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Abelian Functions for Trigonal Curves of Degree Four and Determinantal Formulae in Purely Trigonal Case

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We give the Frobenius-Stickelberger-type and Kiepert-type determinantal formulae for purely trigonal curves of genus three. We explain also general theory of Abelian functions for any trigonal curves of genus three.

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1. Preface

In the theory of elliptic functions, there are two kinds of determinantal formulae of Frobenius-Stickelberger [6] and of Kiepert [7], both of which connect the function $\sigma(u)$ with $\wp(u)$ and its (higher) derivatives through an determinantal expression. These formulae were naturally generalized to hyperelliptic functions by the papers [11], [12], and [13]. Avoiding generality, we restrict the story only for the simplest purely trigonal curve $y^3 = x^4 + \cdots$, where the right hand side is a monic biquadratic polynomial of x. Our main results Theorem 5.3 and Corollary 6.2 are quite natural generalization of those determinantal formulae for such the curves. For more general purely trigonal curve, or for any purely d-gonal curve ($d = 4,\ 5,\ \cdots$), the author would like to publish in other papers, including formulae of Cantor-type (see [13]).

In the case of hyperelliptic functions, we considered only the hyperellptic curves ramified at infinity in [13]. The theta divisor of the Jacobian variety of such a curve is symmetric with respect to the origin of the Jacobian variety. Each of purely trigonal curves considered in this paper is also completely ramified at infinity and it is acted by the third roots of unity. Hence, the theta divisor of the Jacobian variety of a purely trigonal curve has third order symmetry with respect to the origin. Similar symmetry is also possessed by any purely d-gonal curve.

On the other hand, any curve that is not purely d-gonal does not have symmetry with respect to the origin at all. Since this fact is very serious, the author do not imagine whether generalizations of Frobenius-Stickelberger type and of Kiepert type exist or not.

The first two sections are devoted to describe general theory of trigonal curves of genus three. The rest sections are restricted to purely trigonal case.

Conventions. We denote the ring of integers by $\mathbf{Z}$, the field of real numbers by $\mathbf{R}$, the field of complex numbers by $\mathbf{C}$. The transpose of a vector u is denoted by ${}^t u$. The symbol $d^\circ(z_1, \cdots, z_m) \geqq d$ means a power series whose all terms with respect to the specified variables $z_1, \cdots, z_m$ are of total degree bigger than d. This symbol does not mean that it is a power series which contains only the variables $z_1, \cdots, z_m$.

Contents

2. Preliminaries

This and the next Sections are devoted for preliminaries under more general situation than the sections later. Let C be a complete projective algebraic curve defined by

$$\begin{aligned} &f(x,y) = 0, \quad \text{where} \\ &\quad f(x,y) = y^3 - (\lambda_1 x + \lambda_4)y^2 - (\lambda_2 x^2 + \lambda_5 x + \lambda_8)y \\ &\qquad\quad - (x^4 + \lambda_3 x^3 + \lambda_6 x^2 + \lambda_9 x + \lambda_{12}) \ \ (\lambda_j \text{ are constants}), \end{aligned} \tag{2.1}$$

with the unique point ∞ at infinity. The genus of C is 3 if C is non-singular. The three forms of the first kind

$$\omega_1 = \frac{dx}{\frac{\partial}{\partial y} f(x,y)}, \quad \omega_2 = \frac{x dx}{\frac{\partial}{\partial y} f(x,y)}, \quad \omega_3 = \frac{y dx}{\frac{\partial}{\partial y} f(x,y)} \tag{2.2}$$

form a basis of the space of holomorphic 1-forms. We denote the vector whose coordinates are (2.2) by

$$\omega = (\omega_1, \omega_2, \omega_3). \tag{2.3}$$

The general theory of Abelian integrals shows that the integrals

$$\begin{aligned} u &= (u_1, u_2, u_3) \\ &= \int_\infty^{(x_1,y_1)} \omega + \int_\infty^{(x_2,y_2)} \omega + \int_\infty^{(x_3,y_3)} \omega \end{aligned} \tag{2.4}$$

with respect to all the path from ∞ to three variable points (x_1, y_1), (x_2, y_2), (x_3, y_3) on C fill the whole space $\mathbf{C}^3$. In this paper, we denote by the letters u, v, $u^{(j)}$ various points in $\mathbf{C}^3$, and the same letters with subscripts (u_1, u_2, u_3), (v_1, v_2, v_3), $(u_1^{(j)}, u_2^{(j)}, u_3^{(j)})$ denote their canonical coordinates of $\mathbf{C}^3$. We denote by Λ all the values of the integral above with respect to all the closed paths. Then Λ is a lattice of $\mathbf{C}^3$. The $\mathbf{C}$-valued points of the Jacobian variety of C is $\mathbf{C}^3/\Lambda$. We denote this by J. The canonical map given by modulo Λ is denoted by κ:

$$\kappa : \mathbf{C}^3 \to \mathbf{C}^3/\Lambda = J. \tag{2.5}$$

Obviously, $\Lambda = \kappa^{-1}\big((0,0,0)\big)$. We have the standard embedding of C into J given by

$$\begin{aligned} \iota : C &\hookrightarrow J \\ P &\mapsto \int_\infty^P \omega \mod \Lambda. \end{aligned} \tag{2.6}$$

Then $\kappa^{-1}\iota(C)$ is a *universal Abelian covering* of C. More generally, for $0 \leqq k \leqq 3$, the image of the $k^{\rm th}$ symmetric product $\mathrm{Sym}^k(C)$ of C by the map

$$\begin{aligned} \iota :\ \mathrm{Sym}^k(C) &\to J \\ (P_1, \cdots, P_k) &\mapsto \Big(\int_\infty^{P_1}\omega + \cdots + \int_\infty^{P_k}\omega\Big) \mod \Lambda \end{aligned} \tag{2.7}$$

is denoted by $W^{[k]}$. Especially, $W^{[0]} = (0,0,0)$, $W^{[1]} = \iota(C)$, $W^{[3]} = J$. We denote by $[-1]$ the multiplication by (-1) in J, namely

$$[-1](u_1, u_2, u_3) = (-u_1, -u_2, -u_3). \tag{2.8}$$

We define by using this operation $[-1]$ that

$$\Theta^{[k]} = W^{[k]} \cup [-1]W^{[k]}. \tag{2.9}$$

We call $\Theta^{[k]}$ the *standard theta subvarieties* of J. Especially, we have $\Theta^{[0]} = (0,0,0)$ and $\Theta^{[3]} = J$. Note that we have $\Theta^{[1]} \neq W^{[1]}$, $\Theta^{[2]} \neq W^{[2]}$ as our case is different from the case of hyperelliptic curves.

Lemma 2.1. *Suppose $u = (u_1, u_2, u_3) \in \kappa^{-1}\iota(C)$ be near the origin $(0,0,0)$. Then u_1 and u_2 are expanded as a power series with respect to u_3 of the forms*

$$u_1 = \tfrac{1}{5}{u_3}^5 + \cdots, \qquad u_2 = \tfrac{1}{2}{u_3}^2 + \cdots.$$

Proof. Taking a local parameter $t = 1/\sqrt[3]{x}$ on C at ∞, we compute the integral

$$u_j = \int_\infty^{(x,y)} \omega_j, \tag{2.10}$$

then we have

$$u_1 = \tfrac{1}{5}t^5 + \cdots, \qquad u_2 = \tfrac{1}{2}t^2 + \cdots, \qquad u_3 = t + \cdots. \tag{2.11}$$

Hence the statement. □

The result above shows that u_3 is a *local parameter* on $\kappa^{-1}\iota(C)$ at the point $(0,0,0)$. Now we consider the integral

$$u = (u_1, u_2, u_3) = \int_\infty^{(x,y)} \omega, \tag{2.12}$$

and we denote its inverse function on $\kappa^{-1}\iota(C)$ by

$$u \mapsto \big(x(u), y(u)\big). \tag{2.13}$$

Lemma 2.2. *If $u = (u_1, u_2, u_3) \in \kappa^{-1}\iota(C)$, then $x(u)$ and $y(u)$ are expanded as power series with respect to u_3 as follows :*

$$x(u) = \frac{1}{{u_3}^3} + \cdots, \qquad y(u) = \frac{1}{{u_3}^4} + \cdots. \tag{2.14}$$

Proof. This is proved similarly with Lemma 2.1. □

Definition 2.3. We define a weight called *Sato weight* for appeared constants and variables as follows. The Sato weight of variables u_1, u_2, u_3 are 5, 2, 1, respectively, the Sato weight of each the coefficient λ_j in (1.1) is $-j$, The Sato weight of $x(u)$ and $y(u)$ are -3 and -4, respectively. Under this convention, the formulae in this paper are of homogeneous weight.

We define the discriminant of C. Let

$$\begin{aligned} R_1 &= \mathrm{rslt}_x\big(\mathrm{rslt}_y\big(f(x,y), \tfrac{\partial}{\partial x}f(x,y)\big), \mathrm{rslt}_y\big(f(x,y), \tfrac{\partial}{\partial y}f(x,y)\big)\big),\\ R_2 &= \mathrm{rslt}_y\big(\mathrm{rslt}_x\big(f(x,y), \tfrac{\partial}{\partial x}f(x,y)\big), \mathrm{rslt}_x\big(f(x,y), \tfrac{\partial}{\partial y}f(x,y)\big)\big), \end{aligned} \tag{2.15}$$

where rslt_z is the resultant of Sylvester with respect to the variable z. We define *discriminant* D of C by

$$D = \big[\,\mathrm{gcd}(R_1, R_2)\big]^{1/2}. \tag{2.16}$$

Then it is very likely that

$$D \in \mathbf{Z}[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_8, \lambda_9, \lambda_{12}] \tag{2.17}$$

by regarding λ_js indeterminates. Namely, $\mathrm{gcd}(R_1, R_2)$ seems to be a square in the ring above. While this is not obvious, we do not mention about this anymore because of less importance in this paper.

3. Sigma function (General case)

We now define the *sigma function*

$$\sigma(u) = \sigma(u_1, u_2, u_3) \tag{3.1}$$

associating to C by following [3]. We choose a set of generators

$$\alpha_i, \quad \alpha_j \ (1 \leqq i,\ j \leqq 3) \tag{3.2}$$

of $H_1(C, \mathbf{Z})$ such that their intersection numbers are $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = \delta_{ij}$ and $\alpha_i \cdot \beta_j = 0$. We denotes the period matrix obtained from the differentials (2.2) by

$$[\,\omega'\,] = \left[\int_{\alpha_i} \omega_j\right]_{i,j=1,2,3}, \quad [\,\omega''\,] = \left[\int_{\beta_i} \omega_j\right]_{i,j=1,2,3}. \tag{3.3}$$

For indeterminates X, Y, Z, and W, we consider that

$$\Omega\big((X,Y),(Z,W)\big)=\frac{1}{(X-Z)\frac{\partial}{\partial Y}f(X,Y)}\sum_{k=1}^{3}Y^{3-k}\left[\frac{f(Z,W)}{W^{k-1}}\right]_W, \tag{3.4}$$

where $[\ \ ]_W$ means taking only the terms of non-negative powers with respect to w. Let

$$\big((x,y),(z,w)\big)\mapsto R\big((z,w),(x,y)\big)dzdx \tag{3.5}$$

be a 2-form on $C\times C$ satisfying

$$\lim_{(x,y)\to(z,w)}(z-x)^2R\big((z,w),(x,y)\big)=1, \tag{3.6}$$

having poles only along the tiagonal $\{(x,y)=(z,w)\}$ in $C\times C$. The 2-form R is expressed in the form

$$R\big((x,y),(z,w)\big)=\frac{d}{dx}\Omega\big((z,w),(x,y)\big)+\sum_{j=1}^{3}\frac{\omega_j(z,w)}{dz}\frac{\eta_j(x,y)}{dx}. \tag{3.7}$$

Here ω_js are differentials of the first kind in (2.2), Ω the meromorphic function on $C\times C$ given in (3.4), and $\eta_j=\eta_j(x,y)$ ($j=1$, 2, 3) are some differential forms of the second kind on C with poles only at ∞. Moreover the derivation is one with respect to $(x,y)\in C$.

Definition 3.1. If a 2-form $R\big((x,y),(z,w)\big)$ satisfying (3.6) is of homogeneous Sato weight (hence weight 6), and has the symmetricity

$$R\big((z,w),(x,y)\big)=R\big((x,y),(z,w)\big), \tag{3.8}$$

then such a 2-form is called a (*Klein's*) *fundamental* 2*-form of the second kind*.

Lemma 3.2. *Assume $R\big((x,y),(z,w)\big)$ in* (3.7) *be a Klein's fundamental* 2*-form of the second kind. Then the set $\{\eta_j\}$ satisfying* (3.7) *exists and is uniquely determined modulo the space spanned by $\{\omega_j\}$.*

Proof. Under assuming existence of $\{\eta_j\}$, we see the differential (3.7) satisfies the condition on poles. Indeed, we see that, regarding (3.7) as a function of (x,y), it has only pole at $(x,y)=(z,w)$ by 2.2; and that it is similar as a function of (z,w) by (3.8). A fundamental 2-forms of the second kind is obtained similarly as [2], pp.3617–3618 (see also [1], around p.194). □

In the sequel, we merely consider a Klein's fundamental 2-form $R\big((x,y),(z,w)\big)$ of the second kind. It is easily seen that the η_j in (3.2) is written as

$$\eta_j(x,y)=\frac{h_j(x,y)}{\frac{\partial}{\partial y}f(x,y)}dx\quad \textit{where } h_j(x,y)\in\mathbf{Q}[\mu_1,\cdots,\mu_{12}][[x,y]],\ \textit{and} \tag{3.9}$$

h_j *is of homogeneous weight.*

Now we finally define η_j uniquely by requiring

$$\textit{the number of terms in } h_j(x,y) \textit{ is as minimal as possible we could} \tag{3.10}$$

(see [2], p.3618). While it is possible to write down the explicit form of η_js, we do not need such the expressions in this paper. Under the situation above, if we write the 2-form as

$$R\big((x,y),(z,w)\big) = \frac{F\big((x,y),(z,w)\big)}{(x-z)^2\frac{\partial}{\partial y}f(x,y)\frac{\partial}{\partial w}f(z,w)}, \tag{3.11}$$

we see that $F\big((x,y),(z,w)\big)$ is a polynomial of homogeneous Sato weight -8. We define the period matrices of $\{\eta_j\}$ by

$$[\,\eta'\,] = \left[\int_{\alpha_i}\eta_j\right]_{i,j=1,2,3}, \quad [\,\eta''\,] = \left[\int_{\beta_i}\eta_j\right]_{i,j=1,2,3}. \tag{3.12}$$

We concatenate this with (3.3) as

$$M = \begin{bmatrix}\omega' & \omega''\\ \eta' & \eta''\end{bmatrix}. \tag{3.13}$$

Then, M satisfies

$$M\begin{bmatrix} & -1_3\\ 1_3 & \end{bmatrix}{}^t M = 2\pi\sqrt{-1}\begin{bmatrix} & -1_3\\ 1_3 & \end{bmatrix} \tag{3.14}$$

(see [1], p.97(c), [5], Chap.III, [3], p.11, (1.15); Lemma 2.0.1). This is the *generalized Legendre relation* (set of the *Weierstrass relations*). Especially, we see ${\omega'}^{-1}\omega''$ is a symmetric matrix. It is well-known that

$$\mathrm{Im}\,({\omega'}^{-1}\omega'') \quad \text{is positive definite} \tag{3.15}$$

(see [5], Chap.III, for instance). It is known by (2.2) that the canonical divisor class is represented by 4∞. Hence any theta characteristic is given by a 2-torsion point in J ([9], 3.9 and 3.10), because our base point is ∞. Therefore, if

$$\omega'\delta' + \omega''\delta'' \tag{3.16}$$

is the theta characteristic giving the Riemann constant in the sense of Corollary 3.11 in p.166 of [9] for our case, namely with repsect to the base point $\infty \in C$ and to the basis (2.2) of the forms of the first kind (see also [3], p.15, (1.18)), then

$$\delta = \begin{bmatrix}\delta'\\ \delta''\end{bmatrix} \tag{3.17}$$

is an element in $\left(\frac{1}{2}\mathbf{Z}\right)^6$. Under the preparation above, we define[a]

$$\begin{aligned}\sigma(u) &= \sigma(u;M) = \sigma(u_1,u_2,u_3;M) \\ &= c\exp(-\tfrac{1}{2}u\eta'{\omega'}^{-1}\,{}^t\!u)\vartheta[\delta]\,({\omega'}^{-1}\,{}^t\!u;\,{\omega'}^{-1}\omega'') \\ &= c\exp(-\tfrac{1}{2}u\eta'{\omega'}^{-1}\,{}^t\!u)\sum_{n\in\mathbf{Z}^3}\exp\big[2\pi\sqrt{-1}\big\{\tfrac{1}{2}\,{}^t(n+\delta'){\omega'}^{-1}\omega''(n+\delta') \\ &\qquad\qquad + {}^t(n+\delta')({\omega'}^{-1}u+\delta'')\big\}\big],\end{aligned} \tag{3.18}$$

where

$$c = \frac{1}{\sqrt[8]{D}}\left(\frac{\pi^3}{|\omega'|}\right)^{\frac{1}{2}}. \tag{3.19}$$

The series in (3.18) converges by (3.15). Here D is the discriminant defined by (2.16) and (2.17), $\pi = 3.1415\cdots$, and $|\omega'|$ is the determinant of the period matrix ω' defined in (3.4). The roots of (3.19) are explained in 3.6 latter. In this paper, for $u \in \mathbf{C}^3$, we denote by u' and u'' the unique vectors in $\mathbf{R}^3$ such that

$$u = \omega' u' + \omega'' u''. \tag{3.20}$$

We define

$$\begin{aligned} L(u,v) &= {}^t u(\eta' v' + \eta'' v''), \\ \chi(\ell) &= \exp\big\{2\pi\sqrt{-1}\big({}^t\ell'\delta'' - {}^t\ell''\delta' + \tfrac{1}{2}\,{}^t\ell'\ell''\big)\big\}\ (\in\{1,-1\})\end{aligned} \tag{3.21}$$

for u, $v \in \mathbf{C}^3$ and for $\ell\ (= \omega'\ell' + \omega''\ell'') \in \Lambda$.

The following properties are quite important:

Lemma 3.3. *For all $u \in \mathbf{C}^3$, $\ell \in \Lambda$, and $\gamma \in \mathrm{Sp}(6,\mathbf{Z})$, we have the following:*
(1) $\sigma(u+\ell;M) = \chi(\ell)\sigma(u;M)\exp L(u+\frac{1}{2}\ell,\ell)$,
(2) $\sigma(u;\gamma M) = \sigma(u;M)$,
(3) $u \mapsto \sigma(u;M)$ *has zeroes on* $\Theta^{[2]}$ *of order* 1,
(4) $\sigma(u;M) = 0 \iff u \in \Theta^{[2]}$.

Proof. The assertion (1) is a special case of [1], p.286, ℓ.22. The assertion (2) is shown by investigating how the transform of M by γ corresponds to a change of paths α_js and β_js in (2.2) of period integrals. For details, see [3], pp.10–15. The statements (3) and (4) is described in [1], p.252, and partially in [3], p.12, Theorem 1.1 and p.15. □

Remark 3.4. Let M is a matrix satisfying (3.14) and (3.15). Since the Pfaffian of the Riemann form given by $L(\ ,\ )$ is 1 as is seen similarly to [10], Lemma 3.1.2, we see that the entire functions satisfying the equation 3.3(1) form one dimensional space

[a]If we redefine $\sigma(u)$ by using another fundamental 2-form of second kind of 3.2 different from that fixed by (3.10), the $\sigma(u)$ is changed only by the multiplication of exponential of a binary form of u_js.

by [8], p.93, Th.4.1. Any such non-trivial solution has properties 3.3(2), (3), and (4). Namely, 3.3(1) characterizes the sigma function up to a non-zero multiplicative constant.

Using 3.3(1), (3), (4), we have the following equality

$$\left[\frac{\sigma\Big(\int_\infty^{(x,y)}\omega-\sum_{i=1}^{3}\int_\infty^{(x_i,y_i)}\omega\Big)}{\sigma\Big(\int_\infty^{(x,y)}\omega-\sum_{i=1}^{3}\int_\infty^{(z_i,w_i)}\omega\Big)}\right]\left[\frac{\sigma\Big(\int_\infty^{(z,w)}\omega-\sum_{i=1}^{3}\int_\infty^{(x_i,y_i)}\omega\Big)}{\sigma\Big(\int_\infty^{(z,w)}\omega-\sum_{i=1}^{3}\int_\infty^{(z_i,w_i)}\omega\Big)}\right]^{-1}$$
$$=\exp\left[\int_{(z,w)}^{(x,y)}\Big(\sum_{i=1}^{3}\int_{(z_i,w_i)}^{(x_i,y_i)}R\big((x,y),(z,w)\big)dz\Big)dx\right] \tag{3.22}$$

by a similar method in [3], p.36. We define

$$\wp_{ij}(u)=-\frac{\partial^2}{\partial u_i\partial u_j}\log\sigma(u),\quad \wp_{ijk}(u)=-\frac{\partial^3}{\partial u_i\partial u_j\partial u_k}\log\sigma(u),\quad \cdots. \tag{3.23}$$

By 3.3(1), these are periodic functions with the periods lattice Λ. Therefore, we can regard these functions as functions on J. By (3.22), we see that

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\wp_{ij}\Big(\int_\infty^{(x,y)}\omega-\sum_{k=1}^{3}\int_\infty^{(x_k,y_k)}\omega\Big)\frac{\omega_i(x,y)}{dx}\frac{\omega_j(x_k,y_k)}{dx_k}$$
$$=\frac{F\big((x,y),(x_k,y_k)\big)}{(x-x_k)^2\frac{\partial}{\partial y}f(x,y)\frac{\partial}{\partial y_k}f(x_k,y_k)},\qquad (k=1,\,2,\,3). \tag{3.24}$$

Taking a local parameter t at ∞ such that $x=1/t^3$, and expanding both sides of (3.24) as power series of t, we know several equations on $\wp_{j_1j_2\cdots j_n}(u)$s. By using such equations we see the following fact.

Lemma 3.5. (with Definition) *The power series expansion of $\sigma(u)$ at $u=(0,0,0)$ with respect to u_1, u_2, u_3 is of homogeneous Sato weight 5, and it is of the form*

$$\sigma(u)=\varepsilon\left(u_1-u_3{u_2}^2+\tfrac{1}{20}{u_3}^5\right)+(d^\circ(\lambda_1,\cdots,\lambda_{12})\geqq 1),$$

where ε is a non-zero constant. We redefine precisely $\sigma(u)$ such as $\varepsilon=1$ and proportional to (3.18).

Proof. This is proved in Theorem 7.1 in [4]. □

Remark 3.6. It is very plausible that the constant ε in 3.5 is an 8th root of 1. Namely, it seems to correspond only the choice of roots in (2.16) and (3.19).

Lemma 3.7. *The function $\sigma(u)$ is an odd function. Namely, we have*

$$\sigma([-1]u)=-\sigma(u). \tag{3.25}$$

Proof. For the period lattice, we have $[-1]\Lambda = \Lambda$. This is seen by considering the integral of $\ell \in \Lambda$ in the opposite directions. Adding this with 3.4, we see that there is a constant K such that

$$\sigma([-1]u) = K\sigma(u). \tag{3.26}$$

Since the weight 5 is an odd integer, 3.5 implies $K = -1$. □

In the sequel, we write simply

$$\sigma_j(u) = \frac{\partial}{\partial u_j}\sigma(u), \quad \sigma_{ij}(u) = \frac{\partial^2}{\partial u_i \partial u_j}\sigma(u). \tag{3.27}$$

Lemma 3.8. *Let u, $u^{(1)}$, $u^{(2)}$, $v \in \kappa^{-1}\iota(C)$. Then we have the following:*
(1) $\sigma(u^{(1)} + u^{(2)}) = 0\,;$
(2) *The expansion of $v \mapsto \sigma(u^{(1)} + u^{(2)} + v)$ with respect to v_3 is of the form*

$$\sigma(u^{(1)} + u^{(2)} + v) = \sigma_3(u^{(1)} + u^{(2)})v_3 + (d^\circ(v_3) \geqq 2);$$

(3) $\sigma_3(u) = 0\,;$
(4) *The expansion of $v \mapsto \sigma_3(u + v)$ with respect to u_3 is of the form*

$$\sigma_3(u + v) = \sigma_{33}(u)v_3 + (d^\circ(v_3) \geqq 2);$$

(5) *The expansion of $v \mapsto \sigma_{33}(v)$ with respect to v_3 is of the form*

$$\sigma_{33}(v) = {v_3}^3 + (d^\circ(v_3) \geqq 4).$$

Proof. The assertions (1) and (2) are repetition of 3.3(3). For the expansion of (2), by taking $u^{(2)}$ close to $(0, 0, 0)$, we see[b] $\sigma_3(u^{(1)}) = 0$ because of $\sigma(u^{(1)} + v) = 0$ by (1). The assertion (4) is obviously follows from (3). By 3.5, we see that, for $u \in \mathbf{C}^3$,

$$\sigma_{33}(u) = {u_3}^3 + (d^\circ(\lambda_1, \cdots, \lambda_{12}) \geqq 1). \tag{3.28}$$

Since this is still of homogeneous Sato weight, the assertion (5) follows. □

Lemma 3.9. *We have the following translational relations:*
(1) *For $u \in \kappa^{-1}(\Theta^{[2]})$, we have*

$$\sigma_3(u + \ell) = \chi(\ell)\sigma_3(u)\exp L(u + \tfrac{1}{2}\ell, \ell);$$

(2) *For $u \in \kappa^{-1}(\Theta^{[1]})$, we have*

$$\sigma_{33}(u + \ell) = \chi(\ell)\sigma_{33}(u)\exp L(u + \tfrac{1}{2}\ell, \ell).$$

Proof. Differentiating both sides of 3.3(1) by u_3 once or twice, we see the assertions by using 3.8. □

[b] In this situation, for given point $P \in C$ $(\neq \infty)$, the rank of the Brill-Noether matrix $B(P + \infty)$ of the divisor $P + \infty$ on C is $\mathrm{rank}B(P + \infty) = 2$, and $\dim \Gamma(C, \mathcal{O}(P)) = 1$. It is impossible to show directly $\sigma_3(u) = 0$ $(u \in \kappa^{-1}\iota(C))$ from Riemann singularity theorem.

Lemma 3.10. *The function* $u \mapsto \sigma_{33}(u)$ *on* $\kappa^{-1}\iota(C)$ *has only zero of order* 3 *at* $u = (0,0,0)$ *modulo* Λ*, and no zeroes elsewhere.*

Proof. Our proof here is the standard way as in p.149–150 in [9]. Taking into account the equation in 3.9(2), by integrating the form $d\sigma_{33}(u)/\sigma_{33}(u)$ around a regular polygon of the Riemann surface given by C, the standard argument using the principle of arguments shows the total of orders of zeroes of the claimed function is 3, the genus of C. Hence, 3.8(5) implies the assertion. □

4. The sigma functions (Purely trigonal case)

In the rest of this paper, we treat only the curve C

$$y^3 = x^4 + \lambda_3 x^3 + \lambda_6 x^2 + \lambda_9 x + \lambda_{12} \quad (\lambda_j \text{ are constants}) \tag{4.1}$$

that specializing (1.1). Then (1.2) is given by

$$\omega_1 = \frac{dx}{3y^2}, \quad \omega_2 = \frac{xdx}{3y^2}, \quad \omega_2 = \frac{ydx}{3y^2} = \frac{dx}{3y}; \tag{4.2}$$

and (2.4) is

$$\Omega\big((x,y),(x',y')\big) = \frac{y^2 + yy' + y'^2}{(x-x')3y^2}. \tag{4.3}$$

Using the above, we define $\sigma(u)$ as in the previous section. Let $\zeta = e^{2\pi\sqrt{-1}/3}$. Since C has an automorphism defined by $(x,y) \mapsto (x, \zeta y)$, ζ^j acts for $u = (u_1,u_2,u_3) \in \mathbf{C}^3$ by

$$\lceil \zeta^j \rceil u = (\zeta^j u_1, \zeta^j u_2, \zeta^{2j} u_3). \tag{4.4}$$

We see that $W^{[k]}$ and $\Theta^{[k]}$ are stable under the action by ζ^j. Under this situation, the function $\sigma(u) = \sigma(u; M)$ has the following special property.

Lemma 4.1. *We have*

$$\sigma(\lceil \zeta \rceil u) = \zeta \sigma(u).$$

Proof. This follows from 3.4, the fact $\lceil \zeta \rceil \Lambda = \Lambda$, and 3.5. □

The following lemma is the key for our main result.

Lemma 4.2. *For* $u \in \kappa^{-1}\iota(C)$*, the function* $v \mapsto \sigma_3(u + \lceil \zeta \rceil v)$ (*resp.* $v \mapsto \sigma_3(u + \lceil \zeta^2 \rceil v)$) *on* $\kappa^{-1}\iota(C)$ *has only* 3 *zeroes* $(0,0,0)$*,* u*,* $\lceil \zeta^2 \rceil u$ (*resp.* $\lceil \zeta \rceil u$) *of order* 1 *modulo* Λ*, and has no zeroes elsewhere. Its power series expansion at* $(0,0,0)$ *with respect to* v_3 *is of the form*

$$\sigma_3(u + \lceil \zeta \rceil v) = \zeta^2 \sigma_{33}(u) v_3 + (d^\circ(v_3) \geqq 2)$$
$$\big(\textit{resp. } \sigma_3(u + \lceil \zeta^2 \rceil v) = \zeta \sigma_{33}(u) v_3 + (d^\circ(v_3) \geqq 2)\big).$$

Proof. Remark 3.7 shows that $u \mapsto \sigma_3(u)$ is an even function. Becase $u + \lceil\zeta\rceil u + \lceil\zeta^2\rceil u = (0,0,0)$, we have for $u \in \kappa^{-1}\iota(C)$

$$\sigma_3(u + \lceil\zeta\rceil u) = \sigma_3(-\lceil\zeta^2\rceil u) = \sigma_3(\lceil\zeta^2\rceil u) = \zeta^2\sigma_3(u) = 0 \tag{4.5}$$

by 3.8(3) and 4.1. The remaining assertions follows from 3.8(4). □

The following lemma is used in 6.1.

Lemma 4.3. *For $u \in \kappa^{-1}\iota(C)$, we have:*

$$\frac{\sigma_3(2u)}{\sigma_{33}(u)^4} = 3y(u)^2.$$

Proof. Because of 3.5 and 2.1, we have that

$$\sigma_3(2u) = \tfrac{1}{4}(2u_3)^4 - \big(2(\tfrac{1}{2}{u_3}^2 + \cdots)\big)^2 + \cdots = 3{u_3}^4 + \cdots\,,$$

so that

$$\frac{\sigma_3(2u)}{\sigma_{33}(u)^4} = \frac{3{u_3}^4 + \cdots}{\big({u_3}^3 + \cdots\big)^4} = \frac{3}{{u_3}^8} + \cdots\,. \tag{4.6}$$

The left hand side of this is a periodic function with respect to Λ by 3.9. Therefore, its pole is only at $(0,0,0)$ modulo Λ by 3.10. By 4.1 we see

$$\frac{\sigma_3(\lceil\zeta\rceil 2u)}{\sigma_{33}(\lceil\zeta\rceil u)^4} = \zeta^2\frac{\sigma_3(2u)}{\sigma_{33}(u)^4}$$

and that the right hand side of (4.6) should be $3y(u)^2$. □

5. Frobenius-Stickelberger-Type Formula

The initial case of our Frobenius-Stickelberger-type formula for the purely trigonal curve C defined by (4.1) is as follows:

Proposition 5.1. *For u, $v \in \kappa^{-1}\iota(C)$ we have:*

$$\frac{\sigma_3(u+v)\,\sigma_3(u+\lceil\zeta\rceil v)\,\sigma_3(u+\lceil\zeta^2\rceil v)}{\sigma_{33}(u)^3\sigma_{33}(v)^3} = \big((x(u)-x(v)\big)^2 = \begin{vmatrix} 1 & x(u) \\ 1 & x(v)\end{vmatrix}^2.$$

Proof. Lemma 3.9 shows the left hand side is a periodic function of v (resp. u) with the periods Λ. Now, we regard the left hand side as a function of v. Lemma 4.2 states that the second and third factors vanish at $v = u$ modulo Λ, namely the left hand side has zero of order 2 there, and has two zeroes of order 1 at $\lceil\zeta\rceil u$ and $\lceil\zeta^2\rceil u$ modulo Λ. It has no zero elsewhere. Its only pole is at $v = (0,0,0)$ modulo Λ by 3.10. The order is $3\times 3 - 3 = 6$. These situations are exactly the same for the right hand side. Therefore the two sides coincide up to a multiplicative constant depending only on u. If we expand both sides with respect to v_3 we see that the coefficients of the least terms of the two sides coincide exactly by using 3.8 and 2.2. Thus, the proof has completed. □

Remark 5.2. Note that $-u \notin \kappa^{-1}\iota(C)$ for $u \in \kappa^{-1}\iota(C)$ in general. This fact is a reason why the initial formula above is different from the initial formula for the case of hyperelliptic functions (Lemma 8.1 of [13]).

Theorem 5.3. (Frobenius-Stickelberger-type formula) *Let $n \geqq 3$ be an integer. Let $\sigma(u)$, $x(u)$, and $y(u)$ are those defined for the purely trigonal curve $C : y^3 = x^4 + \cdots$ as above. Assume $u^{(1)}, \cdots, u^{(n)}$ are points on $\kappa^{-1}\iota(C)$. Then we have:*

$$\frac{\sigma(u^{(1)}+u^{(2)}+\cdots+u^{(n)})\prod_{i<j}\sigma_3(u^{(i)}+[\zeta]u^{(j)})\,\sigma_3(u^{(i)}+[\zeta^2]u^{(j)})}{\sigma_{33}(u^{(1)})^{2n-1}\cdots\sigma_{33}(u^{(n)})^{2n-1}}$$
$$= \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & yx(u^{(1)}) & y^2(u^{(1)}) & x^3(u^{(1)}) & yx^2(u^{(1)}) & y^2x(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & yx(u^{(2)}) & y^2(u^{(2)}) & x^3(u^{(2)}) & yx^2(u^{(2)}) & y^2x(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & yx(u^{(n)}) & y^2(u^{(n)}) & x^3(u^{(n)}) & yx^2(u^{(n)}) & y^2x(u^{(n)}) & \cdots \end{vmatrix}$$
$$\cdot \begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & \cdots & x^{n-1}(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & \cdots & x^{n-1}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^{n-1}(u^{(n)}) \end{vmatrix}.$$

Remark 5.4. The each row of the first determinant consists of monomials of x and y (they have a pole only at $(0,0,0)$) displayed according to their order of the pole, and their orders are

$$0,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ \cdots$$

respectively.

Proof. We prove the formula by induction. First of all, we see by 3.10 that the left hand side is a periodic function of u and of v with the periods Λ.
(1) For the case $n = 3$, the formula is proved similarly to the case $n \geqq 4$ below (or to [12], p.309). So, we omit the proof of this case.
(2) Suppose $n \geqq 4$. Let $u = u^{(n)}$. We regards both sides as functions of u.
(2-a) We know the divisors of the two sides by 3.10, 3.3(3),(4), 4.2 as follows:
The left hand side. The numerator of the left hand side has zeroes of order 2 at $(n-1)$ points

$$u^{(j)} \text{ modulo } \Lambda \quad (j = 1, \cdots, n-1),$$

zeroes of order 1 at $2(n-1)$ points

$$[\zeta]u^{(j)},\quad [\zeta^2]u^{(j)} \text{ modulo } \Lambda \quad (j = 1, \cdots, n-1),$$

and zeroes of order $2(n-1)$ at $u = (0,0,0)$. It has no zeroes elsewhere. The denominator has only zero at $u = (0,0,0)$ of order $3(2n-1) = 6n-3$. Therefore the left hand side has only pole at $(0,0,0)$ of order $(6n-3) - 2(n-1) = 4n-1$. There are $(4n-1) - 2(n-1) - 2(n-1) = 3$ unknown zeroes of the left hand side.

The right hand side. Both determinants vanish at

$$u^{(j)} \text{ modulo } \Lambda \quad (j = 1, \cdots, n-1),$$

and only the second determinant vanishes at

$$\lceil\zeta\rceil u^{(j)}, \quad \lceil\zeta^2\rceil u^{(j)} \text{ modulo } \Lambda \quad (j = 1, \cdots, n-1).$$

The deepest pole comes from (n, n)-entries of the two determinants. The sum of the pole orders is $(n-2) + 3(n-1) = 4n - 1$.
(2-b) We let α, β, $\gamma \in \mathbf{C}^3$ modulo Λ be the unknown zeroes of the right hand side. Then the Abel-Jacobi theorem shows

$$\begin{aligned} 2(u^{(1)} + \cdots + u^{(n-1)}) + (\lceil\zeta\rceil u^{(1)} \cdots + \lceil\zeta\rceil u^{(n-1)}) + (\lceil\zeta^2\rceil u^{(1)} \cdots + \lceil\zeta^2\rceil u^{(n-1)}) \\ + \alpha + \beta + \gamma \in \Lambda. \end{aligned}$$

Since $u^{(j)} + \lceil\zeta\rceil u^{(j)} + \lceil\zeta^2\rceil u^{(j)} = (0, 0, 0)$, we see

$$u^{(1)} + \cdots + u^{(n-1)} + \alpha + \beta + \gamma \in \Lambda,$$

so that

$$u^{(1)} + \cdots + u^{(n-1)} + u^{(n)} \equiv u - \alpha - \beta - \gamma \mod \Lambda.$$

Observing the first factor in the numerator of the left hand side, we see the left hand side has zeroes of order 1 at α, β, γ, too. Thus the two sides coincide up to multiplicative constant.
(2-c) The coefficients of the lowest term in the Laurent expansion with respect to u, is just the hypothesis of induction, and they coincide exactly. Hence the proof has completed. □

6. Kiepert-Type Formula

We prove the following fundamental formula.

Lemma 6.1. *We have*

$$\lim_{v \to u} \frac{\sigma_3(u + \lceil\zeta\rceil v)\, \sigma_3(u + \lceil\zeta^2\rceil v)}{\sigma_{33}(u)\sigma_{33}(v)(u_3 - v_3)^2} = 3.$$

Proof. Lemma 4.3 and Proposition 5.1 show

$$\begin{aligned} 3y(u)^2 \Big(\lim_{v \to u} & \frac{\sigma_3(u + \lceil\zeta\rceil v)\, \sigma_3(u + \lceil\zeta^2\rceil v)}{\sigma_{33}(u)\sigma_{33}(v)(u_3 - v_3)^2} \Big) \\ &= \lim_{v \to u} \frac{\sigma_3(u + v)\, \sigma_3(u + \lceil\zeta\rceil v)\, \sigma_3(u + \lceil\zeta^2\rceil v)}{\sigma_{33}(u)^3\sigma_{33}(v)^3(u_3 - v_3)^2} \\ &= \lim_{v \to u} \Big(\frac{x(u) - x(v)}{u_3 - v_3} \Big)^2 \\ &= \Big(\frac{dx}{du_3}(u) \Big)^2 \\ &= \big(3y(u)\big)^2. \end{aligned}$$

Here the last equality is seen by (4.2). This yields our desired formula. □

Corollary 6.2. (Kiepert-type formula) *Suppose* $n \geqq 3$ *and* $u \in \kappa^{-1}\iota(C)$. *We have*

$$\psi_n(u) := \frac{\sigma(nu)}{\sigma_{33}(u)^{n^2}} = y^{n(n-1)/2}(u)\times$$

$$\begin{vmatrix} x' & y' & (x^2)' & (yx)' & (y^2)' & (x^3)' & (yx^2)' & (y^2x)' & \cdots \\ x'' & y'' & (x^2)'' & (yx)'' & (y^2)'' & (x^3)'' & (yx^2)'' & (y^2x)'' & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ x^{(n-1)} & y^{(n-1)} & (x^2)^{(n-1)} & (yx)^{(n-1)} & (y^2)^{(n-1)} & (x^3)^{(n-1)} & (yx^2)^{(n-1)} & (y^2x)^{(n-1)} & \cdots \end{vmatrix}(u),$$

where $'$ *means* $\frac{d}{du_3}$ *and the determinant is of size* $(n-1)\times(n-1)$.

Proof. Each factor $x(u^{(i)}) - x(u^{(j)})$ of the Vandermonde determinant gives

$$\lim_{v\to u}\frac{x(u)-x(v)}{u_3-v_3} = \frac{dx}{du_3}(u) = 3y(u)$$

by (4.2). Then 5.3 and 6.1 give the formula of Kiepert-type by similar manipulation to the proof of [11], Theorem 3.3. □

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