \exp L(u + \tfrac{1}{2}\ell, \ell), u + \ell\right)$$

The first space corresponds value space of functions on the second space $\mathbb{C}^2$, and the second space is the space of variable $u = (u_3, u_1)$. The quotient space of this action gives rise to a line bundle on J, which is equivalent to $L(H,\chi)$. In [3], Beauville formulated Coble's theory using basically the cohomology $H^0\big(J, L(H,\chi)^{\otimes 3}\big)$. In our result this space is concretely described as follows.

We set up a projective space of dimension 8 with coordinates corresponding to the 9 functions appearing in (3.1). The locus of these 9 functions is none other than the Jacobian

variety of $\mathscr{C}$. The projective space $\mathbb{P}$ is defined by

$$\mathbb{P} = \big\{ [\, X_0 \,:\, X_2 \,:\, X_4 \,:\, X_6 \,:\, X_3 \,:\, X_5 \,:\, X_7 \,:\, X_9 \,:\, X_8 \,] \\ \big|\ X_j \in \mathbb{C} \ \text{ for } j = 0,\, 2 \leqq j \leqq 9 \,\big\}.$$

In our situation, any base ring is acceptable. The X_i are labeled by their weights according to the correspondences

$$X_2 \longleftrightarrow \wp_{11}(u), \quad X_4 \longleftrightarrow \wp_{13}(u), \quad X_6 \longleftrightarrow \wp_{33}(u), \quad X_3 \longleftrightarrow \wp_{111}(u),$$
$$X_5 \longleftrightarrow \wp_{113}(u), \quad X_7 \longleftrightarrow \wp_{133}(u), \quad X_9 \longleftrightarrow \wp_{333}(u), \quad X_8 \longleftrightarrow \tfrac{1}{2}(\tilde{\wp}(u) + \mu_4 \wp_{13}(u) - \mu_8),$$

Note that we have modified $\tilde{\wp}(u)$ in the definition of X_8 to remain consistent with the notation of Grant [9]. Other *linear* modifications are also possible. The parity of j in X_j coincides with the parity of the corresponding $\wp$-function. The image of the embedding

$$\mathbb{C}^2 \ni u \longmapsto [\, \sigma^3(u) \,:\, \sigma^3(u)\wp_{11}(u) \,:\, \sigma^3(u)\wp_{13}(u) \,:\, \sigma^3(u)\wp_{33}(u) \,:\, \sigma^3(u)\wp_{111}(u) : \\ \sigma^3(u)\wp_{113}(u) \,:\, \sigma^3(u)\wp_{133}(u) \,:\, \sigma^3(u)\wp_{333}(u) : \sigma^3(u)\tfrac{1}{2}(\tilde{\wp}(u) + \mu_4\wp_{13}(u) - \mu_8)\,] \in \mathbb{P}$$

is the Jacobian variety J of $\mathscr{C}_{2,5}$. The restriction of this map on $\kappa^{-1}(\Theta^{[1]})$ ($\subset \mathbb{C}^2$) is the image of $\Theta^{[1]}$, which is also denoted by $\Theta^{[1]}$. The space V spanned by the 9 functions

$$\begin{aligned} &\sigma^3(u)\,,\ \sigma^3(u)\wp_{11}(u)\,,\ \sigma^3(u)\wp_{13}(u)\,,\ \sigma^3(u)\wp_{33}(u)\,,\ \sigma^3(u)\wp_{111}(u)\,, \\ &\qquad \sigma^3(u)\wp_{113}(u)\,,\ \sigma^3(u)\wp_{133}(u)\,,\ \sigma^3(u)\wp_{333}(u)\,,\ \sigma^3(u)\tilde{\wp}(u) \end{aligned} \tag{3.3}$$

coincides with the space of entire functions $\varphi(u)$ satisfying

$$\varphi(u+\ell) = \chi(\ell)\,\varphi(u)\exp L(3u + \tfrac{3}{2}\ell, \ell) \qquad \text{for any} \quad \ell \in \Lambda.$$

Here $L(\ ,\ )$ is the bilinear form defined by (2.3).

For the 3-torsion subgroup $h \in J[3]$ of J, we have

$$\begin{aligned} \varphi(u+\ell+h) &= \chi(\ell)\,\varphi(u+h)\exp 3L(u+h+\tfrac{1}{2}\ell, \ell) \\ &= \chi(\ell)\,\varphi(u+h)\exp L(3u+3h+\tfrac{3}{2}\ell, \ell) \\ &= \chi(\ell)\,\varphi(u+h)\exp\big(L(3u+\tfrac{3}{2}\ell, \ell) + \exp L(3h, \ell)\big). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \varphi(u+\ell+h)\exp L(u+\ell, -3h) &= \varphi(u+\ell+h)\exp\big(L(u,-3h) - L(\ell, 3h)\big) \\ &= \chi(\ell)\varphi(u+h)\exp\big(L(3u+\tfrac{3}{2}\ell, \ell) + \exp L(3h, \ell)\big)\exp\big(L(u,-3h) - L(\ell, 3h)\big) \\ &= \chi(\ell)\varphi(u+h)\exp L(u,-3h)\exp L(3u+\tfrac{3}{2}\ell, \ell)\ \exp\big(L(3h, \ell) - L(\ell, 3h)\big) \\ &= \chi(\ell)\varphi(u+h)\exp L(u,-3h)\exp L(3u+\tfrac{3}{2}\ell, \ell)\ \exp E(3h, \ell) \\ &= \chi(\ell)\varphi(u+h)\exp L(u,-3h)\exp L(3u+\ell, \ell). \end{aligned}$$

Here $E(\ ,\ )$ is the Riemann form of $\mathscr{C}_{2,5}$ defined at (2.4). From this, the map for $h \in \frac{1}{3}\Lambda$

defined by

$$V \ni \varphi(u) \mapsto \varphi(u+h)\exp L(u,-3h)$$

is a linear transformation of V. Namely, for each $h \in \frac{1}{3}\Lambda$ and any $\varphi(u) \in V$, $\sigma(u+h)^3\,\varphi(u)\,\exp L(u,-3h)$ is expressed as a linear combination over $\mathbb{C}$, depending on h, of the 9 functions appeared in (3.3). If we denote, for simplicity,

$$\begin{aligned}\vec{\wp}\,(u) = [\,1 \;\; \wp_{11}(u) \;\; \wp_{13}(u) \;\; \wp_{33}(u) \;\; \wp_{111}(u) \;\; \wp_{113}(u) \\ \wp_{133}(u) \;\; \wp_{333}(u) \;\; \tfrac{1}{2}(\tilde{\wp}(u) + \mu_4\wp_{13}(u) - \mu_8)\,],\end{aligned}$$

there exists, for each $h \in J[3]$, a matrix $T(h)$ of size 9×9 independent of u such that

$$\sigma(u)^3 \exp L(u,-3h)\cdot \vec{\wp}\,(u+h) = \vec{\wp}\,(u)\,T(h).$$

We shall denote the projective coordinate corresponding to

$$(\,X_0\,,\,X_2\,,\,X_4\,,\,X_6\,,\,X_3\,,\,X_5\,,\,X_7\,,\,X_9\,,\,X_8\,)$$

by

$$\begin{aligned}&\mathrm{prpt}(\,X_0\,,X_2\,,\,X_4\,,\,X_6\,,\,X_3\,,\,X_5\,,\,X_7\,,\,X_9\,,\,X_8\,) \\ &\qquad = [\,X_0 : X_2 : X_4 : X_6 : X_3 : X_5 : X_7 : X_9 : X_8\,].\end{aligned}$$

By the linear transformation, we have projective transformation on $\mathbb{P}$

$$\begin{aligned}T(h) \;:\; \mathbb{P} \ni [\,X_0 : X_2 : X_4 : X_6 : X_3 : X_5 : X_7 : X_9 : X_8\,] \\ \longmapsto \mathrm{prpt}\Big([\,X_0\,,\,X_2\,,\,X_4\,,\,X_6\,,\,X_3\,,\,X_5\,,\,X_7\,,\,X_9\,,\,X_8\,]\,T(h)\Big) \in \mathbb{P}.\end{aligned}$$

Here the bracket means a row vector with 9 entries which will be multiplied by $T(h)$. Since, for a given h, h and $h+\ell$ ($\ell \in \Lambda$) give the same transformation, we can regard the group of the 3-torsion points $J[3] \simeq \frac{1}{3}\Lambda/\Lambda$ of J acting on $\mathbb{P}$. Now we recall Coble's theorem.

Theorem 3.4. (Coble [5], pg. 357) *There exists a unique hypersurface, say* $\mathrm{Cb}(\mathscr{C}_{2,5})$, *in the* 8*-dimensional projective space* $\mathbb{P}$ *such that* $\mathrm{Cb}(\mathscr{C}_{2,5})$ *is stable under the action of* $\{T(h)\,|\,h \in J[3]\,\}$ *(i.e. any point in* $\mathrm{Cb}(\mathscr{C}_{2,5})$ *is transformed to itself); its singular locus is the image of the Jacobian variety* J.

The variety denoted by M_2^{18} in pg. 354 of Coble's paper [5] corresponds to our Jacobian variety J.

3.2 Defining equations of the Jacobian for the $(2,5)$-curve

Firstly, we recall a result of Grant [9].

Theorem 3.5. (D. Grant [9]) *The affine part $J-\Theta^{[1]}$ of J is defined using the coordinates X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9 of the previous Section as the following system of equations $f_1, \cdots, f_7$. Here each bracket [ ] at the right hand side indicates the weight of the equation.*

$$
\begin{aligned}
f_1 &= -\mu_8{\mu_4}^2 - \mu_6\mu_{10} + \mu_8\mu_6\mu_2 + (\mu_2\mu_8 - \mu_{10})X_6 - \mu_4(\mu_6\mu_2 - {\mu_4}^2)X_4 \\
&\quad + (\mu_8\mu_6 - \mu_4\mu_{10})X_2 + 2(\mu_6\mu_2 - {\mu_4}^2)X_8 + X_4X_6^2 + {X_2}^2X_4\mu_8 \\
&\quad - (\mu_4\mu_2 - \mu_6)X_4X_6 - (\mu_{10} + \mu_4\mu_6 - \mu_2\mu_8)X_2X_4 - (\mu_6\mu_2 - {\mu_4}^2)X_2X_6 \\
&\quad + \mu_4X_2X_4X_6 - \mu_{10}{X_2}^3 - \mu_2\mu_{10}{X_2}^2 - 2\mu_4X_4X_8 + 2X_8\mu_6X_2 - \mu_6{X_2}^2X_6 \\
&\quad + 2\mu_2X_8X_6 + {X_8}^2 - \mu_2X_2{X_6}^2 \\
&= \begin{vmatrix} 2\mu_{10} & \mu_8 & -X_6 & -\frac{1}{2}X_4 \\ \mu_8 & 2(\mu_6+X_6) & \mu_4+X_4 & -\frac{1}{2}X_2 \\ -X_6 & \mu_4+X_4 & 2(\mu_2+X_2) & \frac{1}{2} \\ -\frac{1}{2}X_4 & -\frac{1}{2}X_2 & \frac{1}{2} & 0 \end{vmatrix} \quad \left(\begin{array}{l}\textit{This is the defin-}\\ \textit{ing polynomial of the}\\ \textit{Kummer surface}\end{array}\right) \quad [16] \\
f_2 &= 2X_8 - X_2X_6 + X_4^2 - \mu_4X_4 + \mu_8, \quad [8] \qquad f_3 = X_7 - X_3X_4 + X_5X_2, \quad [7] \\
f_4 &= X_9 + X_5X_4 + \mu_4X_5 + X_3X_6 - 2X_2X_7 - 2\mu_2X_7, \quad [9] \\
f_5 &= \mu_2\mu_8 - \mu_{10} + \mu_8X_2 - \mu_2\mu_4X_4 + 2\mu_2X_8 - \mu_2\mu_4X_2X_4 - X_2X_6 \\
&\quad + 2X_8X_2 + X_4X_6 + {X_5}^2 - {X_2}^2X_6, \quad [10] \\
f_6 &= {X_3}^2 - {X_2}^3 - X_6 - X_2X_4 - \mu_4X_2 - \mu_2{X_2}^2 - \mu_6, \quad [6] \\
f_7 &= -\mu_4X_4 + X_8 - \mu_2X_4X_2 + X_5X_3 - {X_2}^2X_4. \quad [8]
\end{aligned}
$$

Remark 3.6. (1) The equation $f_2 = 0$ is merely the definition of X_8 above.
(2) The equation $f_1 = 0$ defines Kummer's *quartic* surface (see [10]). The introduction of X_8 has enabled us to write it as a *cubic* equation ([9]).
(3) The Jacobian J is of course a 6-dimensional variety. The 6 equations $f_2, \cdots, f_7$ defines it since $f_1 \in \langle f_5,\ f_6,\ f_7\rangle$, Indeed,

$$
\begin{aligned}
f_1 = &-\tfrac{1}{4}(-2X_8 + 4{\mu_4}^2 + 3\mu_2{X_3}^2 - 3\mu_6\mu_2 + 2\mu_4X_4 - \mu_2X_2X_4 - 4\mu_6X_2 + 4X_2{X_3}^2 \\
&\qquad + {\mu_2}^2{X_2}^2 + \mu_2{X_2}^3 + \mu_2\mu_4X_2 - 3\mu_2X_6 - 2X_3X_5 - 2{X_2}^2X_4)f_2 \\
&+ \tfrac{1}{16}(-X_9 + 20\mu_6X_3 + 2X_2X_7 + 2\mu_2X_7 - X_4X_5 - \mu_4X_5 + 19X_3X_6 - 20{X_3}^3 \\
&\qquad + 20\mu_2{X_2}^2X_3 + 20{X_2}^3X_3 + 20\mu_4X_2X_3 + 20X_2X_3X_4)f_3 \\
&+ \tfrac{1}{16}(X_7 - X_3X_4 + X_2X_5)f_4 \\
&+ \tfrac{1}{8}(3X_2X_4 + 3\mu_4X_2 + 3\mu_2{X_2}^2 + 3{X_2}^3 + 3X_6 + 3\mu_6 + 5{X_3}^2)f_5 \\
&+ \tfrac{1}{8}(10X_3X_7 + 5\mu_{10} - 2\mu_2X_3X_5 - 5X_4X_6 + 3\mu_4X_2X_4 + \mu_2\mu_8 - 3{X_2}^2X_6 \\
&\qquad + 8\mu_2{X_2}^2X_4 + 4X_2X_3X_5 + 6{X_2}^3X_4 + 2{\mu_2}^2X_2X_4 + 3{X_5}^2 - 10{X_3}^2X_4 \\
&\qquad + 8X_2{X_4}^2 + 3\mu_8X_2 + \mu_2\mu_4X_4 - \mu_2X_2X_6 + 6\mu_2{X_4}^2)f_6 \\
&- \tfrac{1}{4}(2\mu_8 - \mu_2{X_3}^2 + \mu_6\mu_2 + 2\mu_4X_4 + X_2X_6 + 2{X_4}^2 + 5\mu_2X_2X_4 + 3\mu_6X_2 - 3X_2{X_3}^2 \\
&\qquad + {\mu_2}^2{X_2}^2 + 4\mu_2{X_2}^3 + 3\mu_4{X_2}^2 + 3{X_2}^4 + \mu_2\mu_4X_2 + \mu_2X_6 + 4X_3X_5 + 7{X_2}^2X_4)f_7.
\end{aligned}
$$

3.3 Main result for the $(2,5)$-Curve

Our main result for the genus two curve $\mathscr{C}_{2,5}$ is as follows:

Theorem 3.7. *Let us define*

$$\begin{aligned}S_{2,5} = {} & X_3X_6X_7 - X_3X_4X_9 + X_4X_5X_7 + X_2X_5X_9 - X_2X_6X_8 - {X_5}^2X_6 + {X_4}^2X_8 \\ & - X_4{X_6}^2 - X_2{X_7}^2 - \mu_8{X_2}^2X_4 + \mu_2{X_4}^2X_6 + \mu_{10}{X_2}^3 - \mu_4{X_4}^3 + \mu_6X_2{X_4}^2 \\ & + X_7X_9 + {X_8}^2 + \mu_4X_5X_7 + \mu_2\mu_{10}{X_2}^2 - \mu_{10}{X_3}^2 - \mu_2\mu_8X_2X_4 - \mu_2{X_7}^2 \\ & - \mu_6{X_5}^2 - \mu_4X_4X_8 + \mu_{10}X_2X_4 + \mu_2\mu_6{X_4}^2 - \mu_6X_4X_6 + \mu_8X_3X_5 \\ & + \mu_8X_8 + \mu_{10}X_6 + \mu_4\mu_{10}X_2 - \mu_4\mu_8X_4 + \mu_6\mu_{10}.\end{aligned}$$

Then the projectivization of the hypersurface defined by $S_{2,5} = 0$ gives the Coble's hypersurface $\mathrm{Cb}(\mathscr{C}_{2,5})$ *in the* 8*-dimensional projective space* $\mathbb{P}$.

This polynomial $S_{2,5}$ is obtained as follows. The key observation is that it must lie in the ideal $\langle f_2,\ f_3,\ f_4,\ f_5,\ f_6,\ f_7 \rangle$ and must be of constant (finite) weight. Once the weight is known, we can write the most general element of the ideal by adding products of general polynomials of suitable weight multiplied by the elements of the Groebner basis of the ideal. These general polynomials will have undetermined coefficients to be found. We then fix these constants by requiring that the derivatives of the general element with respect to the X_i vanish on the ideal. In order to guess the correct weight for $S_{2,5}$, we note that the minimum weight to satisfy the derivative conditions is 16, and indeed this turns out to be the weight required. The resulting calculation is straightforward using a suitable algebraic manipulation system, such as Maple.

We will check the partial derivatives indeed vanish on J. The polynomial $S_{2,5}$ is rewritten as follows:

$$\begin{aligned}S_{2,5} = {} & \tfrac{1}{4}(2X_8 - \mu_2{X_3}^2 + \mu_6\mu_2 - 2\mu_4X_4 - \mu_2X_4X_2 + {\mu_2}^2{X_2}^2 + \mu_2{X_2}^3 + \mu_2\mu_4X_2 \\ & \qquad + \mu_2X_6 + 2X_3X_5 - 2{X_2}^2X_4)f_2 \\ & + \tfrac{1}{16}(9X_9 + 12\mu_6X_3 - 2X_2X_7 - 2\mu_2X_7 + 9X_4X_5 + 9\mu_4X_5 + 21X_3X_6 \\ & \qquad - 12{X_3}^3 - 16\mu_2X_3X_4 + 12\mu_2{X_2}^2X_3 + 12{X_2}^3X_3 + 16\mu_2X_2X_5 \\ & \qquad + 12\mu_4X_2X_3 - 4X_3X_4X_2 + 16{X_2}^2X_5)f_3 \\ & + \tfrac{7}{16}(X_7 - X_3X_4 + X_2X_5)f_4 \\ & - \tfrac{3}{8}({X_2}^3 - {X_3}^2 + X_6 + X_2X_4 + \mu_4X_2 + \mu_2{X_2}^2 + \mu_6)f_5 \\ & - \tfrac{1}{8}(5\mu_{10} - 6X_7X_3 - 2\mu_2X_3X_5 - 5X_4X_6 + 3\mu_4X_2X_4 + \mu_2\mu_8 - 3{X_2}^2X_6 \\ & \qquad + 8\mu_2{X_2}^2X_4 - 12X_2X_3X_5 + 6{X_2}^3X_4 + 2{\mu_2}^2X_2X_4 - 5{X_5}^2 + 6{X_3}^2X_4 \\ & \qquad + 8X_2{X_4}^2 + 3\mu_8X_2 + \mu_2\mu_4X_4 - \mu_2X_2X_6 + 6\mu_2{X_4}^2)f_6 \\ & + \tfrac{1}{4}(2\mu_8 - \mu_2{X_3}^2 + \mu_2\mu_6 + 2\mu_4X_4 + X_2X_6 + 2{X_4}^2 + 5\mu_2X_2X_4 + 3\mu_6X_2 \\ & \qquad - 3{X_3}^2X_2 + {\mu_2}^2{X_2}^2 + 4\mu_2{X_2}^3 + 3\mu_4{X_2}^2 + 3{X_2}^4 + \mu_2\mu_4X_2 \\ & \qquad + \mu_2X_6 - 4X_5X_3 + 7{X_2}^2X_4)f_7.\end{aligned}$$

Hence,

$$S_{2,5} \in \langle f_2,\ f_3,\ f_4,\ f_5,\ f_6,\ f_7 \rangle.$$

The partial derivatives are given by

$$
\begin{aligned}
&\tfrac{\partial}{\partial X_9}S_{2,5} = f_3, \quad \tfrac{\partial}{\partial X_8}S_{2,5} = f_2, \quad \tfrac{\partial}{\partial X_7}S_{2,5} = f_4,\\
&\tfrac{\partial}{\partial X_6}S_{2,5} = (X_2+\mu_2)f_2 + X_4 f_6 - X_2 f_7 + X_3 f_3 - f_5,\\
&\tfrac{\partial}{\partial X_5}S_{2,5} = X_3 f_2 + (2{X_2}^2 + 2\mu_2 X_2 + \mu_4 + X_4)f_3 - 2X_3 f_7 + X_2 f_4 + 2X_5 f_6,\\
&\tfrac{\partial}{\partial X_4}S_{2,5} = -\mu_4 f_2 - (2X_2X_3 - X_5 + 2\mu_2 X_3)f_3 - X_3 f_4 - X_2 f_5\\
&\qquad - (2X_2X_4 - X_6 + 2\mu_2 X_4)f_6 + (2X_4 + \mu_4 + 2{X_2}^2 + 2\mu_2 X_2)f_7,\\
&\tfrac{\partial}{\partial X_3}S_{2,5} = -(2X_2X_3 - X_5 + 2\mu_2 X_3)f_2\\
&\qquad + (5X_6 + 4\mu_6 + 4\mu_2{X_2}^2 + 2X_2X_4 + 4\mu_4 X_2 + 4{X_2}^3 - 2\mu_2 X_4 - 4{X_3}^2)f_3\\
&\qquad - X_4 f_4 + 2X_3 f_5 + 4(X_7 - X_3X_4 + X_5X_2)f_6 - 2X_5 f_7,\\
&\tfrac{\partial}{\partial X_2}S_{2,5} = (2{\mu_2}^2 X_2 - 3\mu_6 - 3X_6 + 2\mu_2{X_2}^2 - 4X_2X_4 - 2\mu_4 X_2 + \mu_2\mu_4 + 3{X_3}^2)f_2\\
&\qquad + (3X_2X_5 - X_7 - X_3X_4 + 2\mu_2 X_5)f_3 + X_5 f_4 - (X_4 + \mu_4 + 2\mu_2 X_2 + 3{X_2}^2)f_5\\
&\qquad - (3\mu_8 + 3\mu_4 X_4 - 3X_2X_6 + 4{X_4}^2 + 6\mu_2 X_2 X_4 - 6X_3X_5 + 6{X_2}^2 X_4)f_6\\
&\qquad + (5X_6 + 6\mu_6 + 2\mu_2 X_4 + 6\mu_2{X_2}^2 + 10X_2X_4 + 6\mu_4 X_2 + 6{X_2}^3 - 6{X_3}^2)f_7.
\end{aligned}
$$

Moreover, the partial derivatives with respect to the coefficients are given by

$$
\begin{aligned}
&\tfrac{\partial}{\partial \mu_{10}}S_{2,5} = f_6, \quad \tfrac{\partial}{\partial \mu_8}S_{2,5} = f_7, \quad \tfrac{\partial}{\partial \mu_6}S_{2,5} = (X_2+\mu_2)f_2 - f_5,\\
&\tfrac{\partial}{\partial \mu_4}S_{2,5} = ({X_2}^2 + \mu_2 X_2 - X_4)f_2 + X_5 f_3 - X_2 f_5 + X_4 f_7,\\
&\tfrac{\partial}{\partial \mu_2}S_{2,5} = -(\mu_4 X_2 + X_6 + \mu_6 + 2X_2X_4 - {X_3}^2)f_2 + (X_5X_2 - X_7 - X_3X_4)f_3\\
&\qquad - {X_2}^2 f_5 - (\mu_8 + \mu_4 X_4 - X_2X_6 + 2{X_4}^2 + 2\mu_2 X_2 X_4 - 2X_3X_5 + 2{X_2}^2 X_4)f_6\\
&\qquad + 2(X_6 + \mu_6 + \mu_2{X_2}^2 + 2X_2X_4 + \mu_4 X_2 + {X_2}^3 - {X_3}^2)f_7.
\end{aligned}
$$

Note that $X_6 + \mu_6 + \mu_2{X_2}^2 + 2X_2X_4 + \mu_4 X_2 + {X_2}^3 - {X_3}^2 = X_2X_4 - f_6$.

Theorem 3.8. *The radical*

$$\sqrt{\left\langle \tfrac{\partial}{\partial X_j}S_{2,5} \,\middle|\, 2 \leqq j \leqq 9 \right\rangle}$$

is the defining ideal of the affine part of J.

Proof. From the proof of 3.7, we have

$$
\begin{aligned}
&\left[\tfrac{\partial}{\partial X_2}S_{2,5} \;\; \tfrac{\partial}{\partial X_3}S_{2,5} \;\; \tfrac{\partial}{\partial X_4}S_{2,5} \;\; \tfrac{\partial}{\partial X_5}S_{2,5} \;\; \tfrac{\partial}{\partial X_6}S_{2,5} \;\; \tfrac{\partial}{\partial X_7}S_{2,5} \;\; \tfrac{\partial}{\partial X_8}S_{2,5} \;\; \tfrac{\partial}{\partial X_9}S_{2,5}\right]\\
&= \left[f_2 \;\; f_3 \;\; f_4 \;\; f_5 \;\; f_6 \;\; f_7\right] {}^tM
\end{aligned}
$$

with some matrix M of size 8×6. Consider the points defined by

$$X_2 = 2, \quad X_3 = 1, \quad X_4 = 1, \quad X_5 = 1, \quad X_6 = 1, \quad \mu_2 = 1, \quad \mu_4 = 1, \quad \mu_6 = 1, \quad \mu_8 = 1$$

the minor obtained from M by removing the first and second rows, and the minor obtained from M by removing the first and third rows are are

$$-20{X_7}^2 + 2672X_7 - 281, \quad 144{X_7}^2 + 2760X_7 - 6480,$$

respectively. Their greatest common divisor is 1. Therefore, all the $\{\frac{\partial}{\partial X_j} S_{2,5} \,|\, 2 \leqq j \leqq 9\}$ are 0 if and only if all the $\{f_j \,|\, 2 \leqq j \leqq 7\}$ are 0. This completes the proof of the theorem. $\square$

From the results above, we have

Corollary 3.9. *The radical ideal of J is given by*

$$\left\langle \frac{\partial S_{2,5}}{\partial X_2}, \cdots, \frac{\partial S_{2,5}}{\partial X_9}, \frac{\partial S_{2,5}}{\partial \mu_{10}}, \frac{\partial S_{2,5}}{\partial \mu_8}, \frac{\partial S_{2,5}}{\partial \mu_6} \right\rangle \ \ (= \langle f_2, f_3, \cdots, f_7 \rangle).$$

4 The theory for the $(3,4)$-curve

We discuss here the curve

$$\mathscr{C}_{3,4} : y^3 + (\mu_2 x^2 + \mu_5 x + \mu_8) y = (x^4 + \mu_6 x^2 + \mu_9 x + \mu_{12}).$$

We use similar notation as in the previous Section. The Jacobian of $\mathscr{C}_{3,4}$ is denoted by J, for example. We shall omit the construction of the function $\sigma(u)$. which is written in [8], for instance. Note that in [8] we consider the more general $(3,4)$ curve

$$y^3 + (\mu_1 x + \mu_4) y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8) y = (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}). \tag{4.1}$$

In this paper we consider the Weierstrass form $\mathscr{C}_{3,4}$ in order to keep the results to a manageable size for display. Full results for the curve 4.1 are given at
`http://www.ma.hw.ac.uk/Weierstrass/Trig34/`.

Let us define

$$Q_{1115} = \wp_{1115} - 6\wp_{15}\wp_{11},$$

which belongs to $\Gamma(J, \mathscr{O}(2\Theta))$.

Lemma 4.2. *We have*

$$\Gamma(J, \mathscr{O}(2\Theta)) = \mathbb{C}1 \oplus \mathbb{C}\wp_{11} \oplus \mathbb{C}\wp_{12} \oplus \mathbb{C}\wp_{15} \oplus \mathbb{C}\wp_{22} \oplus \mathbb{C}\wp_{25} \oplus \mathbb{C}\wp_{55} \oplus \mathbb{C}Q_{1115}.$$

The functions above are even. We prepare projective coordinates

$$X_2,\ X_3,\ X_6,\ X_4,\ X_7,\ X_{10},\ X_8$$

corresponding to the last 7 functions with $\mathrm{wt}(X_j) = -j$. and one more coordinate X_0 corresponding to the constant function 1. We denote by $\mathbb{P}$ the projective space of dimension 7 with these coordinates. In this situation, the group of 2-torsion points $J[2]$ of J acts on $\mathbb{P}$ through the similarly defined matrix $T(h)$ with $h \in J[2]$ as for the curve $\mathscr{C}_{2.5}$ in the previous Section.

The PDEs satisfied by these variables are detailed in [8]. For example we have, in the present notation,

$$P_{111}^2 = {X_2}^3 - 4\,{X_2}^2\mu_2 - 4\,X_2X_4 + {X_3}^2 + 4\,X_6,$$

$$
\begin{aligned}
P_{111}P_{112} &= 4\,{X_2}^2X_3 - 2\,X_2X_3\mu_2 - 2\,X_2\mu_5 - X_3X_4 - 2\,X_7,\\
P_{112}^2 &= 4\,X_2{X_3}^2 - \frac{4}{3}\,X_6\mu_2 + 4\,X_2\mu_6 - \frac{4}{3}\,\mu_8 + {X_4}^2 - \frac{4}{3}\,X_8,\\
\ldots &= \ldots
\end{aligned}
$$

Note that the r.h.s. of these relations is at most cubic in the X_i.

Theorem 4.3. (Coble [6], p.106) *There exists unique quartic hypersurface, say* $\mathrm{Cb}(\mathscr{C}_{3,4})$, *in the space* $\mathbb{P}$ *such that* $\mathrm{Cb}(\mathscr{C}_{3,4})$ *is stable under the action of* $\{T(h)\,|\,h \in J[2]\}$ *and its singular locus is the image of* J. *The hypersurface* $\mathrm{Cb}(\mathscr{C}_{3,4})$ *is the Kummer variety of* J, *namely, it is the quotient variety of* J *by identifying points* P *and* $[-1]P$, *where* $[-1]$ *is the* (-1)*-multiplication on* J.

Now we explain how to get what we will call a *Kummer Relation* (KR), a polynomial identity involving double-pole functions ($\wp_{ij}$ and Q_{1115}). Most KRs are generated from cross products of quadratic 3-index $\wp_{ijk}$ relations. If

$$A = \wp_{ijk}, \quad B = \wp_{\ell mn}, \quad C = \wp_{opq}, \quad D = \wp_{rst},$$

Then

$$(AB)(CD) - (AC)(BD) = 0,$$

is a Kummer Relation, since each of the quadratic 3-index $\wp_{ijk}$ polynomials such as AB can be written as cubics in the 2-pole functions. More specifically such cross-products can be divided into three classes:

$$
\begin{aligned}
&(A^2)(B^2) - (AB)^2 = 0, && A \neq B,\\
&(A^2)(BC) - (AB)(AC) = 0, && A, B, C, \text{ all different},\\
&\begin{cases} (AB)(CD) - (AC)(BD) &= 0,\\ (AB)(CD) - (AD)(BC) &= 0, \end{cases} && A, B, C, D \text{ all different}.
\end{aligned}
$$

We see that KRs formed in this way can be at most sextic. Some KRs can be reduced to quartics or lower by adding suitable multiples of other KRs.

In addition to these KRs, we can find one other polynomial identities in the X_j which *cannot* be derived from this type of formula. This difference follows from a simple weight argument, and from examination of the Gröbner base representations. How do we find these *extra* relations? We take a known bilinear $\wp_{ijk}\wp_{\ell m}$ relation and multiply by a suitable $\wp_{nop}$. After substituting for all known quadratic 3-index products, what we have left should be a known KR derived from a cross-product equation, or something new.

In the (3,4) case, the first "standard" KR is

$$K_{14}^{(3,4)} \equiv \wp_{111}^2\wp_{112}^2 - (\wp_{111}\wp_{112})^2 = 0,$$

which gives a quartic with weight -14. The lowest weight relation of this type is

$$\wp_{555}^2\wp_{255}^2 - (\wp_{555}\wp_{255})^2 = 0,$$

of weight -54. In addition, there is one "special" KR which cannot be generated from the quadratic 3-index $\wp_{ijk}$ equations. This extra relation, which is vital to our calculations, is of weight -12. In the present case we have

$$\begin{aligned}K_{12} \equiv & -3\,X_2{\mu_2}^2{X_3}^2 + 12\,\mu_6{X_2}^3 + 12\,\mu_5{X_2}^2X_3 + 3\,{X_2}^2{X_4}^2 - 4\,{X_2}^2X_6\mu_2 - 12\,\mu_6{X_2}^2\mu_2 \\ & - 6\,X_4X_2{X_3}^2 - 6\,\mu_5X_2\mu_2X_3 - 3\,\mu_2X_2{X_4}^2 + 4\,X_2{\mu_2}^2X_6 + 3\,{X_3}^4 - 3\,X_4\mu_2{X_3}^2 \\ & - 4\,{X_2}^2X_8 - 4\,{X_2}^2\mu_8 + 12\,X_3X_2X_7 - 12\,X_2\mu_6X_4 + 4\,X_2\mu_2X_8 - 2\,X_2\mu_8\mu_2 \\ & - 3\,X_2{\mu_5}^2 + 12\,X_6{X_3}^2 + 3\,{X_3}^2\mu_6 - 3\,X_4X_3\mu_5 - 3\,X_7X_3\mu_2 - 3\,{X_4}^3 - 6\,X_2X_{10} \\ & + 3\,X_3\mu_9 + 3\,X_4X_8 + 6\,{X_6}^2 + 6\,X_6\mu_6 - 3\,X_7\mu_5 + 6\,\mu_{12}.\end{aligned}$$

The first "standard" KR outlined above, K_{14}, is quintic, but can be reduced to a cubic in the X_i coordinates by the use of K_{12}

$$\begin{aligned}K_{14} \equiv & \,6\,\mu_8\mu_2{X_2}^2 - 3\,X_2\mu_2X_3X_7 + 4\,X_4X_2X_6\mu_2 - {X_3}^2X_6\mu_2 + 6\,{X_2}^2X_{10} - 3\,X_2X_3\mu_9 \\ & + X_2X_4X_8 + 4\,X_2\mu_8X_4 - 6\,X_2{X_6}^2 + 6\,X_2X_6\mu_6 - 3\,X_2X_7\mu_5 - {X_3}^2X_8 - \mu_8{X_3}^2 \\ & - 3\,X_4X_3X_7 + 3\,X_6{X_4}^2 - 4\,\mu_2{X_6}^2 - 6\,X_2\mu_{12} - 4\,X_6X_8 - 4\,\mu_8X_6 - 3\,{X_7}^2.\end{aligned}$$

The reduction of K_{14} from quintic to cubic is carried out by taking the normal form of the quintic with respect to K_{12} using graded reverse lexicographic ordering (`tdeg(X10,X8,X7,X6,X4,X3,X2)` in Maple). We can then calculate a Gröbner basis from $\{K_{12},\ K_{14}\}$ and use it to reduce the next element from our set of KR, i.e. K_{15}, etc. Proceeding in this way we build up a set of algebraically independent quartic KR of decreasing weight at weights -12, -14, -15, -16 (two equations), -17 and -18. Note that there may be more than one KR at a particular weight, here we find it necessary to use two of weight -16, but only one from those at weight -17, -18. Hence only 7 KRs are required — *all* the rest can be shown to belong to the ideal $\langle K_{12}, K_{14}, K_{15}, K_{16a}, K_{16b}, K_{17}, K_{18}\rangle$. There are a total of 825 KRs in all, not counting the special K_{12} given above.

We can then examine the Gröbner basis for this ideal, using the `tdeg` ordering as in the genus 2 case. The Gröbner basis consists of eight cubics and fourteen quartics, with weights ranging from -14 to -29. We now attempt to ascertain the weight of the corresponding Coble quartic. As in the genus two case, we argue that the cubics are formed by the derivatives of the Coble quartic by the variables X_i. Since the smallest weight cubic is of weight -14, and the largest weight X_i is of weight -10, this suggests the Coble quartic is of weight -24. We then build up a general quartic using the same techniques as for the cubic in genus two, and solve for the unknown coefficients by requiring that the quartic and its derivatives lie in the Kummer variety described above.

As in the $(2,5)$-curve, we can easily pass to a homogeneous form of the Coble quartic by introducing a homogenizing coordinate X_0. Since we are using a graded mononomial order in our Gröbner base calculations, we can work with homogenized versions of our Gröbner basis (see Cox et al. [7], Chapter 8, Section 4). The calculations for the projective version of the Coble surface using this basis go through with minor modifications to give the projective closure of the affine surface given by Theorem 4.4.

Our second main result is as follows:

Theorem 4.4. *The Coble hypersurface* $\mathrm{Cb}(\mathscr{C}_{3,4})$ *in* 4.3 *in the* 7*-dimensional projective space* $\mathbb{P}$ *is the projectivization of the hypersurface defined by* $S_{3,4} = 0$*, where*

$$
\begin{aligned}
S_{3,4} = {} & 4{X_8}^3 - 108{X_7}^2X_{10} - 144X_6X_8X_{10} + 108{X_6}^4 - 108X_4X_6{X_7}^2 \\
& - 36X_4{X_6}^2X_8 + 108{X_4}^2X_6X_{10} + 108X_3{X_7}^3 + 72X_3X_6X_7X_8 - 108X_3X_4X_7X_{10} \\
& - 36{X_3}^2X_8X_{10} - 36X_2{X_7}^2X_8 - 216X_2{X_6}^2X_{10} + 36X_2X_4X_8X_{10} + 108{X_2}^2{X_{10}}^2 \\
& + 36\,\mu_2{X_7}^2X_8 + 36\,\mu_2X_6{X_8}^2 - 144\,\mu_2{X_6}^2X_{10} - 144\,\mu_2X_4{X_6}^3 + 180\,\mu_2X_3{X_6}^2X_7 \\
& - 36\,\mu_2{X_3}^2X_6X_{10} - 36\,\mu_2X_2X_6{X_7}^2 + 144\,\mu_2X_2X_4X_6X_{10} - 108\,\mu_2X_2X_3X_7X_{10} \\
& + 36\,{\mu_2}^2X_6{X_7}^2 + 96\,{\mu_2}^2{X_6}^2X_8 - 108\,\mu_5{X_6}^2X_7 - 36\,\mu_5X_4X_7X_8 - 12\,\mu_5X_3{X_8}^2 \\
& - 108\,\mu_5X_2X_7X_{10} + 108\,\mu_5X_3X_4{X_6}^2 - 108\,\mu_5{X_3}^2X_6X_7 - 108\,\mu_5X_2X_4X_6X_7 \\
& + 108\,\mu_5X_2X_3{X_7}^2 + 216\,\mu_6{X_6}^3 - 108\,\mu_6X_4{X_7}^2 - 36\,\mu_6X_4X_6X_8 - 36\,\mu_6X_3X_7X_8 \\
& - 12\,\mu_6X_2{X_8}^2 + 216\,\mu_6X_2X_6X_{10} + 108\,\mu_6{X_3}^2{X_6}^2 - 216\,\mu_6X_2X_3X_6X_7 + 108\,\mu_6{X_2}^2{X_7}^2 \\
& + 64\,{\mu_2}^3{X_6}^3 - 36\,\mu_2\mu_5X_4X_6X_7 - 60\,\mu_2\mu_5X_3X_6X_8 - 12\,\mu_8{X_8}^2 - 144\,\mu_8X_6X_{10} \\
& - 144\,\mu_8X_4{X_6}^2 + 36\,\mu_8{X_4}^2X_8 + 288\,\mu_8X_3X_6X_7 - 36\,\mu_8{X_3}^2X_{10} - 36\,\mu_8X_2{X_7}^2 \\
& - 144\,\mu_8X_2X_6X_8 + 144\,\mu_8X_2X_4X_{10} - 144\,\mu_2\mu_6X_4{X_6}^2 - 36\,\mu_2\mu_6X_3X_6X_7 \\
& - 108\,\mu_8{X_3}^2X_4X_6 + 108\,\mu_8{X_3}^3X_7 - 108\,\mu_2\mu_6X_2{X_7}^2 - 96\,\mu_2\mu_6X_2X_6X_8 \\
& + 108\,\mu_8X_2{X_4}^2X_6 - 108\,\mu_8X_2X_3X_4X_7 - 36\,\mu_9X_7X_8 + 108\,\mu_9{X_4}^2X_7 - 324\,\mu_9X_3{X_6}^2 \\
& + 36\,\mu_9X_3X_4X_8 - 216\,\mu_9X_2X_6X_7 - 108\,\mu_9X_2X_3X_{10} - 108\,\mu_9{X_3}^3X_6 \\
& + 108\,\mu_9X_2X_3X_4X_6 + 108\,\mu_9X_2{X_3}^2X_7 - 108\,\mu_9{X_2}^2X_4X_7 - 48\,{\mu_2}^2\mu_5X_3{X_6}^2 \\
& - 72\,\mu_2\mu_8{X_7}^2 - 36\,{\mu_5}^2X_6X_8 - 48\,\mu_2\mu_8X_6X_8 + 144\,\mu_2\mu_8{X_4}^2X_6 - 108\,\mu_2\mu_8X_3X_4X_7 \\
& - 108\,{\mu_5}^2X_2{X_6}^2 - 360\,\mu_2\mu_8X_2{X_6}^2 + 36\,\mu_2\mu_8X_2X_4X_8 + 216\,\mu_2\mu_8{X_2}^2X_{10} \\
& - 192\,{\mu_2}^2\mu_6X_2{X_6}^2 - 108\,\mu_5\mu_6X_6X_7 + 72\,\mu_2\mu_9X_6X_7 + 144\,\mu_2\mu_9X_3X_4X_6 \\
& + 108\,\mu_2\mu_9X_2X_4X_7 + 36\,\mu_2\mu_9X_2X_3X_8 + 216\,\mu_{12}{X_6}^2 + 108\,\mu_{12}X_4X_8 - 216\,\mu_{12}X_2X_{10} \\
& + 108\,{\mu_6}^2{X_6}^2 - 108\,\mu_{12}{X_4}^3 + 432\,\mu_{12}{X_3}^2X_6 + 432\,\mu_{12}X_2X_3X_7 - 144\,\mu_{12}{X_2}^2X_8 \\
& - 36\,\mu_2{\mu_5}^2{X_6}^2 + 108\,\mu_{12}{X_3}^4 - 216\,\mu_{12}X_2{X_3}^2X_4 + 108\,\mu_{12}{X_2}^2{X_4}^2 + 144\,{\mu_2}^2\mu_8X_2X_4X_6 \\
& - 108\,{\mu_2}^2\mu_8X_2X_3X_7 - 36\,\mu_5\mu_8X_4X_7 + 12\,\mu_5\mu_8X_3X_8 + 216\,\mu_5\mu_8X_2X_3X_6 \\
& + 144\,{\mu_2}^2\mu_9X_2X_3X_6 - 144\,\mu_6\mu_8X_4X_6 + 108\,\mu_5\mu_9X_4X_6 + 72\,\mu_6\mu_8X_3X_7 \\
& - 108\,\mu_2\mu_{12}X_3X_7 + 48\,\mu_6\mu_8X_2X_8 - 36\,\mu_5\mu_9X_2X_8 + 144\,\mu_2\mu_{12}X_2X_8 - 108\,\mu_2\mu_{12}{X_3}^2X_4 \\
& - 108\,\mu_2\mu_{12}X_2{X_4}^2 + 432\,\mu_6\mu_8{X_2}^2X_6 - 216\,\mu_5\mu_9{X_2}^2X_6 - 144\,\mu_2\mu_{12}{X_2}^2X_6 \\
& - 108\,\mu_6\mu_9X_3X_6 - 24\,\mu_2\mu_5\mu_8X_3X_6 - 108\,\mu_2\mu_5\mu_8X_2X_7 + 36\,{\mu_8}^2{X_4}^2 - 144\,{\mu_8}^2X_2X_6 \\
& - 24\,\mu_2\mu_6\mu_8X_2X_6 + 72\,\mu_2\mu_5\mu_9X_2X_6 + 144\,{\mu_2}^2\mu_{12}X_2X_6 - 108\,{\mu_8}^2X_2{X_3}^2 \\
& - 108\,{\mu_2}^2\mu_{12}X_2{X_3}^2 + 72\,\mu_8\mu_9X_7 - 108\,\mu_5\mu_{12}X_7 + 36\,\mu_8\mu_9X_3X_4 - 108\,\mu_5\mu_{12}X_3X_4 \\
& - 216\,\mu_8\mu_9{X_2}^2X_3 + 432\,\mu_5\mu_{12}{X_2}^2X_3 - 108\,{\mu_9}^2X_6 + 216\,\mu_6\mu_{12}X_6 - 36\,{\mu_5}^2\mu_8X_6 \\
& - 48\,\mu_2{\mu_8}^2X_6 + 108\,\mu_6\mu_{12}{X_3}^2 + 108\,{\mu_9}^2X_2X_4 - 432\,\mu_6\mu_{12}X_2X_4 + 144\,\mu_2{\mu_8}^2X_2X_4 \\
& - 108\,{\mu_9}^2{X_2}^3 + 432\,\mu_6\mu_{12}{X_2}^3 + 36\,\mu_2\mu_8\mu_9X_2X_3 - 216\,\mu_2\mu_5\mu_{12}X_2X_3 - 144\,\mu_8\mu_{12}{X_2}^2 \\
& + 108\,\mu_2{\mu_9}^2{X_2}^2 - 432\,\mu_2\mu_6\mu_{12}{X_2}^2 + 108\,{\mu_2}^2{\mu_8}^2{X_2}^2 + 108\,\mu_9\mu_{12}X_3 + 24\,\mu_5{\mu_8}^2X_3 \\
& - 48\,\mu_6{\mu_8}^2X_2 + 72\,\mu_5\mu_8\mu_9X_2 - 108\,{\mu_5}^2\mu_{12}X_2 - 72\,\mu_2\mu_8\mu_{12}X_2 + 108\,{\mu_{12}}^2 + 16\,{\mu_8}^3.
\end{aligned}
$$

Proof. As in the genus two case, we are working with the `tdeg` ordering, so we can homogenize

all affine equations using the variable X_0, and return to the affine version by setting $X_0 = 1$. These transformations can be done before or after taking the Gröbner basis and give the same result. By construction, the affine version of the quartic above lies in the ideal $\langle K_{12}, K_{14}, K_{15}, K_{16a}, K_{16b}, K_{17}, K_{18}\rangle$ described above. Also the affine version of its partial first derivatives with respect to X_{10}, X_8, X_7, X_6, X_4, X_3, X_2, X_0 are cubics and lie in the same ideal. The first four correspond to the Kummer relations of weight -14, -16, -17, -18, derived earlier. In addition, the four partial first derivatives of the generating quartic with respect to μ_{12}, μ_9, μ_6, are *quartics* and lie in the same ideal. The first two correspond to the Kummer relations of weight -12, -15, derived earlier. Moreover we can generate an ideal from these derivative relations and it gives exactly the Kummer variety generated earlier (in fact the partial first derivative with respect to μ_6 is not required). Since the Coble quartic is unique, it must be the polynomial $S_{3,4}$. □

The calculation also goes through for the full $(3,4)$-curve

$$y^3 + (\mu_1 x + \mu_4)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y^2 = x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}$$

with the resulting quartic having 461 terms in total. The result is displayed at `http://www.ma.hw.ac.uk/Weierstrass/Trig34/`.

Remark 4.5. (1) *Alternative formulations for the* $(3,4)$*-Kummer.* Buchstaber, Enolski and Leykin [4] (BEL) have put forward a powerful general theory of trigonal curves which is in principle able to generate many of the PDEs involved in the theory for $g \geq 3$. However their approach appears to be restricted to terms involving only $\wp_{ig}$ or $\wp_{ijg}$ $(j \geq g-1)$ so it is not clear how the relation K_{12} would emerge from the theory. The variety generated by the Kummer relations in the BEL theory does not appear to be the same as the one discussed above. However it is related in an interesting way — if we eliminate the variables X_{10}, X_8, and X_7 from the two ideals using resultants, we get the *same* single equation of 1506 terms in X_2, X_3, X_4, X_6, of total degree 15.
(2) The authors do not know if the ideal generated by the partial derivatives is a radical ideal or not.

5 The theory for the $(2,7)$-curve

For the hyperelliptic curve $\mathscr{C}_{2,7}$ of genus 3, we are faced with a degenerate situation in contrast to the case of $\mathscr{C}_{3,4}$.

We define

$$\Delta(u) = \tfrac{1}{2}(\wp_{1155} - 4{\wp_{15}}^2 - 2\wp_{11}\wp_{55})(u).$$

We see easily that $\Delta(u) \in \mathscr{L}(2\Theta^{[2]})$. It is know by [1] that

$$\Delta(u) = (\wp_{13}\wp_{35} - \wp_{15}\wp_{33} + \wp_{15}^2 - \wp_{11}\wp_{55})(u).$$

Lemma 5.1. *We have*

$$\mathscr{L}(\Theta^{[2]})) = \mathbb{C}1,$$
$$\mathscr{L}(2\Theta^{[2]})) = \mathbb{C}1 \oplus \mathbb{C}\wp_{11} \oplus \mathbb{C}\wp_{13} \oplus \mathbb{C}\wp_{33} \oplus \mathbb{C}\wp_{15} \oplus \mathbb{C}\wp_{35} \oplus \mathbb{C}\wp_{55} \oplus \mathbb{C}\Delta.$$

This can be checked by the expansion $\sigma(u) = u_1u_5 - {u_3}^2 + \frac{1}{3}u_3{u_1}^3 + \frac{1}{45}{u_1}^6$.

As in the earlier Sections, we define the following coordinates:

$$\begin{aligned} &X_{12} \ \ (\leftrightarrow \Delta),\ X_{10} \ \ (\leftrightarrow \wp_{55}),\ X_8 \ \ (\leftrightarrow \wp_{35}),\ X_6 \ \ (\leftrightarrow \wp_{33}),\\ &Y_6 \ \ (\leftrightarrow \wp_{15}),\ X_4 \ \ (\leftrightarrow \wp_{13}),\ X_2 \ \ (\leftrightarrow \wp_{11}). \end{aligned} \tag{5.2}$$

In this case, Coble's theorem in 4.3 holds in degenerate situation, and there must exist a cubic or lower degree equation in the functions above. However, checking by Maple in the case with all μ_j being 0 for the basis in 5.1, there are neither cubic nor linear relations and there exists a quadratic relation and no other quadratic ones. The quadratic relation of the functions gives the hypersurface defined by

$$X_{12} + X_8X_4 - X_6Y_6 + Y_6^2 - X_{10}X_2 = 0 \tag{5.3}$$

which is given by rewriting Δ. But, by Coble's theorem, there must be a quartic equation in X_js whose partial derivative vanish on the image of the Jacobian J of $\mathscr{C}_{2,7}$. Therefore the Coble hypersurface in this case must be

$$(X_{12} + X_8X_4 - X_6Y_6 + Y_6^2 - X_{10}X_2)^2 = 0.$$

References

[1] H. F. Baker, *On a system of differential equations leading to periodic functions*, Acta Math. **27** (1903), no. 1, 135–156.

[2] ———, *Multiply periodic functions*, Cambridge Univ. Press, Cambridge, 1907.

[3] A. Beauville, *The Coble hypersurfaces*, Compt. Rendus Math. **337** (2003), no. 3, 189–194.

[4] V. M. Bukhshtaber, D. V. Leikin, and V. Z. Enolski, *Uniformization of Jacobi manifolds of trigonal curves, and nonlinear differential equations*, Functional Anal. **34** (2000), no. 3, 1–16, 96.

[5] A. B. Coble, *Point sets and allied Cremona groups. III*, Trans. Amer. Math. Soc. **18** (1917), no. 3, 331–372.

[6] ———, *Algebraic geometry and theta functions*, A. M. S. Colloquium Publications, vol. 10, A. M. S., 1929.

[7] David Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms*, 3rd ed., Springer-Verlag, New York, 2007.

[8] J. C. Eilbeck, V. Z. Enolski, S. Matsutani, Y. Ônishi, and E. Previato, *Abelian functions for trigonal curves of genus three*, Int. Math. Res. Not. **2008** (2008), no. 1, 102–139.

[9] D. Grant, *Formal groups in genus two*, J. Reine Angew. Math. **411** (1990), 96–121.

[10] R. W. H. T. Hudson, *Kummer's quartic surface*, Cambridge Univ. Press, 1905.

[11] D. Mumford, *Abelian varieties*, Oxford Univ. Press, 1970.