DETERMINANTAL EXPRESSIONS FOR HYPERELLIPTIC FUNCTIONS IN GENUS THREE

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1. *Introduction*

Let $\sigma(u)$ and $\wp(u)$ be the usual functions in the theory of elliptic functions. In the paper [**12**] the author gave a natural generalization to the case of genus two for the two formulae

$$(-1)^{(n-1)(n-2)/2} 1!2!\cdots(n-1)! \frac{\sigma(u^{(1)}+u^{(2)}+\cdots+u^{(n)})\prod_{i<j}\sigma(u^{(i)}-u^{(j)})}{\sigma(u^{(1)})^n\sigma(u^{(2)})^n\cdots\sigma(u^{(n)})^n}$$
$$= \begin{vmatrix} 1 & \wp(u^{(1)}) & \wp'(u^{(1)}) & \wp''(u^{(1)}) & \cdots & \wp^{(n-2)}(u^{(1)}) \\ 1 & \wp(u^{(2)}) & \wp'(u^{(2)}) & \wp''(u^{(2)}) & \cdots & \wp^{(n-2)}(u^{(2)}) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(u^{(n)}) & \wp'(u^{(n)}) & \wp''(u^{(n)}) & \cdots & \wp^{(n-2)}(u^{(n)}) \end{vmatrix} \tag{1.1}$$

discovered by Frobenius and Stickelberger [**8**], and

$$(-1)^{n-1}(1!2!\cdots(n-1)!)^2\frac{\sigma(nu)}{\sigma(u)^{n^2}} = \begin{vmatrix} \wp' & \wp'' & \cdots & \wp^{(n-1)} \\ \wp'' & \wp''' & \cdots & \wp^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \wp^{(n-1)} & \wp^{(n)} & \cdots & \wp^{(2n-3)} \end{vmatrix}(u) \tag{1.2}$$

found earlier than the first one in the paper of Kiepert [**10**].

If we set $y(u) = \frac{1}{2}\wp'(u)$ and $x(u) = \wp(u)$, then we have an equation $y(u)^2 = x(u)^3 + \cdots$, that is a defining equation of the elliptic curve to which the functions $\wp(u)$ and $\sigma(u)$ are attached. Here the complex number u and the coordinates $(x(u), y(u))$ correspond by the equality

$$u = \int_{\infty}^{(x(u),y(u))} \frac{dx}{2y}.$$

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Then (1.1) and (1.2) are easily rewritten as

$$
\begin{aligned}
&(-1)^{(n-1)(n-2)/2}\frac{\sigma(u^{(1)}+u^{(2)}+\cdots+u^{(n)})\prod_{i<j}\sigma(u^{(i)}-u^{(j)})}{\sigma(u^{(1)})^n\sigma(u^{(2)})^n\cdots\sigma(u^{(n)})^n}\\
&\qquad=\begin{vmatrix}
1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & yx(u^{(1)}) & x^3(u^{(1)}) & \cdots\\
1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & yx(u^{(2)}) & x^3(u^{(2)}) & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\\
1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & yx(u^{(n)}) & x^3(u^{(n)}) & \cdots
\end{vmatrix}
\end{aligned}
\tag{1.3}
$$

and

$$
\begin{aligned}
&(-1)^{n-1}1!2!\cdots(n-1)!\frac{\sigma(nu)}{\sigma(u)^{n^2}}\\
&\qquad=\begin{vmatrix}
x' & y' & (x^2)' & (yx)' & (x^3)' & \cdots\\
x'' & y'' & (x^2)'' & (yx)'' & (x^3)'' & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots\\
x^{(n-1)} & y^{(n-1)} & (x^2)^{(n-1)} & (yx)^{(n-1)} & (x^3)^{(n-1)} & \cdots
\end{vmatrix}(u),
\end{aligned}
\tag{1.4}
$$

respectively. The results in [**12**] should be regarded as a generalization of the formulae (1.3) and (1.4) rather than (1.1) and (1.2).

The aim of this paper is to give a quite natural generalization of (1.3) and (1.4) and the results in [**12**] to the case of genus three (see Theorem 3.2 and Theorem 4.2). While our formulae for the case of genus two is attributed in [**6**] to an addition formula known as Schottky-Klein, it is not so easy to transform Fay's famous generalization of Shottky-Klein, namely (44) in p.33 of [**7**], into our formulae presented in this paper. This paper was written before the recent paper [**13**] which extends our line to all of hyperelliptic curves. Although the results of this paper is completely contained in that paper, the author believes that this paper is worthwhile because of its explicitness.

Now we prepare the minimal fundamentals to explain our results. Let $f(x)$ be a monic polynomial of x of degree 7 over the field of complex numbers $\mathbf{C}$. Assume that $f(x)=0$ has no multiple roots. Let C be the hyperelliptic curve defined by $y^2=f(x)$. Then C is of genus 3 and it is ramified over the x-line at infinity. We denote by ∞ the unique point at infinity on C. We regard $\mathbf{C}^3$ as the space of all values of the integrals, with their initial points ∞, of the first kind with respect to a chosen ordered basis $dx/2y$, $xdx/2y$, $x^2dx/2y$ for the differentials of the first kind. Let $\Lambda\subset\mathbf{C}^3$ be the lattice of their periods. So $\mathbf{C}^3/\Lambda$ is the Jacobian variety of C. We have an embedding $\iota: C\hookrightarrow\mathbf{C}^3/\Lambda$ defined by $P\mapsto(\int_\infty^P\frac{dx}{2y},\ \int_\infty^P\frac{xdx}{2y},\ \int_\infty^P\frac{x^2dx}{2y})$. Therefore $\iota(\infty)=(0,0,0)\in\mathbf{C}^3/\Lambda$. We also have a canonical projection $\kappa:\mathbf{C}^3\to\mathbf{C}^3/\Lambda$. An algebraic function on C, which we call a *hyperelliptic function* in this article, is regarded as a function on a universal Abelian covering $\kappa^{-1}\iota(C)$ $(\subset\mathbf{C}^3)$ of C. If $u=(u_1,u_2,u_3)$ is in $\kappa^{-1}\iota(C)$, we denote by $(x(u),y(u))$ the coordinate of the corresponding point on C by

$$u_1 = \int_\infty^{(x(u),y(u))} \frac{dx}{2y}, \quad u_2 = \int_\infty^{(x(u),y(u))} \frac{xdx}{2y}, \quad u_3 = \int_\infty^{(x(u),y(u))} \frac{x^2dx}{2y}$$

with appropriate choice of a path for the integrals.

The most important point of our approach is that we consider $u = (u_1, u_2, u_3)$ not as a variable on $\mathbf{C}^3$ but as a set of dependent variables on $\kappa^{-1}\iota(C)$. More concretely, our generalization of (1.4) is obtained by replacing the sequence of functions in the right hand side by the sequence

$$1,\ x(u),\ x^2(u),\ x^3(u),\ y(u),\ x^4(u),\ yx(u),\ \cdots,$$

consisting of the monomials of $x(u)$ and $y(u)$ displayed in ascending order according to the order of their poles at $u = (0,0,0)$, and by replacing the derivatives with respect to $u \in \mathbf{C}$ by those with respect to u_1 along $\kappa^{-1}\iota(C)$; and then replacing the left hand side of (1.4) by

$$1!2!\cdots(n-1)!\sigma(nu)/\sigma_2(u)^{n^2},$$

where $\sigma(u) = \sigma(u_1, u_2, u_3)$ is an exponential function times a Riemann theta series and $\sigma_2(u) = (\partial\sigma/\partial u_2)(u)$. Therefore, the hyperelliptic function that is the right hand side of the generalization of (1.4) can be naturally extended to a function on $\mathbf{C}^3$ via theta functions. We should note that while the right hand side is no more than a function on $\kappa^{-1}\iota(C)$, the left hand side of this generalization of (1.4) is a function on the whole of $\mathbf{C}^3$. The main result of this paper is to show that the left hand side of the expected generalization of (1.3) should be

$$\frac{\sigma(u^{(1)} + u^{(2)} + \cdots + u^{(n)}) \prod_{i<j} \sigma_3(u^{(i)} - u^{(j)})}{\sigma_2(u^{(1)})^n \sigma_2(u^{(2)})^n \cdots \sigma_2(u^{(n)})^n},$$

where $u^{(j)} = (u_1^{(j)}, u_2^{(j)}, u_3^{(j)})$ are variables on $\kappa^{-1}\iota(C)$ and $\sigma_3(u) = (\partial/\partial u_3)\sigma(u)$. We prove the formula, roughly speaking, by comparing the divisors of the two sides of Theorem 3.2. As the formula (1.4) is obtained by a limiting process from (1.3), our generalization of (1.4) is obtained by similar limiting process from the generalization of (1.3).

Although this paper is almost entirely based on [**12**], several critical facts differ in the genus three case. Sections 3 and 4 are devoted to generalizing (1.3) and (1.4), respectively. We recall in Section 2 the necessary facts for Sections 3 and 4.

We use the following notations throughout the rest of the paper. We denote, as usual, by $\mathbf{Z}$ and $\mathbf{C}$ the ring of rational integers and the field of complex numbers, respectively. In an expression of the Laurent expansion of a function, the symbol $(d^\circ(z_1, z_2, \cdots, z_m) \geq n)$ stands for the terms of total degree at least n with respect to the given variables z_1, z_2, $\cdots$, z_m. When the variables or the least total degree are clear from the context, we simply denote them by $(d^\circ \geq n)$ or by dots "$\cdots$".

For cross references in this paper, we indicate a formula as (1.2), and each of Lemmas, Propositions, Theorems and Remarks also as 3.4.

2. *The Sigma Function in Genus Three*

In this Section we summarize the fundamental facts used in Sections 3 and 4. Detailed treatment of these facts are given in [**1**], [**2**] and [**3**] (see also Section 1 of [**11**]).

Let

$$f(x) = \lambda_0 x^7 + \lambda_1 x^6 + \lambda_2 x^5 + \lambda_3 x^4 + \lambda_4 x^3 + \lambda_5 x^2 + \lambda_6 x + \lambda_7,$$

where $\lambda_0, \dots, \lambda_7$ are fixed complex numbers. Assume that the roots of $f(x) = 0$ are different from each other. Let C be a smooth projective model of the hyperelliptic curve defined by $y^2 = f(x)$. Then the genus of C is 3. We denote by ∞ the unique point at infinity on C. In this paper we suppose that $\lambda_0 = 1$. The set of forms

$$\omega_1 = \frac{dx}{2y},\ \omega_2 = \frac{xdx}{2y},\ \omega_3 = \frac{x^2dx}{2y}$$

is a basis of the space of differential forms of the first kind. We fix generators α_1, α_2, α_3, β_1, β_2, and β_3 of the fundamental group of C such that their intersections are $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$, $\alpha_i \cdot \beta_j = \delta_{ij}$ for i, $j = 1, 2, 3$. If we set

$$\omega' = \begin{bmatrix} \int_{\alpha_1}\omega_1 & \int_{\alpha_2}\omega_1 & \int_{\alpha_3}\omega_1 \\ \int_{\alpha_1}\omega_2 & \int_{\alpha_2}\omega_2 & \int_{\alpha_3}\omega_2 \\ \int_{\alpha_1}\omega_3 & \int_{\alpha_2}\omega_3 & \int_{\alpha_3}\omega_3 \end{bmatrix}, \quad \omega'' = \begin{bmatrix} \int_{\beta_1}\omega_1 & \int_{\beta_2}\omega_1 & \int_{\beta_3}\omega_1 \\ \int_{\beta_1}\omega_2 & \int_{\beta_2}\omega_2 & \int_{\beta_3}\omega_2 \\ \int_{\beta_1}\omega_3 & \int_{\beta_2}\omega_3 & \int_{\beta_3}\omega_3 \end{bmatrix},$$

the lattice of periods of our Abelian functions appearing below is given by

$$\Lambda = \omega' \begin{bmatrix} \mathbf{Z} \\ \mathbf{Z} \\ \mathbf{Z} \end{bmatrix} + \omega'' \begin{bmatrix} \mathbf{Z} \\ \mathbf{Z} \\ \mathbf{Z} \end{bmatrix} (\subset \mathbf{C}^3).$$

Let J be the Jacobian variety of the curve C. We identify J with the Picard group $\mathrm{Pic}^\circ(C)$ of linear equivalence classes of the divisors of degree 0 of C. Let $\mathrm{Sym}^3(C)$ be the symmetric product of three copies of C. Then we have a birational map

$$\begin{aligned} \mathrm{Sym}^3(C) &\to \mathrm{Pic}^\circ(C) = J \\ (P_1, P_2, P_3) &\mapsto \text{the class of } P_1 + P_2 + P_3 - 3 \cdot \infty. \end{aligned} \tag{2.1}$$

We may also identify (the $\mathbf{C}$-rational points of) J with $\mathbf{C}^3/\Lambda$. We denote by κ the canonical map $\mathbf{C}^3 \to \mathbf{C}^3/\Lambda$ and by ι the embedding of C into J given by mapping P to the class of $P - \infty$. The image of the triples of the form (P_1, P_2, ∞), by the birational map (2.1), is a theta divisor of J, and is denoted by Θ. The image $\iota(C)$ is obviously contained in Θ. We denote by O the origin of J. Obviously $\Lambda = \kappa^{-1}(O) = \kappa^{-1}\iota(\infty)$.

LEMMA 2.2. *As a subvariety of J, the divisor Θ is singular only at the origin of J.*

A proof of this fact is seen, for instance, in Lemma 1.7.2(2) of [**12**].

Let

$$\eta_1 = \frac{(\lambda_4 x + 2\lambda_3 x^2 + 3\lambda_2 x^3 + 4\lambda_1 x^4 + 5x^5)dx}{2y},$$
$$\eta_2 = \frac{(\lambda_2 x^2 + 2\lambda_1 x^3 + 3x^4)dx}{2y},$$
$$\eta_3 = \frac{x^3 dx}{2y}.$$

Then η_1, η_2, and η_3 are differential forms of the second kind without poles except at ∞ (see [**1**, p.195, Ex.i] or [**2**, p.314]). We also introduce matrices

$$\eta' = \begin{bmatrix} \int_{\alpha_1}\eta_1 & \int_{\alpha_2}\eta_1 & \int_{\alpha_3}\eta_1 \\ \int_{\alpha_1}\eta_2 & \int_{\alpha_2}\eta_2 & \int_{\alpha_3}\eta_2 \\ \int_{\alpha_1}\eta_3 & \int_{\alpha_2}\eta_3 & \int_{\alpha_3}\eta_3 \end{bmatrix}, \ \eta'' = \begin{bmatrix} \int_{\beta_1}\eta_1 & \int_{\beta_2}\eta_1 & \int_{\beta_3}\eta_1 \\ \int_{\beta_1}\eta_2 & \int_{\beta_2}\eta_2 & \int_{\beta_3}\eta_2 \\ \int_{\beta_1}\eta_3 & \int_{\beta_2}\eta_3 & \int_{\beta_3}\eta_3 \end{bmatrix}.$$

The modulus of C is $Z := {\omega'}^{-1}\omega''$. If we set

$$\delta'' = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \delta' = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix},$$

then the sigma function attached to C is defined, as in [**3**], by

$$\begin{aligned} \sigma(u) = c\exp(-\frac{1}{2}u\eta'{\omega'}^{-1}\,{}^t u) \\ \cdot \sum_{n\in\mathbf{Z}^3} \exp[2\pi\sqrt{-1}\{\frac{1}{2}\,{}^t(n+\delta'')Z(n+\delta'') + {}^t(n+\delta'')({\omega'}^{-1}\,{}^t u + \delta')\}] \end{aligned} \tag{2.3}$$

with a constant c. This constant c is fixed by the following lemma. This function $\sigma(u)$ is an even function ([**2**], p.359).

LEMMA 2.4. *The Taylor expansion of $\sigma(u)$ at $u = (0,0,0)$ is, up to a multiplicative constant, of the form*

$$\begin{aligned} u_1u_3 - {u_2}^2 - \frac{\lambda_7}{3}{u_1}^4 - \frac{\lambda_6}{3}{u_1}^3u_2 - \frac{\lambda_5}{2}{u_1}^2{u_2}^2 - \frac{\lambda_4}{3}u_1{u_2}^3 - \frac{\lambda_3}{3}{u_2}^4 + \frac{2\lambda_5}{3}{u_1}^3u_3 \\ - \frac{\lambda_2}{3}{u_2}^3u_3 - \frac{\lambda_1}{2}{u_2}^2{u_3}^2 + \frac{\lambda_1}{6}u_1{u_3}^3 - \frac{\lambda_0}{3}u_2{u_3}^3 + (d^\circ \geq 6), \quad (\lambda_0 = 1), \end{aligned}$$

with the coefficient of the term ${u_3}^6$ being $\frac{\lambda_0}{45}$.

Lemma 2.4 is proved in Proposition 2.1.1(3) of [**11**] by the same argument of [**11**], p.96. We fix the constant c in (2.3) such that the expansion is exactly of the form in 2.4.

Lemma 2.5. *Let ℓ be an element of Λ. The function $u \mapsto \sigma(u)$ on $\mathbf{C}^3$ satisfies the translational formula*

$$\sigma(u+\ell) = \chi(\ell)\sigma(u)\exp L(u+\tfrac{1}{2}\ell, \ell),$$

where $\chi(\ell) = \pm 1$ is independent of u, $L(u,\ v)$ is a form which is bilinear over the real field and $\mathbf{C}$-linear with respect to the first variable u, and $L(\ell^{(1)},\ \ell^{(2)})$ is $2\pi\sqrt{-1}$ times an integer if $\ell^{(1)}$ and $\ell^{(2)}$ are in Λ.

The detail of 2.5 is given in [**11**], p.286 and Lemma 3.1.2 of [**11**].

We remark here that the mapping $u \mapsto -u$ gives involutions of $\kappa^{-1}(\Theta)$ and of $\kappa^{-1}\iota(C)$.

Lemma 2.6. *Suppose $u \in \kappa^{-1}\iota(C)$.*
(1) *The function $\sigma(u)$ on $\mathbf{C}^3$ vanishes if and only if $u \in \kappa^{-1}(\Theta)$.*
(2) *Suppose that $v^{(1)}$, $v^{(2)}$, $v^{(3)}$ are three points of $\kappa^{-1}\iota(C)$. The function $u \mapsto \sigma(u - v^{(1)} - v^{(2)} - v^{(3)})$ is identically zero if and only if $v^{(1)} + v^{(2)} + v^{(3)}$ is contained in $\kappa^{-1}(C)$. If the function is not identically zero, it vanishes only at $u = v^{(j)}$ modulo Λ for $j = 1,\ 2,\ 3$ of order 1 or of multiple order according to the coincidence of some of the three points.*
(3) *Let v be a fixed point of $\kappa^{-1}\iota(C)$. There exist two points $v^{(1)}$ and $v^{(2)}$ of $\kappa^{-1}\iota(C)$ such that the function $u \mapsto \sigma(u - v - v^{(1)} - v^{(2)})$ on $\kappa^{-1}\iota(C)$ is not identically zero and vanishes at $u = v$ modulo Λ of order 1.*

Proof. The assertions 2.6(1) and (2) are proved in [**1**], pp.252-258, for instance. (This is essentially Riemann's vanishing theorem.) The assertion (3) obviously follows from (2). $\square$

We introduce the functions

$$\wp_{jk}(u) = -\frac{\partial^2}{\partial u_j \partial u_k}\log\sigma(u),\ \ \wp_{jk\cdots r}(u) = \frac{\partial}{\partial u_j}\wp_{k\cdots r}(u)$$

which are defined by Baker. Lemma 2.5 shows that these functions are periodic with respect to the lattice Λ. By 2.6(1) we know that the functions $\wp_{jk}(u)$ and $\wp_{jk\ell}(u)$ have their poles along Θ. We also use the notation

$$\sigma_j(u) = \frac{\partial}{\partial u_j}\sigma(u),\ \ \sigma_{jk\cdots r}(u) = \frac{\partial}{\partial u_j}\sigma_{k\cdots r}(u).$$

Let $u = (u_1, u_2, u_3)$ be an arbitrary point in $\mathbf{C}^3$. Then by the Abel-Jacobi theorem we can find a set of three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) on C such that

$$\begin{aligned} u_1 &= \int_\infty^{(x_1,y_1)} \omega_1 + \int_\infty^{(x_2,y_2)} \omega_1 + \int_\infty^{(x_3,y_3)} \omega_1, \\ u_2 &= \int_\infty^{(x_1,y_1)} \omega_2 + \int_\infty^{(x_2,y_2)} \omega_2 + \int_\infty^{(x_3,y_3)} \omega_2, \\ u_3 &= \int_\infty^{(x_1,y_1)} \omega_3 + \int_\infty^{(x_2,y_2)} \omega_3 + \int_\infty^{(x_3,y_3)} \omega_3 \end{aligned} \tag{2.7}$$

with certain choices for the three paths in the integrals. If (u_1, u_2, u_3) does not belongs to $\kappa^{-1}(\Theta)$, the set of the three points is uniquely determined. In this situation, one can show the following.

Lemma 2.8. *With the notation above, we have*

$$\wp_{13}(u) = x_1x_2x_3, \quad \wp_{23}(u) = -x_1x_2 - x_2x_3 - x_3x_1, \quad \wp_{33}(u) = x_1 + x_2 + x_3.$$

For a proof of this, see [**2**], p.377. This fact depends essentially on the choices we made for the forms ω_j and η_j.

Lemma 2.9. *If $u = (u_1, u_2, u_3)$ is on $\kappa^{-1}\iota(C)$, then we have*

$$u_1 = \frac{1}{5}{u_3}^5 + (d^\circ(u_3) \geq 6), \quad u_2 = \frac{1}{3}{u_3}^3 + (d^\circ(u_3) \geq 4)$$

in a neighborhood of $u_3 = 0$.

This is mentioned in [**11**], Lemma 2.3.2(2). If u is a point on $\kappa^{-1}\iota(C)$, the x- and y-coordinates of $\iota^{-1}\kappa(u)$ will be denoted by $x(u)$ and $y(u)$, respectively. As is shown, for instance, in Lemma 2.3.1 of [**11**], we see the following.

Lemma 2.10. *If $u \in \kappa^{-1}\iota(C)$ then*

$$x(u) = \frac{1}{{u_3}^2} + (d^\circ \geq 0), \quad y(u) = -\frac{1}{{u_3}^7} + (d^\circ \geq -5)$$

in a neighborhood of $u_3 = 0$.

Lemma 2.11. (1) *Let u be an arbitrary point on $\kappa^{-1}\iota(C)$. Then $\sigma_2(u)$ is 0 if and only if u belongs to $\kappa^{-1}(O)$.*
(2) *The Taylor expansion of the function $\sigma_2(u)$ on $\kappa^{-1}\iota(C)$ at $u = (0,0,0)$ is of the form*

$$\sigma_2(u) = -{u_3}^3 + (d^\circ(u_3) \geq 5).$$

Proof. For (1), assume that $u \in \kappa^{-1}\iota(C)$ and $u \notin \kappa^{-1}(O)$. Then with the notation of (2.7) we have

$$\frac{\sigma_1(u)}{\sigma_2(u)} = \frac{\wp_{13}(u)}{\wp_{23}(u)} = \frac{x_1x_2x_3}{-x_1x_2 - x_2x_3 - x_3x_1}\Big|_{x_1=x_2=\infty} = -x(u), \quad \frac{\sigma_3(u)}{\sigma_2(u)} = \frac{\wp_{33}(u)}{\wp_{23}(u)} = 0$$

by using 2.6(1) and 2.8. Hence it must be $\sigma_3(u) = 0$ by the second formula. If $\sigma_2(u) = 0$ then the first formula yields $\sigma_1(u) = 0$. This contradicts to 2.2, 2.6(1) and (2). So it must be $\sigma_2(u) \neq 0$. The assertion (2) follows from 2.4 and 2.9. □

LEMMA 2.12. *Let u be a point on $\kappa^{-1}(\Theta)$. The function $\sigma_3(u)$ vanishes if and only if $u \in \kappa^{-1}\iota(C)$.*

Proof. We have already proved in the proof of 2.10 that if $u \in \kappa^{-1}\iota(C)$ then $\sigma_3(u) = 0$. So we prove the converse. Assume that $u \in \kappa^{-1}(\Theta)$, $u \notin \kappa^{-1}\iota(C)$, and u corresponds to the triplet of points (x_1, y_1), (x_2, y_2), and $(x_3, y_3) = \infty$ by (2.7). Then we have

$$\frac{\sigma_1(u)}{\sigma_3(u)} = \frac{\wp_{13}(u)}{\wp_{33}(u)} = -x_1x_2, \quad \frac{\sigma_2(u)}{\sigma_3(u)} = \frac{\wp_{23}(u)}{\wp_{33}(u)} = -x_1 - x_2$$

by using 2.6(1) and 2.8. If $\sigma_3(u) = 0$, then the second formula says that $\sigma_2(u) = 0$, and the first one says that $\sigma_1(u) = 0$. This contradicts 2.2 by 2.6(1) and (2). So it must be that $\sigma_3(u) \neq 0$. □

LEMMA 2.13. *Let v be a fixed point in $\kappa^{-1}\iota(C)$ different from any point of $\kappa^{-1}(O)$. Then the function*

$$u \mapsto \sigma_3(u - v)$$

defined on $\kappa^{-1}\iota(C)$ vanishes to order 2 at $u = (0, 0, 0)$. Precisely, one has

$$\sigma_3(u - v) = \sigma_2(v){u_3}^2 + (d^\circ(u_3) \geq 3)$$

in a neighborhood of $u_3 = 0$.

Proof. Since $u - v$ is on Θ, we have $\sigma(u - v) = 0$. We assume that u corresponds the triplet (x_1, y_1), ∞, and ∞; and that v corresponds to the triplet (x_2, y_2), ∞, and ∞. Then 2.6(1), 2.8 and 2.10 imply that

$$\begin{aligned}
\frac{\sigma_3(u-v)}{\sigma_2(u-v)} &= \frac{{\sigma_3}^2 - \sigma_{33}\sigma}{\sigma_2\sigma_3 - \sigma_{23}\sigma}(u-v) \\
&= \frac{\wp_{33}}{\wp_{23}}(u-v) \\
&= -\left.\frac{x_1 + x_2 + x_3}{x_1x_2 + x_2x_3 + x_3x_1}\right|_{x_3=\infty} \\
&= -\frac{1}{x_1 + x_2} \\
&= -\frac{1}{\left(\frac{1}{{u_3}^2} + \cdots\right) + x_2} \\
&= -{u_3}^2 + \cdots .
\end{aligned}$$

Since $\sigma_2(-v) = -\sigma_2(v)$, the desired formula follows. □

LEMMA 2.14. *Let v be a fixed point in $\kappa^{-1}\iota(C)$ different from any points in $\kappa^{-1}(O)$. Then the function*

$$u \mapsto \sigma_3(u-v)$$

on $\kappa^{-1}\iota(C)$ has a zero of order 1 at $u=v$.

Proof. We denote by $\frac{du_j}{dx}$ the derivative of the function $u \mapsto u_j$ on $\kappa^{-1}\iota(C)$ by $x(u)$. Since

$$\frac{d(u_j-v_j)}{d(u_1-v_1)} = \frac{d(u_j-v_j)}{du_j}\frac{du_j}{du_1}\frac{du_1}{d(u_1-v_1)} = \frac{du_j}{du_1} = \frac{du_j}{dx}\frac{dx}{du_1} = x^{j-1}(u) \tag{2.15}$$

for $j=2$ and 3, we see

$$u_j - v_j = x^{j-1}(v)(u_1-v_1) + (d^\circ(u_1-v_1) \geq 2).$$

There exist two points $v^{(1)}$ and $v^{(2)}$ in $\kappa^{-1}\iota(C)$ such that the function $u \mapsto \sigma(u - v - v^{(1)} - v^{(2)})$ on $\kappa^{-1}\iota(C)$ is not identically zero and vanishes at $u=v$ of order 1 by 2.6(3). Let m be the vanishing order of the function $u \mapsto u_1 - v_1$. We show that $m=1$ as follows. Then the vanishing orders of $u \mapsto u_j - v_j$ ($j=2$, 3) are equal to or larger than m by (2.15). Furthermore the expansion

$$\begin{aligned}
&\sigma(u-v-v^{(1)}-v^{(2)}) \\
&= \sigma_1(-v^{(1)}-v^{(2)})(u_1-v_1) + \sigma_2(-v^{(1)}-v^{(2)})(u_2-v_2) + \sigma_3(-v^{(1)}-v^{(2)})(u_3-v_3) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + (d^\circ(u_1-v_1, u_2-v_2, u_3-v_3) \geq 2)
\end{aligned}$$

shows that the vanishing order of $u \mapsto \sigma(u-v-v^{(1)}-v^{(2)})$ is higher than or equal to m. Hence m must be 1. On the other hand, 2.4 and (2.15) imply that

$$\sigma_3(u-v) = (u_1-v_1) + (d^\circ(u_1-v_1) \geq 2).$$

Thus the statement follows. □

LEMMA 2.16. *If u is a point of $\kappa^{-1}\iota(C)$, then*

$$\frac{\sigma_3(2u)}{\sigma_2(u)^4} = -2y(u).$$

Proof. We first prove that the left hand side is a function on $\iota(C)$. The function $\sigma(2u)/\sigma(u)^4$ is a function on J by 2.5 and Liouville's theorem for functions of several variables (see for example [**12**], Prop. 3.2.2 or Lemma 4.4.1). For $u \notin \kappa^{-1}\iota(C)$, after multiplying

$$\frac{\wp_{333}(2u)}{\wp_{33}(2u)\wp_{22}(u)^2} = \frac{-2{\sigma_3}^3 + 3\sigma_3\sigma_{33}\sigma - \sigma_{333}\sigma^2}{{\sigma_3}^2 - \sigma_{33}\sigma}(2u) \cdot \left(\frac{\sigma^2}{{\sigma_2}^2 - \sigma_{22}\sigma}\right)^2(u)$$

by the function $\sigma(2u)/\sigma(u)^4$, bringing u first onto $\kappa^{-1}(\Theta)$ and then close to any point of $\kappa^{-1}\iota(C)$, we obtain the left hand side of the desired formula. Here we have used the fact that $u \mapsto \sigma_3(2u)$ does not vanish for generic u, which follows from 2.9. Thus the function $\sigma_3(2u)/\sigma_2(u)^4$ is a function on $\iota(C)$, that is

$$\frac{\sigma_3(2(u+\ell))}{\sigma_2(u+\ell)^4} = \frac{\sigma_3(2u)}{\sigma_2(u)^4}$$

for $u \in \kappa^{-1}(C)$ and $\ell \in \Lambda$. Lemma 2.11(1) states this function has its only pole at $u = (0,0,0)$ modulo Λ. Lemma 2.4 and 2.11(2) give that its Laurent expansion at $u = (0,0,0)$ is

$$\frac{2\left(\frac{1}{5}{u_3}^5\right) - \lambda_0 \cdot 2\left(\frac{1}{3}{u_3}^3\right)(2u_3)^2 + \frac{6\lambda_0}{45}(2u_3)^5 + \cdots}{(-{u_3}^3 + \cdots)^4} = \frac{2}{{u_3}^7} + \cdots .$$

Here we have used the assumption $\lambda_0 = 1$. Because this is an odd function, it must be $-2y(u)$ by 2.10 $\square$

Definition-Proposition 2.17. *Let n be a positive integer. If $u \in \kappa^{-1}\iota(C)$, then*

$$\psi_n(u) := \frac{\sigma(nu)}{\sigma_2(u)^{n^2}}$$

is periodic with respect to Λ. In other words it is a function on $\iota(C)$.

This is proved by a similar argument of 2.16. For details, see Proposition 3.2.2 in [**12**], p.396. By 2.11(2) the function $\psi_n(u)$ has its only pole at $u = (0,0,0)$ modulo Λ. Hence it is a polynomial in $x(u)$ and $y(u)$.

3. *A Generalization of the Formula of Frobenius and Stickelberger*

The following formula is a natural generalization of the corresponding formula for Weierstrass' functions $\sigma(u)$ and $\wp(u)$, that is (1.3) for $n = 1$.

Proposition 3.1. *If u and v are two points in $\kappa^{-1}\iota(C)$, then*

$$\frac{\sigma_3(u+v)\sigma_3(u-v)}{\sigma_2(u)^2\sigma_2(v)^2} = \begin{vmatrix} 1 & x(u) \\ 1 & x(v) \end{vmatrix}.$$

Proof. If we regard u to be a variable on $\mathbf{C}^3$, we see that the function

$$u \mapsto \frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2}$$

is a periodic function with respect to Λ as in the proof of 2.16. After multiplying

$$-\frac{1}{2}\frac{\frac{\wp_{333}}{\wp_{33}}(u+v)\frac{\wp_{333}}{\wp_{33}}(u-v)}{\wp_{22}(u)\wp_{22}(v)}$$

to the function above, bringing u and v close to points on $\kappa^{-1}\iota(C)$, we have the left hand side of the claimed formula because of $\sigma(u \pm v) = \sigma(u) = \sigma(v) = 0$ by 2.6(1) (or (2)) and also $\sigma_3(u) = \sigma_3(v) = 0$. So the left hand side as a function of u is periodic with respect to Λ. Now we compare divisors modulo Λ of the two sides. The left hand side has its only pole at $u = (0, 0, 0)$ modulo Λ by 2.11(1). The two zeroes modulo Λ of the two sides coincide by 2.12 (or 2.14). Lemmas 2.11(2) and 2.13 gives its Laurent expansion at $u = (0, 0, 0)$ as follows:

$$\frac{-\sigma_2(v)({u_3}^2+\cdots)\sigma_2(v)({u_3}^2+\cdots)}{(-{u_3}^3+\cdots)^2\sigma_2(v)^2} = -\frac{1}{{u_3}^2}+\cdots.$$

The leading term of this coincides with that of the right hand side by 2.10. Hence the desired formula holds for all v. $\square$

Our generalization of the formula (1.3) in the Introduction is the following.

THEOREM 3.2. *Let $n \geq 3$ be an integer. Assume that $u^{(1)}$, $u^{(2)}$, $\cdots$, $u^{(n)}$ belong to $\kappa^{-1}\iota(C)$. Then*

$$(-1)^{(n-2)(n-3)/2}\frac{\sigma(u^{(1)}+u^{(2)}+\cdots+u^{(n)})\prod_{i<j}\sigma_3(u^{(i)}-u^{(j)})}{\sigma_2(u^{(1)})^n\sigma_2(u^{(2)})^n\cdots\sigma_2(u^{(n)})^n}$$

is equal to

$$\begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & x^3(u^{(1)}) & y(u^{(1)}) & x^4(u^{(1)}) & yx(u^{(1)}) & \cdots & x^{(n+1)/2}(u^{(1)}) & yx^{(n-5)/2}(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & x^3(u^{(2)}) & y(u^{(2)}) & x^4(u^{(2)}) & yx(u^{(2)}) & \cdots & x^{(n+1)/2}(u^{(2)}) & yx^{(n-5)/2}(u^{(2)}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & x^3(u^{(n)}) & y(u^{(n)}) & x^4(u^{(n)}) & yx(u^{(n)}) & \cdots & x^{(n+1)/2}(u^{(n)}) & yx^{(n-5)/2}(u^{(n)}) \end{vmatrix}$$

or

$$\begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & x^3(u^{(1)}) & y(u^{(1)}) & x^4(u^{(1)}) & yx(u^{(1)}) & \cdots & yx^{(n-6)/2}(u^{(1)}) & x^{(n+2)/2}(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & x^3(u^{(2)}) & y(u^{(2)}) & x^4(u^{(2)}) & yx(u^{(2)}) & \cdots & yx^{(n-6)/2}(u^{(2)}) & x^{(n+2)/2}(u^{(2)}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & x^3(u^{(n)}) & y(u^{(n)}) & x^4(u^{(n)}) & yx(u^{(n)}) & \cdots & yx^{(n-6)/2}(u^{(n)}) & x^{(n+2)/2}(u^{(n)}) \end{vmatrix}$$

according as n is odd or even. Here both of the matrices are of size $n \times n$.

Proof. We prove this Theorem by induction on n. First of all we prove the case of $n = 3$, namely, by denoting $u^{(3)} = u$, the formula

$$\frac{\sigma(u^{(1)}+u^{(2)}+u)\sigma_3(u^{(1)}-u^{(2)})\sigma_3(u^{(1)}-u)\sigma_3(u^{(2)}-u)}{\sigma_2(u)^3\sigma_2(u^{(1)})^3\sigma_2(u^{(2)})^3} = \begin{vmatrix} 1 & x(u) & x^2(u) \\ 1 & x(u^{(1)}) & x^2(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) \end{vmatrix}$$

for $u^{(1)}$, $u^{(2)}$, and u in $\kappa^{-1}\iota(C)$. To prove this, we consider the function

$$u \mapsto \frac{\sigma(u^{(1)}+u^{(2)}+u)\sigma(u^{(1)}-u^{(2)})\sigma(u^{(1)}-u)\sigma(u^{(2)}-u)}{\sigma(u^{(1)})^3\sigma(u^{(2)})^3\sigma(u)^3}$$

on $\mathbf{C}^3$, where $u^{(1)}$ and $u^{(2)}$ are any points *not* on $\kappa^{-1}\iota(C)$. We see that this function of u is a periodic function with respect to the lattice Λ as in the proof of 2.16 and 3.1. After multiplying

$$\frac{\frac{\wp_{333}}{\wp_{33}}(u^{(1)}-u)\frac{\wp_{333}}{\wp_{33}}(u^{(2)}-u)\frac{\wp_{333}}{\wp_{33}}(u^{(1)}-u^{(2)})}{\frac{\wp_{222}}{\wp_{22}}(u)\frac{\wp_{222}}{\wp_{22}}(u^{(1)})\frac{\wp_{222}}{\wp_{22}}(u^{(2)})}$$

by the function above, by bringing $u^{(1)}$, $u^{(2)}$, and u close to points on $\kappa^{-1}\iota(C)$, we have the left hand side of the claimed formula. Here we have used the fact that $\sigma(u-u^{(1)})$, $\sigma(u-u^{(2)})$, and $\sigma(u^{(1)}-u^{(2)})$ vanish for u, $u^{(1)}$, and $u^{(2)}$ on $\kappa^{-1}\iota(C)$ by Lemma 2.6(2). So the left hand side as a function of u on $\kappa^{-1}\iota(C)$ is periodic with respect to Λ. Now we regard both sides to be functions of u on $\kappa^{-1}\iota(C)$. We see the left hand side has its only pole at $u=(0,0,0)$ modulo Λ by 2.11(1), and has its zeroes at $u=\pm u^{(1)}$ and $u=\pm u^{(2)}$ modulo Λ by 2.6(2), 2.12. All these zeroes are of order 1 by 2.6(2) and 2.14. Its Laurent expansion at $u=(0,0,0)$ is given by 2.11(2) and 2.13 and is as follows:

$$\frac{\sigma_3(u^{(1)}+u^{(2)})\sigma_2(u^{(1)})\sigma_2(u^{(2)})\sigma_3(u^{(1)}-u^{(2)})}{\sigma_2(u^{(1)})^3\sigma_2(u^{(2)})^3}(\frac{1}{{u_3}^4}+\cdots).$$

The right hand side has a leading term

$$\begin{vmatrix} 1 & x(u^{(1)}) \\ 1 & x(u^{(2)}) \end{vmatrix}\left(\frac{1}{{u_3}^4}+\cdots\right).$$

Hence the leading terms of these expansions coincide by 3.1, and the sides must be equal. This completes the proof of the case $n=3$.

The proof of the general step of induction is as follows. Assume that $u^{(1)}$, $u^{(2)}$, $\cdots$, $u^{(n)}$ and $u=u^{(n+1)}$ belong to $\iota(C)$. Then we want to prove the equality

$$-\frac{\sigma(u^{(1)}+u^{(2)}+\cdots+u^{(n)}+u)\prod_{j=1}^{n}\sigma_3(u^{(j)}-u)\prod_{i<j}\sigma_3(u^{(i)}-u^{(j)})}{\sigma_2(u^{(1)})^{n+1}\sigma_2(u^{(2)})^{n+1}\cdots\sigma_2(u^{(n)})^{n+1}\sigma_2(u)^{n+1}}$$
$$=\begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & x^3(u^{(1)}) & y(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & x^3(u^{(2)}) & y(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & x^3(u^{(n)}) & y(u^{(n)}) & \cdots \\ 1 & x(u) & x^2(u) & x^3(u) & y(u) & \cdots \end{vmatrix}.$$

We obviously see that the left hand side of the formula above, as a function of u, is periodic with respect to Λ by similar argument as in the case of $n=3$, and that it has its only pole at $u = (0, 0, 0)$ modulo Λ. The order of the pole is $(n+1)\times 3$ coming from $\sigma_2(u)^{n+1}$ minus $n \times 2$ coming from $\sigma_3(u^{(j)} - u)$ for $j = 1, 2, \cdots, n$; and that is equal to $n+3$. We know, by 2.14 that there are n obvious zeroes at $u = u^{(j)}$ modulo Λ of order 1 coming from $\sigma(u^{(j)} - u)$. These are also zeroes of the right hand side. Since the right hand side is a polynomial of $x(u)$ and $y(u)$, it has its only pole at $u = (0, 0, 0)$ modulo Λ. Its order is $n+3$ coming from the $(n+1, n+1)$-entry. So we denote the rest of the zeroes modulo Λ of the right hand side by α, β, and γ. Then the theorem of Abel-Jacobi implies that $u^{(1)} + u^{(2)} + \cdots + u^{(n)} + \alpha + \beta + \gamma = (0, 0, 0)$ modulo Λ. This means $\sigma(u^{(1)} + u^{(2)} + \cdots + u^{(n)} + u)$ is equal to $\sigma(u - \alpha - \beta - \gamma)$ times a trivial theta function. Hence these two sigma functions have the same zeroes. Since the latter function has obviously zeroes at $u = \alpha$, β, and γ modulo Λ by 2.6(2), the divisors modulo Λ of two sides coincide. We can show, as in the proof of the case $n = 3$, that the coefficients of the leading terms of the two sides in their Laurent expansions also coincide by using the inductive hypothesis. Now the proof is completed. □

4. *Determinantal Expression of Generalized Psi-Functions*

In this section we mention a generalization of the formula of (0.2) displayed in Introduction. Our formula is a natural generalization of the formula given in Section 3 of [**13**]. Although we can extend this generalization further to all hyperelliptic curves as in [**11**], we give here the case of genus three by a limiting process from 3.2.

The following formula is analogous to 3.1 in [**13**].

Lemma 4.1. *Let j be 1, 2, or 3. We have*

$$\lim_{u \to v} \frac{\sigma_3(u - v)}{u_j - v_j} = \frac{1}{x^{j-1}(v)}.$$

Proof. Because of 3.1 we have

$$\frac{x(u) - x(v)}{u_j - v_j} = \frac{-\sigma_3(u + v)}{\sigma_2(u)^2 \sigma_2(v)^2} \cdot \frac{\sigma_3(u - v)}{u_j - v_j}.$$

Now we bring u_j close to v_j. Then the limit of the left hand side is

$$\lim_{u \to v} \frac{x(u) - x(v)}{u_j - v_j} = \frac{dx}{du_j}(v).$$

This is equal to $\dfrac{2y}{x^{j-1}}(v)$ by (2.7). The assertion follows from 2.16. □

Since our proof of the following Theorem is obtained by a quite similar argument (by using 4.1) as in the case of genus two (see [**13**]), we leave the proof to the reader.

THEOREM 4.2. *Let $n \geq 3$ be an integer and $j \in \{1, 2, 3\}$. Assume that u belongs to $\kappa^{-1}\iota(C)$. Then the following formula for the function $\psi_n(u)$ of 2.17 holds:*

$$-(1!2!\cdots(n-1)!)\psi_n(u) = x^{(j-1)n(n-1)/2}(u)\times$$
$$\begin{vmatrix} x' & (x^2)' & (x^3)' & y' & (x^4)' & (yx)' & (x^5)' & \cdots \\ x'' & (x^2)'' & (x^3)'' & y'' & (x^4)'' & (yx)'' & (x^5)'' & \cdots \\ x''' & (x^2)''' & (x^3)''' & y''' & (x^4)''' & (yx)''' & (x^5)''' & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ x^{(n-1)} & (x^2)^{(n-1)} & (x^3)^{(n-1)} & y^{(n-1)} & (x^4)^{(n-1)} & (yx)^{(n-1)} & (x^5)^{(n-1)} & \cdots \end{vmatrix}(u).$$

Here the size of the matrix is $n-1$ by $n-1$. The symbols $'$, $''$, $\cdots$, ${}^{(n-1)}$ denote $\frac{d}{du_j}$, $\left(\frac{d}{du_j}\right)^2$, $\cdots$, $\left(\frac{d}{du_j}\right)^{n-1}$, respectively.

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