Theory of Heat Equations for Sigma Functions *

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Abstract

Let e and q be fixed coprime integers satisfying $1 < e < q$. Let $\mathscr{C}$ be the standard deformation of the curve $y^e = x^q$, and $\varDelta$ be the discriminant of $\mathscr{C}$. Following pioneering work by Buchstaber and Leykin (BL), we determine the canonical basis $\{L_j\}$ of the tangent space of the variety $\varDelta = 0$. Moreover, we give a proof of a formula giving $L_j \log \varDelta$ as a polynomial of the moduli parameters of $\mathscr{C}$, which was given by BL without proof. The set $\{L_j\}$ gives rise to a system of heat equations satisfied by the function $\sigma(u)$ associated with $\mathscr{C}$ as a generalisation of Weierstrass' result on his sigma function, and eventually gives its explicit power series expansion. We attempt to give an accessible description of the BL-theory.

2010 *Mathematics subject classification.* 33E99, 14K25, 14H42, 11G05

*Version : September 19, 2021

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1 Introduction

Classically, the Weierstrass function $\sigma(u)$ is defined through the Weierstrass elliptic function $\wp(u)$ as follows:

$$\sigma(u) = u \exp\left(\int_0^u \int_0^u \left(\tfrac{1}{u^2} - \wp(u)\right) du\, du\right).$$

The modern approach is to define the sigma function starting from a general elliptic curve. However, in this introduction, we treat only the curve defined by

$$y^2 = x^3 + \mu_4 x + \mu_6 \quad \text{(Weierstrass form).} \tag{1.1}$$

(We refer the reader to [10] for the case of the most general elliptic curve.) For this curve, we define the function $\sigma(u)$ by

$$\sigma(u) = \Big(\frac{2\pi}{\omega'}\Big)^{1/2} \varDelta^{-\frac{1}{8}} \exp\big(-\tfrac{1}{2}\omega'^{-1}\eta' u^2\big) \cdot \vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\omega'^{-1}u, \omega''/\omega'), \tag{1.2}$$

where $\varDelta = -16(4{\mu_4}^3 + 27{\mu_6}^2)$ is the discriminant of the curve, and ω', ω'', η', and η'' are the periods of the two differential forms

$$\frac{dx}{2y}, \frac{xdx}{2y}$$

with respect to a pair of fixed standard closed paths α_1 and β_1 which represents a symplectic basis of the first homology group, though η'' does not appear explicitly. The last part of (1.2) is Jacobi's theta series defined by

$$\vartheta\begin{bmatrix}b\\ a\end{bmatrix}(z, \tau) = \sum_{n\in\mathbb{Z}} \exp 2\pi i\big(\tfrac{1}{2}\tau(n+b)^2 + (n+b)(z+a)\big) \qquad (a,\ b \in \tfrac{1}{2}\mathbb{Z}). \tag{1.3}$$

From now on, we suppose the function $\sigma(u)$ is defined by (1.2). In using this definition, it is not clear that $\sigma(u)$ is independent of the choice of α_1 and β_1. Indeed, both of the later part (Jacobi's theta series) of (1.2) and former part are not invariant when we choose another pair of α_1 and β_1. However these changes offset each other and $\sigma(u)$ itself is invariant.

Using the Dedekind eta function $\eta(\tau)$ (not to be confused with the η's above and in Section 2.1), the discriminant $\varDelta$ of the curve above is given by $\varDelta = \big(\frac{2\pi}{\omega'}\big)^{12}\eta(\omega''/\omega')^{24}$, and the first terms in (1.2) can be explicitly written as

$$\Big(\frac{2\pi}{\omega'}\Big)^{1/2} \varDelta^{-\frac{1}{8}} = -\frac{\omega'}{2\pi}\eta(\omega''/\omega')^{-3}. \tag{1.4}$$

Although $\varDelta$ is invariant with respect to a change of α_1 and β_1, both sides of (1.4) and ω' are not invariant. The function $\sigma(u)$ has a power series expansion at the origin as follows:

$$\begin{aligned}\sigma(u) &= u \sum_{n_4, n_6 \geqq 0} b(n_4, n_6) \frac{(\mu_4 u^4)^{n_4} (\mu_6 u^6)^{n_6}}{(1 + 4n_4 + 6n_6)!} \\ &= u + 2\mu_4 \frac{u^5}{5!} + 24\mu_6 \frac{u^7}{7!} - 36\mu_4^2 \frac{u^9}{9!} - 288\mu_4\mu_6 \frac{u^{11}}{11!} + \cdots ,\end{aligned} \tag{1.5}$$

where $b(n_4, n_6) \in \mathbb{Z}$. (This expansion also shows the independence of $\sigma(u)$ with respect to the choice of α_1 and β_1). In this work, the family of curves which we will investigate are called (n, s)-curves or *plane telescopic curves* (see Section 2.1 for definitions). One of the motivations of a theory of heat equations for the sigma functions is to know a recurrence relation for $b(n_4, n_6)$. But, there is another motivation as follows. For an arbitrary non-singular curve, especially a plane telescopic curve, there is an intrinsic or axiomatic definition (characterisation) of its sigma function (see 3.13). It would be useful to have a theorem such that the sigma function is expressed in a similar form as (1.2). Indeed it is not so difficult to show that the natural generalisation of the right hand side of (1.2), except for the factor (1.4), satisfies some issues of the characterisation 3.13. Before [7], except for curves of genus one and two, the validity of the expression including the natural generalisation of the factor (1.4) was not yet known. We shall discuss this again at the end of this introduction.

It is well known that the sigma function for a general non-singular algebraic curve is expressed by Riemann's theta series with a characteristic coming from the Riemann constant of the curve multiplied by some exponential factor and some constant factor. This constant factor might be a natural generalisation of (1.4). But, there seems to be no proof of the determination of this constant factor except in genus one and two (we mention this again later). To fix the last constant is another motivation of the theory, which is described around Lemma 4.17 of [6].

We now review the classical theory of the heat equations for $\sigma(u)$. Let z and τ be complex numbers with the imaginary part of τ positive. We define $L = 4\pi i\frac{\partial}{\partial\tau}$ and $H = \frac{\partial^2}{\partial z^2}$. Then Jacobi's theta function (1.3) satisfies the following equation, which is known as the heat equation,

$$(L - H)\,\vartheta\begin{bmatrix} b \\ a \end{bmatrix}(z,\tau) = 0. \tag{1.6}$$

Weierstrass' result in [26], which is displayed as (1.13) below, is regarded as an interpretation of (1.6) in the form attached to his function $\sigma(u)$. Strictly speaking, he did not derive it directly, but by repeated technical integrations from the well-known differential equation $\wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3$ satisfied by the $\wp(u)$, eventually obtaining a recurrence relation (extracted from Subsection 4.3)

$$\begin{aligned} b(n_4,n_6) &= \tfrac{2}{3}(4n_4+6n_6-1)(2n_4+3n_6-1)b(n_4-1,n_6) \\ &\quad - \tfrac{8}{3}(n_6+1)b(n_4-2,n_6+1) + 12(n_4+1)b(n_4+1,n_6-1) \end{aligned} \tag{1.7}$$

for the coefficients in the power series expansion (1.5) of $\sigma(u)$ (see also [21]).

Frobenius and Stickelberger approached (1.7) via a different method. In their paper [12], which was published in the same year as [26], using

$$\wp(u) = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + \frac{{g_2}^2}{1200}u^6 + \cdots, \tag{1.8}$$

where g_2, g_3 are the coefficients of the equation $\wp'(u)^2 = \wp(u)^3 - g_2\wp(u) - g_3$ given by $g_2 = -4\mu_4$, $g_3 = -4\mu_6$, and the corresponding expansion of the Weierstrass function $\zeta(u) = \frac{1}{u} - \int_0^u \big(\wp(u) - \frac{1}{u^2}\big)du$, with

$$g_2 = 60\sum_{(n',n'')\neq(0,0)}\frac{1}{(n'\omega'+n''\omega'')^4},\quad g_3 = 140\sum_{(n',n'')\neq(0,0)}\frac{1}{(n'\omega'+n''\omega'')^6}, \tag{1.9}$$

they obtained the formulae

$$\omega'\frac{\partial g_2}{\partial\omega'} + \omega''\frac{\partial g_2}{\partial\omega''} = -4g_2, \qquad \omega'\frac{\partial g_3}{\partial\omega'} + \omega''\frac{\partial g_3}{\partial\omega''} = -6g_3,$$

$$\eta'\frac{\partial g_2}{\partial \omega'}+\eta''\frac{\partial g_2}{\partial \omega''}=-6g_3,\quad \eta'\frac{\partial g_3}{\partial \omega'}+\eta''\frac{\partial g_3}{\partial \omega''}=-\frac{1}{3}{g_2}^2,$$

where $\eta'=\zeta(u+\omega')-\zeta(u)$, $\eta''=\zeta(u+\omega'')-\zeta(u)$ which are independent of u, and (see (4.4))

$$\begin{aligned}\omega'\frac{\partial}{\partial \omega'}+\omega''\frac{\partial}{\partial \omega''}&=-4g_2\frac{\partial}{\partial g_2}-6g_3\frac{\partial}{\partial g_3},\\ \eta'\frac{\partial}{\partial \omega'}+\eta''\frac{\partial}{\partial \omega''}&=-6g_3\frac{\partial}{\partial g_2}-\frac{1}{3}{g_2}^2\frac{\partial}{\partial g_3}.\end{aligned}\tag{1.10}$$

Moreover, they obtained (p.318 in [12])

$$\omega'\frac{\partial \eta'}{\partial \omega'}+\omega''\frac{\partial \eta'}{\partial \omega''}=\eta',\quad \omega'\frac{\partial \eta''}{\partial \omega'}+\omega''\frac{\partial \eta''}{\partial \omega''}=\eta''.$$

On p.326 of [12], they gave the same system of heat equations as in [26]. Observing the work of Weierstrass from the viewpoint of the paper [7], the left hand sides of (1.10) correspond to $L=4\pi i\frac{\partial}{\partial\tau}$, namely the operations with respect to the period integrals τ or $\{\omega',\ \omega'',\ \eta',\ \eta''\}$ of the curve, which fit the expression (1.2); while, the right hand sides of (1.10) are an interpretation of such operations in order to fit the expansion (1.5) of (1.2) given by [26]. Although we suspect there are fruitful correspondences between Weierstrass, Frobenius, and Stickelberger, the authors have no details of these.

It appears difficult to generalise the Weierstrass method to the higher genus cases. There is some hint in the work of Frobenius-Stickelberger to generalise the result to these cases. In order to do so, it seems necessary to have generalisation of relations (1.10). But we do not have naive generalisations of (1.8) and (1.9).

Recently Buchstaber and Leykin were able to generalise the above results to the sigma functions of higher genus curves ([7], see also [4, 5, 6]). In [7], Buchstaber and Leykin generalise (1.10) to higher genus curves by using the first de Rham cohomology H^1_{dR} of the curve over the base ring, that is the space of the differential forms of the second kind modulo the exact forms (see Section 3.4). The paper [7] is our main reference for our work. Understanding that paper requires some background on the basic theory of heat equations and singularity theory, so we will summarise their arguments and those of Frobenius and Stickelberger, with hopefully accessible explanations.

We shall explain their method by taking the curve (1.1) as an example. Firstly, we introduce a certain heat equation (primary heat equation) satisfied by a certain function defined in (1.11) below, which is a generalisation of the individual terms of the series appearing in the definition (1.2) of $\sigma(u)$. Let us take arbitrary element

$$L\in\mathbb{Q}[\mu_4,\mu_6]\frac{\partial}{\partial\mu_4}\oplus\mathbb{Q}[\mu_4,\mu_6]\frac{\partial}{\partial\mu_6}.$$

Thanks to a lemma due to Chevalley (Lemma 3.18), we see that $\frac{\partial}{\partial\mu_j}$ acts on H^1_{dR}, so that the operator L acts naturally on H^1_{dR}. Take a symplectic basis $(\frac{dx}{2y},\ \frac{xdx}{2y})$

of H^1_{dR} with respect to a naturally defined inner product in H^1_{dR} (See (2.7)). Let

$$\Gamma^L = \begin{bmatrix} -\beta & \alpha \\ -\gamma & \beta \end{bmatrix}$$

be the representation matrix of this action of L (see (3.19)), which is called a *Gauss-Manin connection* in [7]. Then we see that α, β, and γ belong to $\mathbb{Q}[\mu_4, \mu_6]$. Taking integrals along the set of closed paths α_1 and β_1 which make a symplectic homology basis of the curve, we see the action of L gives a linear transformation of the period matrix (see (3.2))

$$\varOmega = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}$$

with respect to the basis $(\frac{dx}{2y},\ \frac{xdx}{2y})$, α and β, which is also represented by Γ^L as $L(\varOmega) = \Gamma^L\,\varOmega$ (see (3.20)). Moreover, we introduce another operator

$$H^L = \tfrac{1}{2}\,[\,\tfrac{\partial}{\partial u}\ \ u\,]\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}\begin{bmatrix} \frac{\partial}{\partial u} \\ u \end{bmatrix} = \frac{1}{2}\Big(\alpha\,\frac{\partial^2}{\partial u^2} + 2\beta\, u\frac{\partial}{\partial u} + \gamma\, u^2 + \beta\Big).$$

Then the function

$$\begin{aligned} G(b, u, \varOmega) = {} & \left(\frac{2\pi}{\omega'}\right)^{\frac{1}{2}} \exp\big(-\tfrac{1}{2}\eta'\omega'^{-1}u^2\big) \\ & \times \exp\Big(2\pi i\big(\tfrac{1}{2}\omega'^{-1}\omega'' b''^2 + b''(\omega'^{-1}u + b')\big)\Big), \end{aligned} \tag{1.11}$$

where $b = [\,b'\ \ b''\,]$ is an arbitrary constant vector, satisfies the heat equation

$$(L - H^L)G(b, u, \varOmega) = 0. \tag{1.12}$$

We call this and its generalisation (Theorem 3.35) the *primary heat equation.* To check the validity of (1.12) is complicated, no details are given by Buchstaber and Leykin, and the description of this equation in [7] is not entirely consistent. We denote the expression of the right hand side (1.2) without $\varDelta^{-\frac{1}{8}}$ by $\tilde{\sigma}(u)$ (see (3.40)). According to the above equation and the fact that $\tilde{\sigma}(u)$ is an infinite sum of the $G(b, u, \varOmega)$s for various b' and b'', we see that $(L - H^L)\,\tilde{\sigma}(u) = 0$.

At the next stage, we shall find suitable Ls which give rise to a system of heat equations which are satisfied by the right hand side of (1.2), including the factor $\varDelta^{-\frac{1}{8}}$. It is easy to see that such an operator must belong to the tangent space of the singular locus given by $\varDelta = 0$ (see the former part of Subsection 3.6). We import techniques of calculation from singularity theory to get $\varDelta$ (Lemma 3.44(3)) as well as its tangent space. This stage is carried out in Subsections 3.6 and 3.7 and the result is given in (3.59). For the curve (1.1), the operators finally obtained are

$$L_0 = 4\mu_4\frac{\partial}{\partial\mu_4} + 6\mu_6\frac{\partial}{\partial\mu_6}, \qquad L_2 = 6\mu_6\frac{\partial}{\partial\mu_4} - \frac{4}{3}{\mu_4}^2\frac{\partial}{\partial\mu_6},$$

which are, of course, exactly same as those in [26] and [12]. The paper [7] uses knowledge of singularity theory and succeeds in generalising nicely the result of

Weierstrass and Frobenius-Stickelberger.

In summary, the system of heat equations for (1.1) obtained by Weierstrass is

$$
\begin{aligned}
(L_0 - H^{L_0})\,\sigma(u) &= \left(4\mu_4\frac{\partial}{\partial\mu_4} + 6\mu_6\frac{\partial}{\partial\mu_6} - u\frac{\partial}{\partial u} + 1\right)\sigma(u) = 0,\\
(L_2 - H^{L_2})\,\sigma(u) &= \left(6\mu_6\frac{\partial}{\partial\mu_4} - \frac{4}{3}{\mu_4}^2\frac{\partial}{\partial\mu_6} - \frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{6}\mu_4 u^2\right)\sigma(u) = 0,
\end{aligned}
\tag{1.13}
$$

which is reproved as (4.8). For the general curves, the corresponding results are given as Theorem 3.71 in the text.

It is very important to determine whether the obtained system of heat equations characterise the sigma function. For the genus one case, it was seen by Weierstrass that the recurrence (1.7) determines all the coefficients if we give an arbitrary value for $b(0,0)$. That is, the solution space of (1.7), as well as (1.13), is of dimension one.

For a general non-singular curve, we consider the multivariate function $\sigma(u)$ defined similarly to (1.2). We can check that the operators we obtain (of Theorem 3.71) kill $\sigma(u)$. However, *it is not clear whether the solution space is of dimension one* over the base field. The authors could not find any reason which suggests that the solution space is one dimensional. Nevertheless, we shall show that, for any curve of genus less than or equal to three, the solution space is one dimensional by giving an explicit recurrence relation from the system of heat equations obtained, which is described in Section 4.

In the last paragraph of Section 2 in [7], there is some description on the positive modality case. In such case we do not know how define the analogy of the operator L_0 beause of lack of some μ_j (see (3.59).

Although our main results are in Subsections 4.6, 4.5, 4.8, and 4.7, we give a number of useful results, which may be known by specialists but are not well known to other researchers, with detailed proofs in Section 3 and Subsection 4.2. Subsection 4.3 reproduces the classical result and it would be helpful to read the following Subsections. Subsection 4.4 is rewritten in a slightly different formulation (Hurwitz-type series expansion of $\sigma(u)$) from [6].

We shall explain here a notion called *modality* which was introduced by Arnol'd (see 2.4 for details). For any coprime positive integers (e,q) with $e<q$, we consider the deformation of the singularity at the origin of the curve $y^e = x^q$. Then any deformed curve is called a plane telescopic curve. The number of parameters necessary for such deformation is less than or equal to $(e-1)(q-1)$ that is twice the genus of the generic deformed curve. The difference between this number and $(e-1)(q-1)$ is called the *modality* of this deformation. For instance, the curve (1.1) is regarded as a deformation of $y^2 = x^3$ with two parameters μ_4 and μ_6, which is equal to twice its genus (i.e. $2 = 1\times 2$). So that the modality is 0 in this case. It is known that the hyperelliptic curve given by deforming in the case $e=2$ is of

modality 0. There are only two kinds of non-hyperelliptic plane telescopic curves of modality 0, which are the trigonal quartic curve, the $(3,4)$-curve (genus three), and the trigonal quintic curve, the $(3,5)$-curve (genus four). Concerning these two curve, we treat only the former one, the $(3,4)$-curve, in this paper, and we hope to treat the $(3,5)$-curve elsewhere. For a general hyperelliptic curve, we give its corresponding system of heat equations in Lemma 4.1. For any plane telescopic curve, we gave a simple formula for its modality in Proposition 2.12.

We shall mention two additional consequences of this theory. Firstly, for hyperelliptic curves of genus less than or equal to three, we again prove partially the result of [23] on Hurwitz integrality of the expansion of the sigma function. For example it is obvious from (1.7) that $b(n_4, n_6) \in \mathbb{Z}[\frac{1}{3}]$. Similar results are shown for the hyperelliptic curves of genus two and three. This idea was suggested to Y.Ô. by Buchstaber. Secondly, this theory of heat equations in turn helps the construction of the sigma function, as explained in Lemma 4.17 of [6] (see also Section 3.8). The formula (1.2) is well-known for the curve (1.1), and its generalisation (see (3.16)) is proved for the genus two hyperelliptic curve by Grant [14] by using Thomae's formula. For any curve in the family we have investigated, there is a rough explanation in Lemma 2.3 in p.98 of [5], but without using Thomae's formula.

We explain here that BL-theory indeed gives a method to prove such a formula as (1.2) on the sigma functions, at least, for our curves of genus less than or equal to three. Firstly, assuming the expression (1.2) to be the correct sigma function, we show that it satisfies the system of heat equations. On the other hand, as we mentioned above, the solution space of the system is one dimensional and the system of heat equations gives a recursion relation, by which we have the power series expansion of the solution as shown in Section 3. Especially, we see the solution space is of dimension one. Therefore, we have a proof that the assumed expression of the sigma indeed gives the sigma function up to a non-zero absolute constant. (see Theorem 3.72).

Finally, one of the authors S.Y. wishes to point out to the reader that his contribution on this paper is limited to the proof of the case $e = 2$ in Proposition 3.62.

Acknowledgements: We are grateful to Christophe Ritzenthaler who explained Sylvester's algorithm, which is very useful to compute the discriminants for curves of genus less than or equal to three. We would also like to thank Julia Bernatska who explained some of the details of the practical implementation of the algorithm behind the genus two example in [7]. We would like to thank Toshizumi Fukui for bringing Zakalyukin's paper to our attention. We are grateful to an anonymous referee for many helpful suggestions and corrections to improve the paper. This research is supported by JSPS grant 25400010, and 16K05082.

2 Preliminaries

2.1 The curves

We shall use e and q instead of n and s of (n,s)-curves, respectively. Although the name (n,s)-curve (which comes from singularity theory) is used by Buchstaber and Leykin in their papers, we wish to avoid confusion with the many n used as subscripts in sections from Section 4 onward, and the use of s for Schur polynomials in Subsection 3.1. Let e and q be two fixed positive integers such that $e<q$ and $\gcd(e,q)=1$. We define, for these integers, a polynomial of x and y

$$f(x,y) = y^e - p_1(x)y^{e-1} - \cdots - p_{e-1}(x)y - p_e(x), \tag{2.1}$$

where $p_j(x)$ is a polynomial of x of degree $\lceil \frac{jq}{e} \rceil$ or smaller and its coefficients are denoted by

$$\begin{aligned} p_j(x) &= \sum_{k:jq-ek>0} \mu_{jq-ek}\, x^k \quad (1 \le j \le e-1), \\ p_e(x) &= x^q + \mu_{e(q-1)}x^{q-1} + \cdots + \mu_{eq}. \end{aligned} \tag{2.2}$$

Please note that the sign at the front of each $p_j(x)$ with $j \ne e$ in $f(x,y)$ is different from previous papers written by some of the authors. The base ring over which we work is quite general. For simplicity the reader may start by taking the field $\mathbb{C}$ of complex numbers and assume the μ_is to be constants belonging to this field. Let $\mathscr{C} = \mathscr{C}_\mu^{e,q}$ be the projective curve defined by

$$f(x,y) = 0 \tag{2.3}$$

having a unique point ∞ at infinity.

Our situation is that all the coefficients μ_j of $f(x,y)$ should be indeterminates. We denote by $\mathbb{Q}[\mu]$ the ring generated over the rationals $\mathbb{Q}$ by all the μ_js. Then we shall treat $\mathscr{C}$ as a scheme over the $\operatorname{Spec}\mathbb{Q}[\mu]$. As an abuse of language, we freely switch the standing position where μ_js are assumed to be complex numbers or indeterminates.

This $\mathscr{C}$ should be called an *(e,q)-curve* following Buchstaber, Enolskii, and Leykin [8], or a *plane telescopic curve* after the paper [18]. Assuming all the μ_js are complex numbers, the genus of $\mathscr{C}$ is $(e-1)(q-1)/2$ provided that it is non-singular. We will use g to denote this quantity throughout this paper whether the curve $\mathscr{C}$ is non-singular or singular as well as in the case of the μ_js being indeterminates: $g=(e-1)(q-1)/2$.

As the general elliptic curve is defined by an equation of the form

$$y^2 - (\mu_1 x + \mu_3)y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6,$$

the curves $\mathscr{C}$ discussed here are a natural generalisation of elliptic curves.

Now we introduce a weight function as follows. For the curve $\mathscr{C}$ given by (2.1), we define the weight wt on $\mathbb{Q}[\mu][x,y]$ by

$$\mathrm{wt}(\mu_j) = -j, \quad \mathrm{wt}(x) = -e, \quad \mathrm{wt}(y) = -q. \tag{2.4}$$

Then all the equations for functions, power series, differential forms, and so on in this paper are of homogeneous weight. So that, we have $\mathrm{wt}(f(x,y)) = -eq$.

For the fixed pair (e,q) satisfying $q > e > 0$ and $\gcd(e,q) = 1$, we call the number $g = \frac{(e-1)(q-1)}{2}$ the *genus* of $\mathscr{C}$ in this paper, which is the genus (in usual sense) of $\mathscr{C}$ if it is non-singular. We will extend the notion wt in Subsection 3.2.

2.2 Differential forms of the curve

We denote the Weierstrass gap sequence at ∞ by

$$w_1(=2g-1),\ w_2,\ \cdots,\ w_g(=1). \tag{2.5}$$

Namely, this is the unique finite decreasing sequence of positive integers which cannot be written in the form $ae + bq$ with non-negative integers a, b. It is well-known that each term of the sequence is written in the form

$$\omega_j = 2g - 1 - a_j e - b_j q,$$

with non-negative integers a_j, b_j. Especially, $a_1 = b_1 = 0$. Then we define, for each of $j = 1, \cdots, g$, a differential form

$$\omega_{w_j} = \frac{x^{a_j} y^{b_j}}{f_2(x,y)} dx,$$

where $f_2 = \frac{\partial}{\partial y} f$, of the first kind and of weight w_j. We define g more differential forms of the second kind η_{-w_j} $(j = g, \cdots, 1)$ of weight $-w_j$, such that these $2g$ forms give rise to a symplectic basis of the space $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\mu])$ which is explained below. We denote $\boldsymbol{\omega} = (\omega_{w_1}, \cdots, \omega_{w_g}, \eta_{-w_g}, \cdots, \eta_{-w_1})$. Throughout this paper, we denote $\mathbb{Q}[\mu] = \mathbb{Q}[\{\mu_k\}]$. In this paper, we should consider the curve $\mathscr{C}$ and other objects arising from $\mathscr{C}$ to be defined over the ring $\mathbb{Q}[\mu]$, in which the period matrix Ω is exceptional and is defined over only the field $\mathbb{C}$. It is known that the η_{-w_j}s as well as the ω_{w_j}s are defined over $\mathbb{Q}[\mu]$, namely, they are of the form $\frac{h(x,y)}{f_2(x,y)} dx$ with $h(x,y) \in \mathbb{Q}[\mu, x, y]/(f)$. Assuming x, y and all the μ_j are indeterminates, then, it is natural for us to define

$$H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\mu]) = \frac{\left\{ \frac{h(x,y)}{f_2(x,y)} dx \,\middle|\, h(x,y) \in \mathbb{Q}[\mu][x,y]/(f(x,y)) \right\}}{d\big(\mathbb{Q}[\mu][x,y]/(f(x,y))\big)}. \tag{2.6}$$

We define

$$M(x,y) = \{\, x^i y^j \,|\, 0 \le i \le q-2,\ 0 \le j \le e-2 \,\}.$$

Then the set $\left\{ \frac{h(x,y)}{f_2(x,y)} dx \,|\, h(x,y) \in M(x,y) \right\}$ forms a basis of $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\mu])$ as a $\mathbb{Q}[\mu]$-module. Here we note that the order of pole of $x^{q-2} y^{e-2}$ at ∞, which is the highest in $M(x,y)$, is

$$e(q-2)+q(e-2)=2(e-1)(q-1)-2=2g-2,$$

and $H^1_{\rm dR}(\mathscr{C}/\mathbb{Q}[\mu])$ is a free $\mathbb{Q}[\mu]$-module of rank $2g$.

The space $H^1_{\rm dR}(\mathscr{C}/\mathbb{Q}[\mu])$ is equipped with the antisymmetric inner product given by

$$\omega\star\eta=\operatorname*{Res}_{P=\infty}\Big(\int_\infty^P\omega\Big)\eta(P)$$

for any ω, $\eta\in H^1_{\rm dR}(\mathscr{C}/\mathbb{Q}[\mu])$. This is formally defined using the formal expansion with respect to a local parameter at ∞ (see [23]), which is a formal interpretation of the product $\omega\star\eta=\operatorname*{Res}_{P\in\mathscr{C}^\circ}\big(\int_\infty^P\omega\big)\eta(P)$ if we regard the μ_js as complex numbers and $\mathscr{C}$ as a non-singular curve (i.e. a compact Riemann surface). Here $\mathscr{C}^\circ$ the regular polygon obtained from the Riemann surface attached to the curve $\mathscr{C}$ with respect to the paths $\{\alpha_j,\,\beta_j\,|\,j=1,\cdots,g\}$ which form a symplectic basis of the homology group $H_1(\mathscr{C},\mathbb{Z})$ as usual.

We choose η_{-w_i} to satisfy the symplectic relations

$$\omega_{w_i}\star\omega_{w_j}=0,\quad \eta_{-w_i}\star\eta_{-w_j}=0,\quad \omega_{w_i}\star\eta_{-w_j}=-\eta_{-w_j}\star\omega_{w_i}=\delta_{ij}. \tag{2.7}$$

The choice of η_{-w_j} is not unique. For a more concrete construction of these forms, we refer the reader to [23]. Especially, as a $\mathbb{Q}[\mu]$-module, ω_{w_j}s and η_{-w_j}s form a basis of $H^1_{\rm dR}(\mathscr{C}/\mathbb{Q}[\mu])$.

In this paper we frequently switch between regarding the μ_js as indeterminates or complex numbers. After Lemma 3.18, we see that $H^1_{\rm dR}(\mathscr{C}_\mu/\mathbb{Q}[\mu])$ is a $\mathbb{Q}[\mu,\frac{\partial}{\partial\mu}]$-module, where $\frac{\partial}{\partial\mu}$ stands for the set of $\frac{\partial}{\partial\mu_j}$s.

2.3 Definition of the discriminant

We shall define the discriminant of the curve $\mathscr{C}$.

Definition 2.8. *The discriminant Δ of the form $f(x,y)$ or of the curve $\mathscr{C}$ defined by $f(x,y)=0$ is the polynomial (up to the signs $\pm$) of the least degree in the μ_js with integer coefficients such that the greatest common divisor of the coefficients is 1, and every zero of Δ corresponds exactly to the case that $\mathscr{C}$ has a singular point.*

For instance, if $(e,q)=(2,3)$, then

$$f(x,y)=y^2-(\mu_1x+\mu_3)y-(x^3+\mu_2x^2+\mu_4x+\mu_6)$$

and its discriminant is given by

$$\begin{aligned}\Delta={}&-\mu_6{\mu_1}^6+\mu_3\mu_4{\mu_1}^5+((-{\mu_3}^2-12\mu_6)\mu_2+{\mu_4}^2){\mu_1}^4\\&+(8\mu_3\mu_4\mu_2+{\mu_3}^3+36\mu_6\mu_3){\mu_1}^3+((-8{\mu_3}^2-48\mu_6){\mu_2}^2+8{\mu_4}^2\mu_2\\&+(-30{\mu_3}^2+72\mu_6)\mu_4){\mu_1}^2+(16\mu_3\mu_4{\mu_2}^2+(36{\mu_3}^3+144\mu_6\mu_3)\mu_2-96\mu_3{\mu_4}^2)\mu_1\\&+(-16{\mu_3}^2-64\mu_6){\mu_2}^3+16{\mu_4}^2{\mu_2}^2+(72{\mu_3}^2+288\mu_6)\mu_4\mu_2\\&-64{\mu_4}^3-27{\mu_3}^4-216\mu_6{\mu_3}^2-432{\mu_6}^2.\end{aligned}$$

Specializing $\mu_1 = \mu_3 = \mu_2 = 0$, we have the discriminant $\Delta = -16(4{\mu_4}^3 + 27{\mu_6}^2)$ for the Weierstrass form which appeared in (1.2). For $(e, q) = (2, 2g+1)$, as in Section 2.4, we rewrite the equation as $y^2 = x^{2g+1} + \cdots$, where the right hand side is a polynomial of x only. Then the discriminant of this curve is a non-zero integer multiple of the discriminant of the right hand side as a polynomial of x only. For the $(3, 4)$-curve, we have Sylvester's method as described in [13] pp.118-120, as explained to the authors by C. Ritzenthaler. However, Lemma 3.44 below gives a much more general method, which seems to cover the $(3, 5)$-curve and more. We do have explicit forms of the discriminants of the curves with $(e, q) = (2, 3)$, $(2, 5)$, $(2, 7)$, $(3, 4)$ which we treat in this paper. Using the resultant of two forms, we mention here an alternative (but conjectural) construction for the interest of the reader, though it is not used in this paper.

Definition 2.9. *Let the coefficients μ_j of* (2.1) *be indeterminates, and define*

$$\begin{aligned} R_1 &= \mathrm{rslt}_x\Big(\mathrm{rslt}_y\big(f(x,y), \tfrac{\partial}{\partial x} f(x,y)\big), \mathrm{rslt}_y\big(f(x,y), \tfrac{\partial}{\partial y} f(x,y)\big)\Big), \\ R_2 &= \mathrm{rslt}_y\Big(\mathrm{rslt}_x\big(f(x,y), \tfrac{\partial}{\partial x} f(x,y)\big), \mathrm{rslt}_x\big(f(x,y), \tfrac{\partial}{\partial y} f(x,y)\big)\Big), \\ R &= \gcd(R_1, R_2) \quad \text{in } \mathbb{Z}[\mu]. \end{aligned}$$

Here rslt_z *is the Sylvester resultant with respect to* z.

Now we recall the conjecture from the paper [11].

Conjecture 2.10. *Defining R by 2.9, we have the following*:
(1) *R is always a perfect square in $\mathbb{Z}[\mu]$ and $R = \Delta^2$*;
(2) *The discriminant Δ of the (e, q)-curve is of weight $-2eqg = -eq(e-1)(q-1)$.*

Remark 2.11. It can be confirmed that (1) of 2.10 is correct for the cases $(d, q) = (2, 3)$, $(2, 5)$, $(2, 7)$, $(3, 4)$, and $(3, 5)$. Actually, computation by `Maple` for these cases shows that R is a square of some $\Delta' \in \mathbb{Z}[\mu]$. It is easy to check Δ' is irreducible. Then, from the definition of R, we see Δ' must be Δ up to the sign, and we checked (2) of 2.10 for these cases. We prove that (2) of 2.10 is true if $\gcd(e-1, q-1) = 1$ in 3.57. However, the authors have no proof of the conjecture in general.

2.4 The Weierstrass form of the curve and its modality

As explained in [6], the method we use here works only for hyperelliptic curves and for some trigonal curves as explained below.

Starting from the equation $f(x, y) = 0$ in (2.1) and removing the terms of y^{e-1} and x^{q-1} by replacing y by $y + \frac{1}{e}\,p_1(x)$, and x by $x + \frac{1}{q}\mu_{(q-1)e}$, respectively, we get a new equation $f(x, y) = 0$ which is called the *Weierstrass form* of the original one. After making such transformations, we re-label the coefficients by μ_j.

For example, if $(e, q) = (2, 2g+1)$, the new equation is

$$f(x, y) = y^2 - (x^{2g+1} + \mu_{4g-2}x^{2g-1} + \mu_{4g-4}x^{2g-2} + \cdots + \mu_{4g+2}) = 0;$$

and if $(e,q)=(3,4)$, the new one is

$$f(x,y)=y^3-(\mu_2x^2+\mu_5x+\mu_8)y-(x^4+\mu_6x^2+\mu_9x+\mu_{12})=0.$$

In these cases, the number of remaining μ_js is $2g$. However, in general, we can have cases such that this number is less than $2g$. The difference

$$2g-\text{“the number of }\mu_j\text{”}$$

is called the *modality*[1] of the (e,q)-curve. We give here a simple formula giving modalities and, especially, determine all the curves of modality 0.

Proposition 2.12. *The modality of an (e,q)-curve is given by $\frac{1}{2}(e-3)(q-3)+\lfloor\frac{q}{e}\rfloor-1$. The only curves of modality 0 are the $(2,2g+1)$-, $(3,4)$-, and $(3,5)$-curves.*

Proof. The number of μ_js appearing in the Weierstrass form is

$$\begin{aligned}&\sum_{j=1}^{e-2}\left(\left\lfloor\frac{(e-j)q}{e}\right\rfloor+1\right)+(q-1)\\ &=\frac{1}{2}\sum_{j=1}^{e-1}\left(\left\lfloor\frac{(e-j)q}{e}\right\rfloor+\left\lfloor\frac{jq}{e}\right\rfloor\right)-\left\lfloor\frac{q}{e}\right\rfloor+(e-2)+(q-1)\\ &=\frac{1}{2}\sum_{j=1}^{e-1}(q-1)-\left\lfloor\frac{q}{e}\right\rfloor+e+q-3=\frac{1}{2}(e-1)(q-1)+e+q-3-\left\lfloor\frac{q}{e}\right\rfloor.\end{aligned}$$

Therefore the modality is

$$2g-\left(\frac{1}{2}(e-1)(q-1)+e+q-3-\left\lfloor\frac{q}{e}\right\rfloor\right)=\frac{1}{2}(e-3)(q-3)+\left\lfloor\frac{q}{e}\right\rfloor-1.$$

The latter part follows directly from this. This completes the proof. □

On the case for a curve with positive modality, there is some description in [7] (the last of Section 2). Since it is not clear for us how positive modality causes difficulty, we do not discuss this theme here, though we give an example of positive modality in 3.43.

[1] A terminology used in singularity theory.

3 Theory of heat equations

3.1 Materials for construction of the sigma functions

In this section, we recall the definition of the sigma function (1.2) for the general curve $\mathscr{C}$. This is now a function of g variables $u = {}^t(u_{w_g}, \cdots, u_{w_1})$ and written as $\sigma(u) = \sigma(u_{w_g}, \cdots, u_{w_1})$. It is called, analogously, the sigma function for $\mathscr{C}$. To define it precisely, we need to introduce the Schur polynomial, period matrices, and others.

We use the classical notation of matrices concerning theta series, so our notation is transposed from the notation of [7]. Specifically, we will denote the period matrix of a curve as (3.2) whereas Buchstaber and Leykin's papers use the transpose of this matrix. Other differences between their notation and ours will follow from this.

Letting $\boldsymbol{T} = \sum_{j=1}^{g} u_{w_j} T^{w_j}$, we define $\{p_k\}$ by $p_k = 0$ for negative k and

$$\sum_{k=0}^{\infty} p_k T^k = \sum_{n=0} \frac{\boldsymbol{T}^n}{n!}.$$

Then we define $s(u) = s(u_{w_g}, \cdots, u_{w_1})$ (see Section 4 in [19]) by

$$s(u) = \det([\, p_{w_{g+1-i}+j-g} \,]_{1\leq i\leq g,\ 1\leq j\leq g}). \tag{3.1}$$

This is the *Schur polynomial* corresponding to the sequence $(w_g, \cdots, w_1)$.

From now on in this Section, we assume all the μ_js are complex numbers and $\mathscr{C}$ is non-singular. The period matrices are defined by $\mathit{\Omega} = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}$ with

$$\begin{aligned} \omega' &= \left[\int_{\alpha_j} \omega_{w_{g-i+1}}\right], \quad \omega'' = \left[\int_{\beta_j} \omega_{w_{g-i+1}}\right], \\ \eta' &= \left[\int_{\alpha_j} \eta_{w_{g-i+1}}\right], \quad \eta'' = \left[\int_{\beta_j} \eta_{w_{g-i+1}}\right]. \end{aligned} \tag{3.2}$$

We introduce the g-dimensional space $\mathbb{C}^g$ with the coordinates

$$u = {}^t(u_{w_g}, u_{w_{g-1}}, \cdots, u_{w_1}) \tag{3.3}$$

for the domain on which the sigma function is defined. We define a lattice in this space by

$$\mathit{\Lambda} = \omega'\mathbb{Z}^g + \omega''\mathbb{Z}^g. \tag{3.4}$$

For any $u \in \mathbb{C}^g$, we define u', $u'' \in \mathbb{R}^g$ by $u = \omega' u' + \omega'' u''$. Likewise, for $\ell \in \mathit{\Lambda}$, we write $\ell = \omega'\ell' + \omega''\ell''$. In addition we write ${\omega'}^{-1}\,{}^t(\omega_{w_g}, \cdots, \omega_{w_1}) = {}^t(\hat{\omega}_1, \cdots, \hat{\omega}_g)$, ${\omega'}^{-1}\omega'' = \big[\tau_{ij}\big]$, and define

$$\delta_j = -\tfrac{1}{2}\tau_{jj} - \int_{\infty}^{P_j} \hat{\omega}_j + \sum_{i=1}^{g} \int_{\alpha_i} \left(\int_{\infty}^{P} \hat{\omega}_j\right) \hat{\omega}_i(P), \quad \delta = \omega'\, {}^t(\delta_1, \cdots, \delta_g).$$

Here P_j is a fixed initial point of the path α_j. Then we define the Riemann constant by $[\delta' \ \delta'']$. It is well known that, for our curve $\delta', \delta'' \in (\frac{1}{2}\mathbb{Z})^g$. The $[\delta' \ \delta'']$ can be taken independent of the values of μ_js.

Using the above notation, we define a linear form $L(\ ,\ )$ on $\mathbb{C}^g \times \mathbb{C}^g$ by

$$L(u, v) = {}^t u\,(\eta' v' + \eta'' v''). \tag{3.5}$$

This is $\mathbb{C}$-linear with respect to the first variable, and $\mathbb{R}$-linear with respect to the second one. Moreover, we define

$$\chi(\ell) = \exp\,\big(2\pi i({}^t\delta'\ell'' - {}^t\delta'\ell' + \frac{1}{2}\,{}^t\ell'\ell'')\big) \in \{1, -1\} \tag{3.6}$$

for any $\ell \in \Lambda$. Let

$$\kappa\,:\,\mathbb{C}^g \longmapsto \mathbb{C}^g/\Lambda \tag{3.7}$$

be the map given by modulo Λ, and $\mathrm{Sym}^k\mathscr{C}$ be the k-th symmetric product of $\mathscr{C}$. Then we have the Abel-Jacobi mapping

$$\iota\,:\,\mathrm{Sym}^k\mathscr{C} \longrightarrow \mathbf{C}^g/\Lambda, \quad (P_1, \cdots, P_k) \longmapsto \sum_{j=1}^k \left(\int_\infty^{P_j} \omega_{w_g}, \cdots, \int_\infty^{P_j} \omega_{w_1}\right), \tag{3.8}$$

whose image is denoted by $\Theta^{[k]}$. We denote $\Theta = \Theta^{[g-1]}$, which is called the standard theta divisor of the Jacobian variety $\mathbf{C}^g/\Lambda$ of $\mathscr{C}$. For $k = 1$, the map ι is an isomorphism from $\mathscr{C}$ to $\Theta^{[1]}$, by which we identify these two.

3.2 Weight revisited

In this subsection, we assume all μ_js in (2.1) are complex numbers. We denote by $\mathscr{C}_\mu$ the curve given by (2.1) and assume it is non-singular, namely, we assume it gives a Riemann surface. Here, we shall extend the notion of the weight wt as follows. Firstly, we define the weight of u_j, a coordinate of u in (3.3), by

$$\mathrm{wt}(u_j) = j. \tag{3.9}$$

Let ε be arbitrary non-zero complex number and $\mathscr{C}_{\varepsilon\mu}$ be the curve defined by (2.1) with every μ_j being replaced $\varepsilon^{-j}\mu_j$. Then the map

$$\boldsymbol{\varepsilon}\,:\,\mathscr{C}_\mu \longrightarrow \mathscr{C}_{\varepsilon\mu}, \quad (x, y) \longmapsto (\varepsilon^{-e}x, \varepsilon^{-q}y)$$

is an isomorphism (i.e. an isomorphism of compact Riemann surfaces). Now we extend the defining domain of wt to the space $\mathbb{C}^g$ of (3.3) as follows. We take a fixed reference point $(\mu, u) = (\{{\mu_j}^{(0)}\}, u^{(0)})$ of $(\mu, u) = (\{\mu_j\}, u)$ of (2.1) and (3.3), and assume $\mathscr{C}_{\mu^{(0)}}$ is also non-singular. Let

$$\mathscr{M} = \mathscr{M}(\mu^{(0)}, u^{(0)}) \tag{3.10}$$

be the germ of the analytic functions of (μ, u) at $(\mu^{(0)}, u^{(0)})$, where we regard them as complex variables. Let $\Phi(\mu, u) \in \mathscr{M}$. We denote by $\Phi(\varepsilon\mu, \varepsilon u)$ the function obtained from $\Phi(\mu, u)$ by replacing all μ_j by $\varepsilon^{-j}\mu$ and all u_i by $\varepsilon^i u_i$ for $i = w_g$,

$\cdots$, w_1. We say that the obtained function is induced by the mapping $\boldsymbol{\varepsilon}$. If there is a constant integer w such that

$$\Phi(\varepsilon\mu, \varepsilon u) = \varepsilon^w \Phi(\mu, u)$$

for any $\varepsilon \in \mathbb{C}$, then we say the function $\Phi(\mu, u)$ is of weight w, and denote as

$$\mathrm{wt}(\Phi(\mu, u)) = w.$$

Remark 3.11. Now we explain that the definition of wt is actually an extension of weight defined at (2.4). The coordinates x and y of $\mathscr{C}_\mu$ are naturally seen as functions of u_g of u in (3.3) on certain restricted domain in $\mathbb{C}^g$ for any $\mu_j \in \mathbb{C}$, which should be denoted by $\kappa^{-1}\iota(\mathscr{C})$ by using the notation (3.7) and (3.8). Then, observing the weight of x and y as power series of u_g, their weight are $-e$ and $-q$, respectively. The function $u_g \mapsto x$ above is seen as restriction to $\kappa^{-1}\iota(\mathscr{C})$ of the Abelian function

$$u \longmapsto \frac{x_1 \cdots x_g}{\displaystyle\sum_{1 \leq i_1 < \cdots < i_{g-1} \leq g} x_{i_1} \cdots x_{i_{g-1}}}, \tag{3.12}$$

where $u = \iota((x_1, y_1), \cdots, (x_g, y_g)) \mod \Lambda$. The function (3.12) is easily checked to be of weight $-e$. Under similar observation, the function $u_g \mapsto y$ is seen of weight $-q$. Summarizing the above, the notion of weight of (2.4) is completely compatible with the former notion above as well as one on $\mathscr{M}$.

Take any circuit integrals of an entry of one of the matrices in (3.2) and consider the map $\boldsymbol{\varepsilon}$. For example, we choose $\displaystyle\int_{\alpha_j} \omega_{w_i}$. Note that it is an element in $\mathscr{M}$ which is a function on μ_js and independent of u. While the map $\boldsymbol{\varepsilon}$ changes ω_{w_i} to $\varepsilon^{w_i}\omega_{w_i}$ as $\mathrm{wt}(\omega_{w_i}) = w_i$, the deformation of the path α_j of the integral no longer affects this change of value integral because of Cauchy's theorem. So that, the mapping $\boldsymbol{\varepsilon}$ induces change of the integral

$$\int_{\alpha_j} \omega_{w_i} \longmapsto \varepsilon^{w_i} \int_{\alpha_j} \omega_{w_i},$$

for any non-zero ε. Therefore, we have

$$\mathrm{wt}\left(\int_{\alpha_j} \omega_{w_i} \right) = w_i.$$

By the argument above, the weight of such a circuit integral does not depend on its path of integral as well as the choice of the reference points ${\mu_j}^{(0)}$s and $u^{(0)}$.

3.3 Construction of the sigma functions

Using the notions defined in the previous subsections, we define the sigma function by the following characterization. Now we fix the curve $\mathscr{C} = \mathscr{C}_\mu^{e,q}$.

Theorem 3.13. *There exists a unique function* $(\mu, u) \mapsto \sigma(u) = \sigma(\mu, u)$ *having the following properties:*

(1) *$\sigma(u)$ is an entire function on $\mathbb{C}^g$ for any fixed μ_js in $\mathbb{C}$;*
(2) *Supposing that the $\{\mu_j\}$ are constants in $\mathbb{C}$ and $\varDelta$ is not zero, we have*

$$\sigma(u+\ell) = \chi(\ell)\,\sigma(u)\exp L(u+\tfrac{1}{2}\ell,\ell)$$

for any $u \in \mathbb{C}^g$ and $\ell \in \varLambda$, where $\varLambda$, L, and χ are defined in (3.4), (3.5), *and* (3.6), *respectively;*
(3) *Viewing $\sigma(u)$ as an element in $\mathscr{M}$ with arbitrary reference point* (*see* (3.10)),

$$\mathrm{wt}(\sigma(u)) = \tfrac{1}{24}(e^2-1)(q^2-1);$$

Moreover, $\sigma(u)$ is expanded as a power series around the origin $u=(0,\cdots,0)$ with coefficients in $\mathbb{Q}[\mu]$ of homogeneous weight $(e^2-1)(q^2-1)/24$;
(4) *$\sigma(u)|_{\mu=\mathbf{0}}$ is the Schur polynomial $s(u)$ of* (3.1);
(5) $\sigma(u)=0 \iff u \in \kappa^{-1}(\Theta)$.

Definition 3.14. *We call the function $\sigma(u)$ whose existence is granteeed by* 3.13 *the sigma function of the curve $\mathscr{C}$.*

The theorem 3.13 was proved in various ways in the literature, each version has a slightly different point of view. It is convenient to summarise these versions from our point of view. We define

$$\begin{aligned}\tilde{\sigma}(u) = \tilde{\sigma}(u,\varOmega) &= \Big(\frac{(2\pi)^g}{\det\omega'}\Big)^{\frac{1}{2}}\exp\big(-\tfrac{1}{2}\,{}^t u\,\eta'\omega'^{-1}u\big)\\ &\cdot\sum_{n\in\mathbb{Z}^g}\exp\big(\tfrac{1}{2}\,{}^t(n+\delta'')\,\omega'^{-1}\omega''(n+\delta'')+{}^t(n+\delta'')(\omega'^{-1}u+\delta')\big),\end{aligned} \tag{3.15}$$

where det denotes the determinant. Note that the arguments in any exponential are of weight 0, so that the infinite series part of (3.15) is of weight 0 as well. It is obvious that this function has property (1), and it is easy to show that this function satisfies (2) using (3.21). Frobenius' method shows that the solutions of the equation (2) form a one dimensional space (see p.93 of [15]). Although this function is constructed by using $\varOmega$, it is independent of $\varOmega$. Namely, the function $\tilde{\sigma}(u,\varOmega)$ is invariant under a modular transform, that is the transform of $\varOmega$ by $\mathrm{Sp}(2g,\mathbb{Z})$. Therefore, it is expressed as a power series of u with coefficients being functions of the μ_js. On the latter part of (3) and (4), we refer the reader to [19]. In that paper, $\tilde{\sigma}(u)$ times some constant is expressed as a determinant of infinite size (see also [23]). See also the paper [20] by Nakayashiki, in which he proved the sigma function is no other than (3.15) times a non-zero constant depending on μ_js with emphasizing the later part of (3). It is well known that $\tilde{\sigma}(u)$ has the property (5).

Using notation we have explained until now, we define

$$\hat{\sigma}(u) = \varDelta^{-\frac{1}{8}}\tilde{\sigma}(u), \tag{3.16}$$

with an apropriate choice of the $\frac{1}{8}$th root of the discriminannt $\varDelta$. It is known for $g=1$ and 2 that this function exactly satisfies all the properties in 3.13, namely, we have $\sigma(u) = \hat{\sigma}(u)$. For $g = 1$, it is shown as in [22] by a transformation formulae for $\eta(\tau)$ and the theta series described in pp.176–180 of [24], and for $g = 2$ the paper [14] by D. Grant, in which the property (4) is shown by using Thomae's formula.

However, it is not know for $g \geq 3$ if the function (3.16) is the sigma function. The paper [7] is the first one seriously attacked this problem.

In this paper, we show, following the idea of [7], that $\sigma(u) = \hat{\sigma}(u)$ for the genus 3 curves i.e. for the $(2,7)$-curve and the $(3,4)$-curve This means (3.16) satisfies especially (4) up to an absolutely numerical multiplicative constant. The strategy of proof is as follows: We construct a system of heat equations for $\hat{\sigma}(u)$ and show that the solution space is of dimension one (over the base field) by *explicit construction* of a recursive system for the coefficients of the power series expansion of any unknown solution and *showing the uniqueness* of the solution of this system. Then we see $\sigma(u) = \hat{\sigma}(u)$ up to a non-zero multiplicative absolutely numerical constant. This is seen as a generalisation of *Thomae's formula* ([25]). Section 4 is devoted to these last steps.

We here mention a lemma which is used later.

Lemma 3.17. *The constant* $\big((2\pi)^g/(\det\omega')\big)^{\frac{1}{2}}\Delta^{-\frac{1}{8}}$ *is of weight* $(e^2-1)(q^2-1)/24$*, which is equal to the weight of* $\sigma(u)$*.*

Proof. The weight of $\Delta^{\frac{1}{8}}$ is $-eq(e-1)(q-1)/8$. The weight of $\det(\omega')$ is $\sum_{j=1}^{g} w_j$, which equals

$$\sum_{j=1}^{g} w_j = \frac{eq(e-1)(q-1)}{4} - \frac{(e^2-1)(q^2-1)}{12}$$

by p.97 of [5]. Hence the weight of the constant is $\frac{(e^2-1)(q^2-1)}{24}$. □

3.4 Generalisation of the Frobenius-Stickelberger theory

This and the following section are devoted to explaining the theory of Buchstaber and Leykin [7], on the differentiation of Abelian functions with respect to their parameters, as clearly as we can. That generalises the work of Frobenius and Stickelberger [12], discussed above, on the elliptic case of this problem.

For higher genus cases, we do not have a naive generalisation of (1.8) and (1.9), which are mentioned in the Introduction. However, we can give a natural generalisation of the relations (1.10) to the curve $\mathscr{C}_\mu$ as explained in the next section.

We shall consider operators in $\mathbb{Q}[\mu][\partial_\mu]$. Here we denote by ∂_μ the set $\{\frac{\partial}{\partial\mu_j}\}$. We first explain the symbol $\frac{\partial}{\partial\mu_j}$. To do so, we recall Lemma 3.18 below due to Chevalley which is essentially the same as Lemma 1 in [16].

Let K be a field and R be a function field of one variable with K the field of coefficients. We will apply this lemma for $K = \mathbb{Q}(\mu)$. Take a transcendental element ξ in R over K and fix it. Let D be a derivation of K over $\mathbb{Q}$. By [9], p.112, Lemma 2, there exists unique derivation D_ξ on R satisfying

$$D_\xi(a) = 0 \quad \text{for } a \in \mathbb{Q}, \ \ D_\xi(\xi) = 0.$$

We extends D_ξ to the space of differentials of R via

$$D_\xi(\omega) = D_\xi\Big(\frac{\omega}{d\xi}\Big)d\xi \quad \text{for any } \omega \in R\,d\xi.$$

Lemma 3.18. (Chevalley [9], p.125, Lemma 3) *Under the notation above, let ξ and ζ be two transcendental elements in R over K. Then we have the following relation between D_ξ and D_ζ. For any $w \in R$, we have*

$$D_\xi(wd\xi) - D_\zeta(wd\xi) = d(-wD_\zeta\xi).$$

Proof. (From Manin [16]) The operator $D_\xi - D_\zeta + (D_\zeta\xi)\frac{d}{d\xi}$ is a derivative on R which vanishes on $\mathbb{Q}(\mu)$ and also kills ξ. Hence this vanishes on R, so that

$$(D_\xi - D_\zeta)w + (D_\zeta\xi)\frac{dw}{d\xi} = 0.$$

Moreover $(D_\xi - D_\zeta)(wd\xi) = (D_\xi - D_\zeta)w \cdot d\xi + w \cdot (D_\xi - D_\zeta)d\xi$. Since ζ is transcendental, we see $\frac{d}{d\zeta}D_\zeta = D_\zeta\frac{d}{d\zeta}$ by [9], p.125, Lemma 1, and

$$D_\xi d\xi = D_\xi\Big(\frac{d\xi}{d\xi}\Big)d\xi = 0, \quad D_\zeta d\xi = D_\zeta\Big(\frac{d\xi}{d\zeta}\Big)d\zeta = \frac{d}{d\zeta}(D_\zeta\xi)d\zeta = d\big(D_\zeta(\xi)\big).$$

Therefore

$$(D_\xi - D_\zeta)(wd\xi) = -(D_\zeta\,\xi)\frac{dw}{d\xi}d\xi - w{\cdot}d(D_\zeta\,\xi) = -d(wD_\zeta\,\xi)$$

as desired. □

For a sample usage of $\frac{\partial}{\partial\mu_j}$, see Section 4.3. By Lemma 3.18, any element L in $\mathbb{Q}[\mu][\partial_\mu]$ operates linearly on the space $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\mu])$. For complex variables μ_js, we let L operate firstly on the forms with *variables* μ_js of representative in $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\mu])$, then we restore μ_js to the original values in $\mathbb{C}$. Let L be a first order operator, namely an element in $\bigoplus_j \mathbb{Q}[\mu]\frac{\partial}{\partial\mu_j}$. We define $\Gamma^L \in \mathrm{Mat}(2g, \mathbb{Q}[\mu])$ as the representation matrix for the following action of L:

$$L(\boldsymbol{\omega}) = \boldsymbol{\omega}\ {}^t\Gamma^L \quad \text{for } \boldsymbol{\omega} = (\omega_{w_g}, \cdots, \omega_{w_1}) \in H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\mu]). \tag{3.19}$$

In [7], the matrix $-{}^t(\Gamma^L)$ defined by (3.19) is called the *Gauss-Manin connection* for the vector field L.

From here to the end of Section 3, we assume that all the μ_js are complex variables and $\mathscr{C}$ is non-singular. We can then use the periods ω_{ij}s and η_{ij}s.

By integrating (3.19) along each element in the chosen symplectic basis of $H_1(\mathscr{C}_\mu, \mathbb{Z})$, we get the natural action

$$L(\Omega) = \Gamma^L\,\Omega \quad \text{of } L \text{ on } \Omega = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}. \tag{3.20}$$

So, we see how L operates on the field $\mathbb{Q}(\{\omega'_{ij}\}, \{\omega''_{ij}\})$. Of course, since Ω is the period matrix of a symplectic basis, these elements must satisfy the constraint

$$(3.21)\qquad {}^t\Omega J \Omega = 2\pi i J, \quad \text{where } J = \begin{bmatrix} & 1_g \\ -1_g & \end{bmatrix}$$

by (2.7). This is none other than the generalisation of Frobenius-Stickelberger's relation (1.10), and is to say that $(2\pi i)^{-\frac{1}{2}}\Omega \in \mathrm{Sp}(2g, \mathbb{C})$. It follows immediately that

$$(3.22)\qquad \Omega^{-1} = \frac{1}{2\pi i} J^{-1}\,{}^t\Omega J = \frac{1}{2\pi i}\begin{bmatrix} {}^t\eta'' & -{}^t\omega'' \\ -{}^t\eta' & {}^t\omega' \end{bmatrix}.$$

After operating L on both sides of (3.21), using (3.20) and (3.21), we see that the matrix Γ^L satisfies

$$(3.23)\qquad {}^t\Gamma^L J + J\Gamma^L = 0, \quad \text{i.e.} \quad {}^t(\Gamma^L J) = \Gamma^L J$$

because $L({}^t\Omega) = {}^t\Omega\,{}^t\Gamma^L$, which is to say that $\Gamma^L \in \mathfrak{sp}(2g, \mathbb{Q}[\mu])$, the Lie algebra of $\mathrm{Sp}(2g, \mathbb{Q}[\mu])$. Thus we may write

$$(3.24)\qquad -\Gamma^L J = \begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix}, \quad \Gamma^L = \begin{bmatrix} -\beta & \alpha \\ -\gamma & {}^t\beta \end{bmatrix}$$

with ${}^t\alpha = \alpha$ and ${}^t\gamma = \gamma$.

Remark 3.25. (1) We use a different notation for $D(x, y, \lambda)$ compared to p.273 of [7], and for Ω, Γ and β compared to p.274 of loc. cit. Our ${}^t\boldsymbol{\omega}$ equals $D(x, y, \lambda)$ by transposing and changing the sign on the latter half entries. The others are naturally modified according to this difference and taking transposes. We will give a detailed comparison of our notation with theirs in 3.74 at the end of Subsection 3.8.
(2) In general, it requires some work to write down the given operator L as a partial differential operator with respect to the periods ${\omega'}_{ij}$ and ${\omega''}_{ij}$ similar to the LHS of (1.10). However, we do not use such an expression in the present paper.

Conversely, starting from a matrix

$$\Gamma = \begin{bmatrix} -\beta & \alpha \\ -\gamma & {}^t\beta \end{bmatrix} \in \mathfrak{sp}(2g, \mathbb{Q}[\mu]),$$

with ${}^t\alpha = \alpha$ and ${}^t\gamma = \gamma$, we get uniquely an operator $L \in \bigoplus_j \mathbb{Q}[\mu]\frac{\partial}{\partial \mu_j}$ such that $\Gamma^L = \Gamma$. So far, this is a natural generalisation of the situation investigated by Frobenius-Stickelberger [12].

3.5 Primary heat equations

In this Section, we review the general heat equations satisfied by the sigma functions.

If we want to find second-order linear partial differential equations (heat equations) satisfied by the sigma function, we should proceed in as general a way as possible. Here, note that the equation (1.6) is satisfied not only by the Jacobi theta function (1.3) but also by each individual term of the sum in (1.3). This corresponds a statement in the proof of Theorem 13 in page 274 of [7]. Here we will review their theory of such equations, correcting a few minor errors, and apply it explicitly to more general curves than considered in [7]. We shall start from this point of view.

In this paper, we start with a first order operator as in 3.19: $L \in \bigoplus_j \mathbb{Q}[\mu]\frac{\partial}{\partial \mu_j}$. Then we have the symmetric matrix $-\Gamma^L J = \begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix}$, and we also associate with it a second-order differential operator H^L, given by

$$
\begin{aligned}
H^L &= \frac{1}{2}[{}^t\partial_u \quad {}^tu]\begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix}\begin{bmatrix} \partial_u \\ u \end{bmatrix} \\
&= \sum_{i=1}^{g}\sum_{j=1}^{g}\Big(\tfrac{1}{2}\,\alpha_{ij}\,\frac{\partial^2}{\partial u_i \partial u_j} + \beta_{ij}\, u_i\frac{\partial}{\partial u_j} + \tfrac{1}{2}\gamma_{ij}\, u_i u_j\Big) + \tfrac{1}{2}\operatorname{Tr}\beta.
\end{aligned}
\tag{3.26}
$$

Here ∂_u denotes the column vector with g components $\frac{\partial}{\partial u_i}$, and u the column vector with g components u_is. The very last term comes from the commutation relation $\frac{\partial}{\partial u_i}u_j = u_j\frac{\partial}{\partial u_i} + \delta_{ij}$. It is then straightforward to verify the following

Lemma 3.27. *If we define a Green's function* $G_0(u,\varOmega)$ *by*

$$G_0(u,\varOmega) = \Big(\frac{(2\pi)^g}{\det\omega'}\Big)^{\frac{1}{2}} \exp\big(-\tfrac{1}{2}\,{}^tu\,\eta'\omega'^{-1}u\big),$$

then the heat equation

$$(L - H^L)G_0 = 0 \tag{3.28}$$

holds.

Proof. We have

$$
\begin{aligned}
L\,G_0 &= -\frac{1}{2}\operatorname{Tr}(\omega'^{-1}L(\omega'))G_0 - \frac{1}{2}({}^tu\,L(\eta')\omega'^{-1}u)G_0 + \frac{1}{2}({}^tu\,\eta'\omega'^{-1}L(\omega')\omega'^{-1}u)G_0 \\
&= \frac{1}{2}\Big[\operatorname{Tr}(\omega'^{-1}(\beta\omega' - \alpha\eta')) + ({}^tu\,((\gamma\omega' - {}^t\beta\eta')\omega'^{-1} - \eta'\omega'^{-1}(\beta\omega' - \alpha\eta')\omega'^{-1})\,u)\Big]\,G_0 \\
&= \frac{1}{2}\Big[\operatorname{Tr}(\beta - \alpha\eta'\omega'^{-1}) + ({}^tu\,(\gamma - {}^t\beta\eta'\omega'^{-1} - \eta'\omega'^{-1}\beta + \eta'\omega'^{-1}\alpha\eta'\omega'^{-1})\,u)\Big]\,G_0
\end{aligned}
$$

by using

$$\Gamma^L\varOmega = \begin{bmatrix} -\beta & \alpha \\ -\gamma & {}^t\beta \end{bmatrix}\begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix} = \begin{bmatrix} -\beta\omega' + \alpha\eta' & -\beta\omega'' + \alpha\eta'' \\ -\gamma\omega' + {}^t\beta\eta' & -\gamma\omega'' + {}^t\beta\eta'' \end{bmatrix}. \tag{3.29}$$

This is the same as

$$
\begin{aligned}
H^{L_0}G_0 &= \frac{1}{2}\big[{}^t\partial_u \quad {}^tu\big]\begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix}\begin{bmatrix} \partial_u \\ u \end{bmatrix}G_0 = \frac{1}{2}\big[{}^t\partial_u \quad {}^tu\big]\begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix}\begin{bmatrix} -\eta'\omega'^{-1}u \\ u \end{bmatrix}G_0 \\
&= \frac{1}{2}\operatorname{Tr}({}^t\beta - \alpha\eta'\omega'^{-1})\,G_0 + \frac{1}{2}\big[-{}^tu\eta'\omega'^{-1} \quad u\big]\begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix}\begin{bmatrix} -\eta'\omega'^{-1}u \\ u \end{bmatrix}G_0.
\end{aligned}
$$

Here we have used the generalized Legendre relation (3.21) and the symmetry of

$\eta'\omega'^{-1}$. □

Now we recall that the different terms in the expansion of the theta function are periodic translates of one another. Analogously, to construct the different terms appearing in the expansion of the sigma function, we act on G_0 by iterating an element of the Heisenberg group. For a variable $z \in \mathbb{Q}(\{\omega'_{ij}\},\ \{\omega''_{ij}\})$, and a two g-component column vectors p, q whose components also belong to $\mathbb{Q}(\{\omega'_{ij}\},\ \{\omega''_{ij}\})$, we introduce

$$F(z,p,q) = \exp(z)\exp({}^t p u)\exp({}^t q\partial_u).$$

We write its inverse operator as

$$F^{-1}(z,p,q) = \exp(-\,{}^t q\partial_u)\exp(-\,{}^t p u)\exp(-z).$$

Lemma 3.30. *Defining $F(z,p,q)$, L, and H^L as above, the operator equality*

$$F^{-1}(z,p,q)(L - H^L)F(z,p,q) = L - H^L \tag{3.31}$$

holds if and only if

$$L\begin{bmatrix} q \\ p \end{bmatrix} = \Gamma^L \begin{bmatrix} q \\ p \end{bmatrix} \quad \text{and} \quad L(z) = \frac{1}{2}\left({}^t p\alpha p - {}^t q\gamma q\right), \tag{3.32}$$

for any p, q, and z. Here Γ^L is that of (3.24).

Proof. We calculate directly that

$$\begin{aligned} F^{-1}(z,p,q)\,LF(z,p,q) &= F^{-1}(z,p,q)\big(L(z) + L({}^t p)\,u + L({}^t q)\,\partial_u\big)F(z,p,q) + L \\ &= L + \big(L(z) + L({}^t p)(u-q) + L({}^t q)\,\partial_u\big). \end{aligned}$$

Similarly, we find

$$\begin{aligned} F^{-1}(z,p,q)\,H^L F(z,p,q) &= \frac{1}{2}\big[{}^t\partial_u + {}^t p \quad {}^t u - {}^t q\big]\begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix}\begin{bmatrix} \partial_u + p \\ u - q \end{bmatrix} \\ &= H^L + \big[{}^t p \quad -{}^t q\big]\begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix}\begin{bmatrix} \partial_u \\ u \end{bmatrix} + \frac{1}{2}\big[{}^t p \quad -{}^t q\big]\begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix}\begin{bmatrix} p \\ -q \end{bmatrix}. \end{aligned}$$

Matching coefficients of ∂_u, u, and 1, and transposing and rearranging, we see that, respectively,

$$L(q) = [\,-\beta \quad \alpha\,]\begin{bmatrix} q \\ p \end{bmatrix}, \quad L(p) = [\,-\gamma \quad {}^t\beta\,]\begin{bmatrix} q \\ p \end{bmatrix}, \quad L(z) = \frac{1}{2}\left({}^t p\alpha p - {}^t q\gamma q\right)$$

as desired. □

Corollary 3.33. *The formula* (3.32) *holds if*

$$\begin{bmatrix} q \\ p \end{bmatrix} = \Omega\begin{bmatrix} b' \\ b'' \end{bmatrix}, \quad z = z_0 + \tfrac{1}{2}\,{}^t pq, \tag{3.34}$$

where b' and b'' are arbitrary numerical constant vectors, and z_0 is an irrelevant numerical constant, which we set to zero below.

Proof. We have $L\Big(\begin{bmatrix} q \\ p \end{bmatrix}\Big) = L(\Omega)\begin{bmatrix} b' \\ b'' \end{bmatrix} = \Gamma^L\Omega\begin{bmatrix} b' \\ b'' \end{bmatrix} = \Gamma^L\begin{bmatrix} p \\ q \end{bmatrix}$, which is the former relation in (3.32). The latter one is checked easily. □

Denoting the constant vector ${}^t[b'\ b'']$ simply by b, we denote p, q, and z with $z_0 = 0$ in (3.34) by $p(b)$, $q(b)$, and $z(b)$, respectively. We define

$$G(b,u,\Omega) = F\big(z(b),p(b),q(b)\big)\,G_0(u,\Omega).$$

Using the Legendre relations in the form (3.22), we note that ${}^tp\omega' - {}^tq\eta' = 2\pi i\,{}^tb''$, and hence we obtain

$$\begin{aligned} G(b,u,\Omega) &= F\big(z(b),p(b),q(b)\big)\,G_0(u,\Omega) \\ &= \exp(\tfrac{1}{2}\,{}^tpq)\ \exp({}^tpu)\ \exp(-\,{}^tq\eta'\omega'^{-1}u)\,\exp(-\tfrac{1}{2}\,{}^tq\eta'\omega'^{-1}q)\,G_0 \\ &= \exp(-\tfrac{1}{2}(2\pi i\,{}^tb''\omega'^{-1}q)\,\exp(2\pi i\,{}^tb''\omega'^{-1}u)\,G_0 \\ &= \exp(-\tfrac{1}{2}(2\pi i\,{}^tb''\omega'^{-1}(\omega'b'+\omega''b'')\,\exp(2\pi i\,{}^tb''\omega'^{-1}u)\,G_0 \\ &= \left(\frac{(2\pi)^g}{\det(\omega')}\right)^{\frac{1}{2}}\ \exp\big(-\tfrac{1}{2}\,{}^tu\eta'\omega'^{-1}u\big) \\ &\quad\cdot\exp\Big(2\pi i\big(\tfrac{1}{2}\,{}^tb''\omega'^{-1}\omega''b'' + {}^tb''(\omega'^{-1}u+b')\big)\Big). \end{aligned}$$

Now, the following theorem, which is the foundation of BL theory, is obvious from (3.28) and (3.31).

Theorem 3.35. (Primary heat equation) *For the function $G(b,u,\Omega)$ above, one has*

$$(L - H^L)\,G(b,u,\Omega) = 0 \tag{3.36}$$

for any L in $\bigoplus_j \mathbb{Q}[\mu]\frac{\partial}{\partial\mu_j}$.

3.6 The algebraic heat operators

For the coordinates of the space in which the sigma function is defined, we do not use $(u_1,\cdots,u_g)$ for subscripts of the variable u, but denote instead, as in (3.3)

$$u = (u_{w_g},\cdots,u_{w_1}).$$

That is, the components of u are labelled by their weights, which are the Weierstrass gaps. As in Section 3.5, we suppose $L \in \bigoplus_j \mathbb{Q}[\mu]\frac{\partial}{\partial\mu_j}$. Then we have the representation matrix $\Gamma^L \in \mathfrak{sp}(2g,\mathbb{Q}[\mu])$. As a corollary to Theorem 3.35, we have

Corollary 3.37. *Let*

$$\rho(u) = \sum_b G(b,u,\Omega), \tag{3.38}$$

where b runs through the elements of any set $\subset \mathbb{C}^{2g}$ such that the sum converges absolutely. Then we have

$$(L - H^L)\,\rho(u,\Omega) = 0$$

for any L in $\bigoplus_j \mathbb{Q}[\mu]\frac{\partial}{\partial\mu_j}$.

Proof. Since both of L and H^L are independent of b, each term of $\rho(u)$ satisfies (3.36). $\square$

Remark 3.39. Assume that $\mathscr{C}_\mu$ is non-singular. Let $\varOmega$ be the usual period matrix defined by (3.2), and δ be its Riemann constant. The function defined at (3.15) is written as

$$\tilde{\sigma}(u) = \tilde{\sigma}(u, \varOmega) = \sum_{n \in \mathbb{Z}^g} G({}^t[\delta'\ n + \delta''], u, \varOmega). \tag{3.40}$$

Since the imaginary part of $\omega'^{-1}\omega''$ is positive definite, this series converges absolutely. This is a special case of $\rho(u)$ of (3.37).

Because both L and H^L are independent of b, there are infinitely many linearly independent entire functions $\rho(u)$ on $\mathbb{C}^g$ satisfying $(L - H^L)\rho(u) = 0$. Moreover, since, for a fixed b, the function $G(b, u, \varOmega)$ is independent of L, we see that, by switching the choice of L, there are infinitely many linearly independent operators of the form $(L - H^L)$ which satisfy $(L - H^L)\,\rho(b, u, \varOmega) = 0$ for some fixed $\rho(u)$.

However, because our aim is to find a method to calculate the power series expansion of the sigma function, we need a more detailed discussion. For our purpose,

(A1) we need to find operators in the ring $\mathbb{Q}[\mu][\partial_\mu, \partial_u]$ which annihilate the sigma function (3.16), and

(A2) we require that any function killed by all such operators belongs to the ring $\mathbb{Q}[\mu][[u_{w_g}, \cdots, u_{w_1}]]$.

Since L is a derivation with respect to the μ_js but H^L is a differential operator with respect to the u_{w_j}s, we see that, for some function $\varXi$ depending only on the μ_js,

$$L\,\varXi\tilde{\sigma}(u) = (L\varXi)\,\tilde{\sigma}(u) + \varXi(L\tilde{\sigma}(u)), \qquad H^L\,\varXi\tilde{\sigma}(u) = \varXi(H^L\tilde{\sigma}(u)).$$

Therefore, $\varXi\tilde{\sigma}(u)$ satisfies

$$(L - H^L)(\varXi\tilde{\sigma}(u)) = \tfrac{L\varXi}{\varXi}\,\varXi\,\tilde{\sigma}(u) = (L\log\varXi)\varXi\,\tilde{\sigma}(u). \tag{3.41}$$

If $\varXi\,\tilde{\sigma}(u)$ is the correct sigma function, the left hand side of the above is in $\mathbb{Q}[\mu][u]]$, So, if we suppose that the correct $\sigma(u)$ is equal to $\hat{\sigma}(u)$ of (3.16), then we should narrow down the choice of L to one such that $L\log\varDelta \in \mathbb{Q}[\mu]$.

3.7 The operators tangent to the discriminant

Throughout this section, we suppose that all the μ_js, x, and y are variables or indeterminates. In view of the approach in [7], we require that $L\log\varDelta$ belongs to $\mathbb{Q}[\mu]$ because of conditions (A1), (A2) and equation (3.41). Now we explain suitable choices for Γ for which L satisfies the condition. Let $M(x, y)$ be a vector consisting of $2g$ monomials of x and y displayed in descending order of weight (i.e. ascending order of absolute value of weight) in the first g entries and in the ascending order (i.e. descending order of absolute value of weight) in the remaining g entries.

For $\{w_1,\ \cdots,\ w_g\}$ in (2.5) with respect to (e,q), we define a sequence $\{v_j\}_{j=1}^{2g}$ by

$$v_j=\begin{cases}2g-1-w_{g-j+1} & \text{if}\ \ 1\le j\le g,\\ 2g-1+w_{j-g} & \text{if}\ \ g+1\le j\le 2g.\end{cases}$$

We denote by $M_j(x,y)$ the monomial in x and y of weight $-v_j$. We define

$$\begin{aligned}M(x,y)&={}^t[\,M_j(x,y)\ \ (j=1,2,\cdots,2g)\,],\\ \check{M}(x,y)&={}^t[\,M_j(x,y)\ \ (j=2g,2g-1,\cdots,1)\,].\end{aligned}$$

This is equivalent to saying that $M(x,y)$ is the vector whose entries are displayed as the elements of the set

$$M(x,y)=\{\,x^iy^j\,|\,0\le i\le e-2,\ 0\le j\le q-2\,\}$$

in the order as above. We see that all the terms $\{M_j(x,y)\}$ appear in $f(x,y)$ provided the *modality* of the curve is 0. We shall give here explicit values of the data above not only for the convenience of the reader who following the detailed calculation in Section 4, but also for those considering higher genus curves outside the scope of this paper.

Example 3.42. If $(e,q)=(2,2g+1)$, then M_js are given as follows:

j	1	2	$\cdots$	g	$g+1$	$g+2$	$\cdots$	$2g$	
w_{g-j+1}	$2g-1$	$2g-3$	$\cdots$	1	1	3	$\cdots$	$2g-1$	w_{j-g}
v_j	0	2	$\cdots$	$2g-2$	$2g$	$2g+2$	$\cdots$	$4g-2$	
$M_j(x,y)$	1	x	$\cdots$	x^{g-1}	x^g	x^{g+1}	$\cdots$	x^{2g-1}	

If $e=3$ and $q=4$ or 5, then M_js are given as follows:

$(3,4)$-curve :

j	1	2	3	4	5	6	
w_{3-j+1}	5	2	1	1	2	5	w_{j-3}
v_j	0	3	4	6	7	10	
$M_j(x,y)$	1	x	y	x^2	xy	x^2y	

$(3,5)$-curve :

j	1	2	3	4	5	6	7	8	
w_{4-j+1}	7	4	2	1	1	2	4	7	w_{j-4}
v_j	0	3	5	6	8	9	11	14	
$M_j(x,y)$	1	x	y	x^2	xy	x^3	x^2y	x^3y	

All these examples, and only these, are of modality zero.

Example 3.43. In contrast to the cases above, we have for the $(3,7)$-curve :

j	1	2	3	4	5	6	7	8	9	10	11	12	
w_{6-j+1}	11	8	5	4	2	1	1	2	4	5	8	11	w_{j-6}
v_j	0	3	6	7	9	10	12	13	15	16	19	22	
$M_j(x,y)$	1	x	x^2	y	x^3	xy	x^4	x^2y	x^5	x^3y	x^4y	x^5y	

for$(e,q)=(3,7)$. However, the Weierstrass equation of the $(3,7)$-curve is given by

$$\begin{aligned} &y^3-(\mu_2x^4+\mu_5x^3+\mu_8x^2+\mu_{11}x+\mu_{14})y \\ &\quad = x^7+\mu_6x^5+\mu_9x^4+\mu_{12}x^3+\mu_{15}x^3+\mu_{18}x+\mu_{21} \end{aligned}$$

and this equation does not include a term in $M_{12}(x,y)=x^5y$. This curve is of modality 1. We will not discuss this curve further in this paper.

Lemma 3.44. *Let x and y be indeterminates and $f_1=\frac{\partial}{\partial x}f$, $f_2=\frac{\partial}{\partial y}f$.*
(1) *As a $\mathbb{Q}[\mu]$-module, $\mathbb{Q}[\mu][x,y]/(f_1(x,y),f_2(x,y))$ is of rank $2g$ and spanned by $M(x,y)=\{\,x^iy^j\,|\,0\le i\le e-2,\ 0\le j\le q-2\,\}$.*
(2) *The linear map given by the multiplication by $-eq\,f(x,y)$*

$$\cdot(-eq)\,f(x,y)\ :\ \mathbb{Q}[\mu][x,y]/(f_1,f_2)\longrightarrow \mathbb{Q}[\mu][x,y]/(f_1,f_2)$$

is of rank $2g$.
(3) *Let T be the representation matrix of the map above (with respect to $M(x,y)$). Then its determinant $\det(T)$ is a constant multiple of a power of Δ.*

Proof. (From Theorem A8 in [3]) We assume $\mu\subset\mathbb{C}$. We have $\det(T)=0$ if and only if the rank[2] of the co-kernel $\mathbb{Q}[\mu][x,y]/(f,f_1,f_2)$ of the map is positive. This is exactly the case that the ideal (f,f_1,f_2) does not contain $1\in\mathbb{Q}[\mu][x,y]$. By Hilbert's Nullstellensatz (Theorem 5.4(i) in [17], for instance), we see this is equivalent to saying that there exists a set $(x,y)\in\mathbb{C}^2$ such that

$$f(x,y)=f_1(x,y)=f_2(x,y)=0.$$

Therefore the zeroes of the discriminant Δ and ones of $\det(T)$ coincide. So that $\det(T)$ must be non-zero integer multiple of a power of the discriminant Δ. □

Here we shall explain how we explicitly calculate T. To plug smoothly with the later calculation, we shall write down the definition of the entries in

$$T=[\,T_{ij}\,]_{i,j=1,\cdots,2g}.$$

Namely, we define T_{ij}s by the equalities

$$-eq\,f(x,y)\,M_i(x,y)\equiv\sum_{j=1}^{2g}T_{ij}\,M_{2g+1-j}(x,y)\ \bmod\ (f_1,f_2) \tag{3.45}$$
$$\left(\text{ i.e. }\ -eq\,f(x,y)\,M(x,y)\equiv T\,\check{M}(x,y)\ \bmod\ (f_1,f_2)\ \right).$$

Then $\mathrm{wt}(T_{ij})=-(eq+v_i-v_{2g+1-j})$. The factor $-eq$ makes later calculation simpler. Note that the signs of the first row and column of T is negative. We have the expression of T_{ij} by using $f_1(x,y)=\cdots-qx^{q-1}+\cdots$ and $f_2(x,y)=ey^{e-1}+\cdots$. We give the explicit ratio, an integer, of $\det(T)/\Delta$ for the cases we look at in Sections 4.3, 4.4, 4.5, and 4.7

Now we explain another method to construct the *discriminant* Δ and a basis of the space of the vector fields tangent to the variety defined by $\Delta=0$ by following

[2] the Tjurina number at μ.

a method known in singularity theory.

However, following p.112 of [6], we also define a function $H = H\big((x,y),(z,w)\big)$, which is defined by[3]

$$H = \frac{1}{2}\begin{vmatrix} \dfrac{f_1(x,y)-f_1(z,w)}{x-z} & \dfrac{f_2(x,y)-f_2(z,w)}{x-z} \\ \dfrac{f_1(z,y)-f_1(x,w)}{y-w} & \dfrac{f_2(z,y)-f_2(x,w)}{y-w} \end{vmatrix}.$$

For any $F \in \mathbb{Q}[\mu][x,y]$, we define

$$\text{Hess}\, F = \begin{vmatrix} \frac{\partial^2}{\partial x^2}F & \frac{\partial^2}{\partial x \partial y}F \\ \frac{\partial^2}{\partial y \partial x}F & \frac{\partial^2}{\partial y^2}F \end{vmatrix}. \tag{3.46}$$

Lemma 3.47. (Buchstaber-Leykin [6], p.64) *Let I be the ideal in $\mathbb{Q}[\mu][x,y,z,w]$ generated by $f_1(x,y)$, $f_2(x,y)$, $f_1(z,w)$, and $f_2(z,w)$. The determinant H has the following properties.*
(1) $H\big((x,y),(x,y)\big) = \text{Hess}\, f(x,y)$.
(2) $H\big((x,y),(z,w)\big) = H\big((z,w),(x,y)\big)$.
(3) *We have*

$$H\big((z,w),(x,y)\big)\, F\big((x,y),(z,w)\big) \equiv H\big((x,y),(z,w)\big)\, F\big((z,w),(x,y)\big) \mod I$$

for any $F\big((x,y),(z,w)\big) \in \mathbb{Q}[\mu][x,y,z,w]$.

Proof. (1) Taking limit $z \to x$ after subtracting the second row times $\frac{y-w}{x-z}$ from the first row in H, we have

$$\frac{1}{2}\begin{vmatrix} f_{11}(x,y)+f_{11}(x,w) & f_{21}(x,y)+f_{21}(x,w) \\ \dfrac{f_1(x,y)-f_1(x,w)}{y-w} & \dfrac{f_2(x,y)-f_2(x,w)}{y-w} \end{vmatrix},$$

where $f_{11}(x,y) = \frac{\partial^2}{\partial x^2}(x,y)$, etc. Then, by taking limit $y \to w$ we get $\text{Hess}\, f(x,y)$.
(2) is trivial.
(3) By expanding the matrix, we see that the numerator

$$\begin{aligned} &\big(f_1(x,y)f_2(z,y) - f_1(x,y)f_2(x,w) - f_1(z,w)f_2(z,y) + f_1(z,w)f_2(x,w)\big) \\ &\quad - \big(f_1(z,y)f_2(x,y) - f_1(z,y)f_2(z,w) - f_1(x,w)f_2(x,y) + f_1(x,w)f_2(z,w)\big) \\ &= \big(f_1(x,y)f_2(z,y) - f_1(z,y)f_2(x,y)\big) - \big(f_1(x,y)f_2(x,w) - f_1(z,y)f_2(z,w)\big) \\ &\quad - \big(f_1(z,w)f_2(z,y) - f_1(x,w)f_2(x,y)\big) + \big(f_1(z,w)f_2(x,w) - f_1(x,w)f_2(z,w)\big) \\ &= \big(f_1(x,y)f_2(z,y) - f_1(x,w)f_2(z,w)\big) - \big(f_1(x,y)f_2(x,w) - f_1(x,w)f_2(x,y)\big) \\ &\quad - \big(f_1(z,w)f_2(z,y) - f_1(z,y)f_2(z,w)\big) + \big(f_1(z,w)f_2(x,w) - f_1(z,y)f_2(x,y)\big) \end{aligned} \tag{3.48}$$

is divisible by $(z-x)(w-y)$, because the second expression is clearly divisible by $(z-x)$, while the third expression is divisible by $(w-y)$. Hence, $H\big((x,y),(x,y)\big) \in \mathbb{Q}[\mu][x,y]$. Moreover, the second expansion is equal to

$$\big(f_1(x,y)f_2(z,y) - f_1(x,y)f_2(x,y) + f_1(x,y)f_2(x,y) - f_1(z,y)f_2(x,y)\big)$$

[3] Do not confuse this with the heat operator H.

$$
\begin{aligned}
&- \big(f_1(x,y)f_2(x,w) - f_1(x,y)f_2(z,w) + f_1(x,y)f_2(z,w) - f_1(z,y)f_2(z,w)\big)\\
&- \big(f_1(z,w)f_2(z,y) - f_1(z,w)f_2(x,y) + f_1(z,w)f_2(x,y) - f_1(x,w)f_2(x,y)\big)\\
&+ \big(f_1(z,w)f_2(x,w) - f_1(z,w)f_2(z,w) + f_1(z,w)f_2(z,w) - f_1(x,w)f_2(z,w)\big)\\
&= f_1(x,y)\big(f_2(z,y) - f_2(x,y)\big) + \big(f_1(x,y) - f_1(z,y)\big)f_2(x,y)\\
&- f_1(x,y)\big(f_2(x,w) - f_2(z,w)\big) - \big(f_1(x,y) - f_1(z,y)\big)f_2(z,w)\\
&- f_1(z,w)\big(f_2(z,y) - f_2(x,y)\big) - \big(f_1(z,w) - f_1(x,w)\big)f_2(x,y)\\
&+ f_1(z,w)\big(f_2(x,w) - f_2(z,w)\big) + \big(f_1(z,w) - f_1(x,w)\big)f_2(z,w),
\end{aligned}
$$

which implies that $H\big((x,y),(z,w)\big)(w-y)$ already belongs to I. A similar calculation shows that $H\big((x,y),(z,w)\big)(z-x) \in I$. For $F\big((z,w),(x,y)\big) = x^a y^b$, by using (2), we see

$$
\begin{aligned}
&H\big((z,w),(x,y)\big)\, x^a y^b - H\big((x,y),(z,w)\big)\, z^a w^b\\
&\quad = H\big((z,w),(x,y)\big)\,(x^a y^b - z^a w^b)\\
&\quad = H\big((z,w),(x,y)\big)\,(x^a y^b - x^a w^b + x^a w^b - z^a w^b)\\
&\quad = H\big((z,w),(x,y)\big)\,\big(x^a(y^b - w^b) + (x^a - z^a)w^b\big) \in I.
\end{aligned}
$$

Hence, (3) has been proved. □

Below, we will use, instead of T, the *symmetric* $2g \times 2g$ matrix

$$V = [V_{ij}]$$

with entries in $\mathbb{Q}[\mu]$ defined by the equation

$$ {}^t\check{M}(x,y)\, V\, \check{M}(z,w) = f(x,y)\, H \tag{3.49}$$

in the ring

$$\mathbb{Q}[\mu][x,y,z,w]\,/\,I.$$

We define $[\,H_{ij}\,]$ as the matrix given by

$$H\big((x,y),(z,w)\big) = {}^t\check{M}(x,y)\,[\,H_{ij}\,]\,\check{M}(z,w).$$

Lemma 3.50. *The matrix* $[\,H_{ij}\,]$ *is of the form*

$$[\,H_{ij}\,] = \begin{bmatrix} * & \cdots & * & -eq \\ * & \cdots & -eq & \\ \vdots & \iddots & & \\ -eq & & & \end{bmatrix}. \tag{3.51}$$

If $e = 2$, *then we have explicitly*

$$[\,H_{ij}\,] = \begin{bmatrix} -4\mu_{2(q-2)} & -6\mu_{2(q-3)} & -8\mu_{2(q-4)} & \cdots & -2(q-2)\mu_4 & 0 & -2q \\ -6\mu_{2(q-3)} & -8\mu_{2(q-4)} & -10\mu_{2(q-5)} & \cdots & 0 & -2q & \\ -8\mu_{2(q-4)} & -10\mu_{2(q-5)} & -12\mu_{2(q-6)} & \cdots & -2q & & \\ \vdots & \vdots & \vdots & \iddots & & & \\ -2(q-2)\mu_4 & 0 & -2q & & & & \\ 0 & -2q & & & & & \\ -2q & & & & & & \end{bmatrix}. \tag{3.52}$$

Proof. Any entry H_{ij} belongs to $\mathbb{Q}[\mu]$ by consideration of 3.48 in the proof of 3.47. Setting all the μ_j to be 0, we have

$$\frac{1}{2}\begin{vmatrix} \frac{q(-x^{q-1}+z^{q-1})}{x-z} & \frac{e(y^{e-1}-w^{e-1})}{x-z} \\ \frac{q(x^{q-1}-z^{q-1})}{y-w} & \frac{e(y^{e-1}-w^{e-1})}{y-w} \end{vmatrix} = -eq(x^{q-2}+x^{q-3}z+\cdots+z^{q-2}) \cdot (y^{e-2}+y^{e-3}w+\cdots+w^{e-2}).$$

It follows that the counter-diagonal entries of $[\,H_{ij}\,]$ are $-eq$. From the definitions, the weight of H is $-2(eq-e-q)$ and $\mathrm{wt}(M_{2g}(x,y)) = -2g-1+w_g = -2(eq-q-e)$. Therefore the entries below the counter-diagonal must be 0. For the case $e=2$, we have

$$\begin{aligned}\frac{1}{2}&\begin{vmatrix} -\frac{q(x^{q-1}-z^{q-1})+(q-2)\mu_4(x^{q-3}-z^{q-3})+\cdots+\mu_{eq-e}(x-z)}{x-z} & \frac{2y-2w}{x-z} \\ -\frac{q(x^{q-1}-z^{q-1})+(q-2)\mu_4(x^{q-3}-z^{q-3})+\cdots+\mu_{eq-e}(x-z)}{y-w} & \frac{2y-2w}{y-w} \end{vmatrix} \\ &= -2\,\frac{q(x^{q-1}-z^{q-1})+(q-2)\mu_4(x^{q-3}-z^{q-3})+\cdots+\mu_{eq-e}(x-z)}{x-z} \\ &= -2\,\big(q(x^{q-2}+x^{q-3}z+\cdots+z^{q-2}) \\ &\quad +(q-2)\mu_4(x^{q-3}+x^{q-4}z+\cdots+z^{q-4})+\cdots+\mu_{eq-e}\big),\end{aligned}$$

giving the desired form of $[\,H_{ij}\,]$. □

Lemma 3.53. *We have* $\det(V) = \det(T)$.

Proof. Since

$$\begin{aligned} f(x,y)\,H\big((x,y),(z,w)\big) &= f(x,y)\,{}^t\check{M}(z,w)\,[\,H_{jk}\,]\,\check{M}(x,y) \\ &\equiv {}^tM(z,w)\,[\,-\tfrac{1}{eq}\,T_{ij}\,][\,H_{jk}\,]\,\check{M}(x,y) \bmod I \end{aligned}$$

by (3.45), and the entries in $M(x,y)$ form a basis of $\mathbb{Q}[\mu][x,y]/(f_1(x,y),f_2(x,y))$, we see that V equals $-\frac{1}{eq}T[H_{jk}]$ with sorted rows in reverse order. Since $[H_{jk}]$ is a skew-upper-triangular matrix of the form (3.51), we have demonstrated $\det(V) = \det(T)$ as desired. □

Remark 3.54. *It is easy to see that* $\mathrm{wt}(V_{ij}) = \mathrm{wt}(T_{ij}) = eq - v_i - v_j$.

We now compute the weight of T_{ij} and V_{ij}. If $1 \le i \le g$ and $1 \le j \le g$, then, by the definition,

$$\begin{aligned}\mathrm{wt}(V_{ij}) = \mathrm{wt}(T_{ij}) &= -(eq+v_i-v_{2g+1-j}) \\ &= -eq-(2g+1-w_{g+1-i})+(2g+1+w_{g+1-j}) \\ &= -eq+w_{g+1-i}+w_{g+1-j}.\end{aligned}$$

If $g+1 \le i \le 2g$ and $1 \le j \le g$, then

$$\begin{aligned}\mathrm{wt}(V_{ij}) = \mathrm{wt}(T_{ij}) &= -(eq+v_i-v_{2g+1-j}) \\ &= -eq-(2g+1+w_{i-g})+(2g+1+w_{g+1-j}) \\ &= -eq-w_{i-g}+w_{g+1-j}.\end{aligned}$$

For the other i and j, we have similar formulae.

Lemma 3.55. *We have* $\mathrm{wt}(\det V) = \mathrm{wt}(\det T) = -eq(e-1)(q-1)$.

Proof. This follows from $\mathrm{wt}(T_{ii}) + \mathrm{wt}(T_{2g-i,2g-i}) = -2eq$. □

Example 3.56. If $(e, q) = (2, 2g + 1)$, we see, for $1 \leq i \leq g$ and $1 \leq j \leq g$, that

$$\begin{aligned}\mathrm{wt}(V_{ij}) = \mathrm{wt}(T_{ij}) &= -eq + w_{g+1-i} + w_{g+1-j} \\ &= -2(2g+1) + 2(g+1-i) - 1 + 2(g+1-j) - 1 = -2(i+j).\end{aligned}$$

If $g + 1 \leq i \leq 2g$ and $1 \leq j \leq g$, then

$$\begin{aligned}\mathrm{wt}(V_{ij}) = \mathrm{wt}(T_{ij}) &= -eq - w_{i-g} + w_{g+1-j} \\ &= -2(2g+1) - 2(i-g) + 1 + 2(g+1-j) - 1 = -2(i+j).\end{aligned}$$

For the other i and j, we have the same result.

Lemma 3.57. *We have* $\mathrm{wt}(\varDelta) = -eq(e-1)(q-1)$ *and* $\det T = \det V = \pm\varDelta$ *if* $\gcd(e-1, q-1) = 1$.

Proof. Letting all the coefficients μ_j of $p_j(x)$ for $1 \leq j \leq e-1$ to be zero, the discriminant $\varDelta$ becomes a power of the square of the difference of all the roots of $p_e(x) = 0$. Since the weight of any root is $-e$, the weight of the square of the difference is

$$2 \cdot \binom{q}{2} \cdot (-e) = -eq(q-1).$$

Similar arguments shows that $\mathrm{wt}\,\varDelta$ is $-qe(e-1)$ times an positive integer. By the assumption $\gcd(e-1, q-1) = 1$, we have $\mathrm{wt}\,\varDelta$ is $-eq(e-1)(q-1)$ multiple of a positive integer. The statement follows from 3.55 combined with 3.44 (3). □

Remark 3.58. *The condition* $\gcd(e-1, q-1) = 1$ *in* 3.57 *holds if* $e = 2$ *or* $(e, q) = (3, 4)$, $(3, 5)$, *for which we already know* $\varDelta$ *explicitly as mentioned in* 2.11.

From now on we assume the modality of C to be 0. For any v_i, the coefficient μ_{eq-v_i} appears in the Weierstrass form, which is is the reason why the modality is so important. Let

$$L_{v_i} = \sum_{j=1}^{2g} V_{ij} \frac{\partial}{\partial \mu_{eq-v_j}} \quad (i = 0, \cdots, 2g). \tag{3.59}$$

It is natural to define $\mathrm{wt}\big(\frac{\partial}{\partial \mu_j}\big) = j$ and $\mathrm{wt}(L_{v_i}) = v_i$. Then by definition (3.26), we have

$$\begin{aligned} H^{L_{v_j}} &= \frac{1}{2}\begin{bmatrix} \frac{\partial}{\partial u_{w_g}} & \cdots & \frac{\partial}{\partial u_{w_1}} & u_{w_g} & \cdots & u_{w_1}\end{bmatrix}\begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma\end{bmatrix}\begin{bmatrix} \frac{\partial}{\partial u_{w_g}} \\ \vdots \\ \frac{\partial}{\partial u_{w_1}} \\ u_{w_g} \\ \vdots \\ u_{w_1}\end{bmatrix} \\ &= \sum_{i=1}^{g}\sum_{j=1}^{g}\Big(\tfrac{1}{2}\,\alpha_{ij}\,\frac{\partial^2}{\partial\,u_{w_{g+1-i}}\,\partial\,u_{w_{g+1-j}}} \\ &\qquad + \beta_{ij}\,u_{w_{g+1-i}}\frac{\partial}{\partial u_{w_{g+1-j}}} + \tfrac{1}{2}\,\gamma_{ij}\;u_{w_{g+1-i}}u_{w_{g+1-j}}\Big) + \tfrac{1}{2}\,\mathrm{Tr}\,\beta. \end{aligned} \tag{3.60}$$

On the operators (3.59) and the discriminant, we have the following.

Proposition 3.61. *Any L_{v_i} is tangent to the discriminant Δ, and $(L_{v_i}\Delta)/\Delta \in \mathbb{Q}[\mu]$. Moreover, any operator $D \in \mathbb{Q}[\mu][\partial_\mu]$ which is tangent to Δ is a linear combination over $\mathbb{Q}[\mu]$ of the L_{v_i}s.*

Proof. This follows from Kyoji Saito's theorem (see Theorem A4 in [3]). See also Corollary 3 to Theorem (p.2716) of [27] of Zakalyukin, and Corollary 3.4 in [1]. However, the statement for the cases $(e,q) = (2,q)$, $(3,4)$ is contained in 3.62. □

From 3.61 and the Frobenius integrability theorem, we see that the set $\{L_{v_j}\}$ of operators form a basis of a Lie algebra. The subvariety defined by $\Delta = 0$ is the maximal integral manifold of that set. The structure constants of this algebra are polynomials in the μ_i, so it is a polynomial Lie algebra, as discussed in [4]. The corresponding fundamental relations of $\{L_{v_j}\}$ for $(e,q) = (2,3)$, $(2,5)$, $(2,7)$, and $(3,4)$ are available on request.

We prove the following proposition for which there is no proof in [7].

Proposition 3.62. (Buchstaber-Leykin) *Let*

$$L = M(x,y)\ {}^t[\,L_{v_1}\ L_{v_2}\ \cdots\ L_{v_{2g}}\,].$$

Then, in the ring $\mathbb{Q}[\mu][x,y]/(f_1,f_2)$, we have

$$L(\Delta) = \mathrm{Hess} f \cdot \Delta. \tag{3.63}$$

Proof. For the case $e = 2$, $f(x,y)$ is of the form $y^2 - p_2(x)$. We denote $p_2(x) = p(x)$ for simplicity. Moreover, we denote $p'(x) = \frac{\partial}{\partial x}p(x)$ and $p''(x) = \frac{\partial^2}{\partial x^2}p(x)$. In this case, $\mathbb{Q}[\mu][x,y]/(f_1,f_2)$ is identified with $\mathbb{Q}[\mu][x]/(p'(x))$ since $f_1(x,y) = \frac{\partial}{\partial y}f(x,y) = 2y$. Let F be a splitting field of $p(x)$. We write the factorisation of $p(x)$ in F as $p(x) = (x-a_1)\cdots(x-a_q)$. Then μ_{2i} is $(-1)^i$ times the fundamental symmetric function of $a_1, \cdots, a_q$ of degree i. Of course the ring $\mathbb{Q}[\mu]$ is a sub-ring of $\mathbb{Q}[a_1, \cdots, a_q]$. The main idea is to consider $\frac{\mathrm{Hess}\, f}{f} = -\frac{p''(x)}{p(x)}$ in the localised ring

$$\big(F[x]/(p'(x))\big)_{p(x)}$$

of $F[x]/(p'(x))$ with respect to the multiplicative set $\{1, p(x), p(x)^2, \cdots\}$ (see [17], Section 4). The following calculation is done in the localised ring above, by which we see $\big(F[x]/(p'(x))\big)_{p(x)} = F[x]/(p'(x))$.

$$\begin{aligned}
\frac{p''(x)}{p(x)} &= \sum_{(i,j),i<j} \frac{2}{(x-a_i)(x-a_j)} = \sum_{(i,j),i<j} \frac{2}{a_i - a_j}\Big(\frac{1}{x-a_i} - \frac{1}{x-a_j}\Big) \\
&= 2\sum_{i=1}^{q}\Big(\sum_{j\neq i}\frac{1}{a_i-a_j}\Big)\frac{1}{x-a_i} = -2\sum_{i=1}^{q}\Big(\sum_{j\neq i}\frac{1}{a_i-a_j}\Big)\frac{1}{p'(a_i)}\frac{-p'(a_i)}{x-a_i} \\
&= -2\sum_{i=1}^{q}\Big(\sum_{j\neq i}\frac{1}{a_i-a_j}\Big)\frac{1}{p'(a_i)}\frac{p'(x)-p'(a_i)}{x-a_i} = -2\sum_{i=1}^{q} c_i\,\frac{p'(x)-p'(a_i)}{x-a_i},
\end{aligned}$$

where

$$c_i = \Big(\sum_{j\neq i}\frac{1}{a_i-a_j}\Big)\frac{1}{p'(a_i)}.$$

Since $\mathrm{Hess} f(x,y) = -p''(x)$ and $p(x) = -f(x,y)$ in the localised ring, it is sufficient to show that

$$\frac{\partial}{\partial \mu_{2i}} \log \varDelta = \sum_{j=1}^{q} c_j {a_j}^{q-i}$$

up to a non-zero constant multiple. Indeed, if we have the formula above, we have

$$\begin{aligned}\frac{L(x)\varDelta}{\varDelta} &= \sum_{i,k} M_i(x,y) V_{ik} \frac{\partial}{\partial \mu_{2i}} \log \varDelta = \sum_{i,k} M_i(x,y) V_{ik} \sum_{j=1}^{q} c_j {a_j}^{q-k} \\ &= \sum_{j=1}^{q} c_j \sum_{i,k} M_i(x,y) V_{ik} {a_j}^{q-k} = \sum_{j=1}^{q} c_j\, f(x,y)\, H\big((x,y),(a_j,0)\big) \\ &= f(x,y) \sum_{j=1}^{q} c_j\, 2\, \frac{p'(x)-p'(a_j)}{x-a_j} = f(x,y)\,\frac{p''(x)}{p(x)} = -p''(x) = \mathrm{Hess}\, f(x,y).\end{aligned}$$

To calculate $\frac{\partial \log(\varDelta)}{\partial \mu_{2i}}$, we remove the assumption $\mu_2 = 0$. Since $\varDelta$ is some non-zero constant multiple of

$$\prod_{i<j} (a_i - a_j)^2,$$

we easily get the $q \times q$-matrix $\left(\frac{\partial \mu_{2i}}{\partial a_j}\right)$, and then we get $\frac{\partial \log(\varDelta)}{\partial \mu_{2i}}$ by using its inverse matrix. For $(e,q) = (3,4)$, $(3,5)$, we know only a proof by direct calculation with Maple by using explicit $\varDelta$ and the operators L_{v_j}s. □

3.8 Analytic expression of the sigma function

Before showing that the function (3.16) is exactly the sigma function $\sigma(u)$ (see Lemma 4.17 in [7]), we shall first describe some heuristic arguments supporting this result.

From the definition of L_0 and 3.54, we see that

$$L_0 = \sum_j (eq - v_j) \mu_{eq-v_j} \frac{\partial}{\partial \mu_{eq-v_j}} \tag{3.64}$$

and $L_0\big(F(\mu)\big) = -\mathrm{wt}\big(F(\mu)\big) F(\mu)$ for any homogeneous form $F(\mu) \in \mathbb{Q}[\mu]$.

Lemma 3.65. *We have*

$$L_0\, {}^t\boldsymbol{\omega} = \left[\begin{array}{ccc|ccc} -w_g & & & & & \\ & \ddots & & & & \\ & & -w_1 & & & \\ \hline & & & w_g & & \\ & & & & \ddots & \\ & & & & & w_1 \end{array}\right] {}^t\boldsymbol{\omega} = \Gamma_0\, {}^t\boldsymbol{\omega}$$

on $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\mu])$.

Proof. For a power series expansion at ∞ of any 1-form of homogeneous weight w

$$\omega = \sum_j c_j\, t^{j+w}\, dt,$$

where t is a local parameter at ∞ of weight 1, $c_j = c_j(\mu) \in \mathbb{Q}[\mu]$ is homogeneous and $\mathrm{wt}(c_j) = -j-1$, we see

$$\begin{aligned} L_0\Big(\sum_j c_j t^{j+w} dt\Big) + w \sum_j c_j t^{j+w} dt &= \sum_j (j+1) c_j t^{j+w} dt + \sum_j w c_j t^{j+w} dt \\ &= \sum_j (j+w+1) c_j t^{j+w} dt = d\Big(\sum_j c_j\, t^{j+w+1}\Big). \end{aligned}$$

If ω is any one of ω_{w_i}, which is of the form $\dfrac{h(x,y)}{f_2(x,y)}\,dx$ with $h(x,y) \in \mathbb{Q}[\mu][x,y]$, we see the last above is

$$d\Big(\frac{h(x,y)}{f_2(x,y)} \cdot t\,\frac{dx}{dt}\Big). \tag{3.66}$$

Since we can choose t as a quotient of monomials of x and y (see [23], Section 3), (3.66) is an exact form. So that

$$L_0(\omega) = -\mathrm{wt}(\omega)\,\omega \quad \text{in} \quad H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\mu]).$$

As $\mathrm{wt}(\omega_i) = i$ and $\mathrm{wt}(\eta_{-i}) = -i$, the statement is now obvious. $\square$

The function $\sigma(u)$, characterized in 3.13, is a power series of homogeneous weight, which must be written as

$$\begin{aligned} \sigma(u) = {u_1}^{(e^2-1)(q^2-1)/24} \cdot \sum_{\{n_{eq-v_j},\,\ell_{w_i}\}} &\Big[a(\ell_{w_2},\cdots,\ell_{w_g}, n_{eq-v_1},\cdots,n_{eq-v_{2g}}) \\ &\prod_{j=1}^{2g} (\mu_{eq-v_j} {u_1}^{eq-v_j})^{n_{eq-v_j}} \prod_{i=2}^{g} \Big(\frac{u_{w_i}}{{u_1}^{w_i}}\Big)^{\ell_{w_i}}\Big], \end{aligned} \tag{3.67}$$

where the $a(\ell_{w_2},\cdots,\ell_{w_g}, n_{eq-v_1},\cdots,n_{eq-v_{2g}})$'s are absolute constants and the set of $3g-1$ variables $\{n_{eq-v_j},\, \ell_{w_i}\}$ runs through the non-negative integers such that

$$\frac{(e^2-1)(q^2-1)}{24} - \sum_{i=2}^{g} \ell_{w_i} + \sum_{j=1}^{2g} n_{eq-v_j} \geq 0.$$

It is obvious that the operator

$$\begin{aligned} &\sum_{j=1}^{g} (eq-v_j)\mu_{eq-v_j} \frac{\partial}{\partial \mu_{eq-v_j}} - \sum_{j=1}^{g} w_j\, u_{w_j} \frac{\partial}{\partial u_{w_j}} + \mathrm{wt}(\sigma(u)) \\ &\quad = L_0 - \sum_{j=1}^{g} w_j\, u_{w_j} \frac{\partial}{\partial u_{w_j}} + \frac{(e^2-1)(q^2-1)}{24} \end{aligned} \tag{3.68}$$

kills the series (3.67).

Here, we shall give a proof along the lines of our reconstruction of BL theory for Lemma 4.17 of [6], that is if $\sigma(u)$ can be written as $\sigma(u) = \varDelta^{-M}\tilde{\sigma}(u)$ with a numerical constant M, then $M = \frac{1}{8}$. From (3.41), we have

$$(L_0 - H^{L_0} + M L_0 \log \varDelta)\,\sigma(u) = 0. \tag{3.69}$$

By Lemma 3.65 and (3.26) (or by 3.55 and (3.64)), we have

$$H^{L_0} = -\sum_{j=1}^{g} w_j\, u_{w_j} \frac{\partial}{\partial u_{w_j}} - \frac{1}{2}\sum_{j=1}^{g} w_j.$$

On the other hand, (3.63) and (3.51) yield that

$$L_0 \varDelta = \mathrm{wt}(\varDelta)\varDelta = eq(e-1)(q-1)\cdot\varDelta.$$

Then

$$\begin{aligned} & L_0 - H^{L_0} + M L_0 \log\varDelta = L_0 - H^{L_0} + M\,\mathrm{wt}(\varDelta) \\ & = L_0 - \sum_{j=1}^{g} w_j\, u_{w_j}\frac{\partial}{\partial u_{w_j}} \\ & -\frac{1}{2}\left(\frac{eq(e-1)(q-1)}{4} - \frac{(e^2-1)(q^2-1)}{12}\right) + Meq(e-1)(q-1). \end{aligned} \tag{3.70}$$

Therefore the operator (3.70) should equal the operator (3.68). Hence,

$$-\frac{1}{2}\left(\frac{eq(e-1)(q-1)}{4} - \frac{(e^2-1)(q^2-1)}{12}\right) + Meq(e-1)(q-1) = \frac{(e^2-1)(q^2-1)}{24},$$

and it follows that $M = \frac{1}{8}$ as desired.

In the rest of the paper, we use the notation

$$H_{v_i} = H^{L_{v_i}} - \tfrac{1}{8}L_{v_i}\log\varDelta,$$

where $H^{L_{v_i}}$ is defined by (3.60). Then 3.39 and (3.41) imply the following.

Theorem 3.71. *We have* $(L_{v_j} - H_{v_j})\,\hat\sigma(u) = 0$ *for* $j = 1, \cdots, 2g$.

The following theorem is one of the important consequences of the BL-theory.

Theorem 3.72. *The function $\sigma(u)$ is equal to $\hat\sigma(u)$ up to a non-zero absolute constant.*

Proof. In Section 4, for $(e,q) = (2,3)$, $(2,5)$, $(2,7)$, and $(3,4)$, we will solve the system of equations

$$(L_{v_i} - H_{v_i})\,\varphi(u) = 0 \quad (\, i = 1, \ \cdots, \ 2g\ ) \tag{3.73}$$

for an unknown holomorphic function $\varphi(u)$, and below we will show that the solution space of this system is of dimension 1, and any solution satisfies the properties of $\sigma(u)$ in (3.13). Hence we have proved that $\sigma(u)$ is equal to $\hat\sigma(u)$ up to a non-zero absolute constant. □

From now on, throughout this paper, we denote

$$\Gamma_{v_j} = \Gamma^{L_{v_j}}.$$

Especially, $\Gamma_0 = \Gamma^{L_0} = \Gamma^{L_{v_1}}$.

Remark 3.74. As noted above in 3.25, our notation differs from that of Buchstaber and Leykin; we denote the matrix Γ_j in p.274 of [7] by Γ_j^{BL} and we define the sub-matrices of $-J\Gamma_j^{\rm BL}$ and $\Gamma_{v_j}J$ by

$$-J\Gamma_j^{\rm BL} = \begin{bmatrix} \alpha_j^{\rm BL} & {}^t(\beta_j^{\rm BL}) \\ \beta_j^{\rm BL} & \gamma_j^{\rm BL} \end{bmatrix}, \quad \Gamma_{v_j}J = \begin{bmatrix} \alpha_{v_j} & \beta_{v_j} \\ {}^t(\beta_{v_j}) & \gamma_{v_j} \end{bmatrix}$$

by following the notation of [7] and the present paper. Then we have

$$\alpha_j^{\rm BL} = \alpha_{v_j}, \quad \beta_j^{\rm BL} = {}^t(\beta_{v_j}), \quad \gamma_j^{\rm BL} = \gamma_{v_j}, \quad \Gamma_j^{\rm BL} = {}^t(\Gamma_{v_j})$$

for any j.

4 Solving the heat equations

4.1 The initial conditions

For the rest of the paper, we shall solve (3.73) for $(2,3)$-, $(2,5)$-, $(2,7)$-, and $(3,4)$-curves. we frequently switch from regarding the μ_js as indeterminates to regarding them as elements in $\mathbb{C}$. We use the following two initial conditions for solving (3.73):
IC1. $\varphi(u) \in \mathbb{Q}[\mu][[u_{w_g}, \cdots, u_{w_1}]]$, and
IC2. $\mathrm{wt}(\varphi(u)) = (e^2-1)(q^2-1)/24$.
Since the condition **IC2** is milder than the property (4) in 3.13, there may be a possibility to reduce the characterization in 3.13 of the sigma function, in general.

It is not clear to the authors which part of [7] shows that the space of the solutions $\varphi(u) = \varphi(\mu, u_{w_g}, \cdots, u_{w_2}, u_{w_1}) \in \mathbb{Q}[\mu][[u_{w_g}, \cdots, u_{w_2}, u_{w_1}]]$ of (3.73) is one dimensional. The following is a partial answer to this question.

4.2 General results for the $(2,q)$-curve

In this section, we discuss the hyperelliptic case $e=2$. Firstly, we give the explicit expression for the entries of the matrix V of (3.49). The authors know that the issue in this subsection is described in p.566 in V.I.Arnol'd's [1] and p.65 in [2]. Since they do not know any source which contains a proof, we shall give a detailed proof here.

Lemma 4.1. *We have*

$$\begin{aligned} V_{ij} &= -\frac{2i(q-j)}{q}\mu_{2i}\mu_{2j} + \sum_{m=1}^{m_0} 2(j-i+2m)\,\mu_{2(i-m)}\,\mu_{2(j+m)} \\ &= -\frac{2i(q-j)}{q}\mu_{2i}\mu_{2j} + \sum_{\ell=\ell_0}^{i-1 \text{ or } j} 2(i+j-2\ell)\,\mu_{2\ell}\,\mu_{2(i+j-\ell)}, \end{aligned}$$

where $\mu_0 = 1$, $\mu_2 = 0$, $m_0 = \min\{i,\, q-j\}$, *and* $\ell_0 = \max\{0, i+j-q\}$.

Proof. First of all, assuming the first equality, we show the second equality. To change the first expression to the second with summation to $i-1$, we use the substitution $\ell = i-m$. It is obvious that the second equality with summation to

$i-1$ is equal to one with summation to j for $i=j,\ j+1$. For the case of $i<j$, the difference of the two is expressed as

$$\sum_{\ell=i}^{j} 2(i+j-2\ell)\,\mu_{2\ell}\,\mu_{2(i+j-\ell)} = -\sum_{\ell'=i}^{j} 2(i+j-2\ell')\,\mu_{2(i+j-\ell')}\,\mu_{2\ell'}$$

setting $\ell'=i+j-\ell$, it is clear that this vanishes. We see the case $j<i$ in a similar way. The matrix $V=[V_{ij}]$ is symmetric by definition. However, if the Lemma is proved, we see this directly, by subtracting the term for $\ell=j$ from the first term. Now, noting that in the hyperelliptic case, the $M_j(x,y)$ are independent of x, we define $M^{(i)}=M^{(i)}(x)\in\mathbb{Z}[\mu][x]$ by using $[H_{ij}]$ of (3.52):

$$M^{(i)}(x)=\sum_{j=1}^{q-1} H_{q-i,j}M_j(x,y)=\sum_{m=0}^{i-1} 2(q+1+m-i)\,\mu_{2(i-m-1)}\,x^m.$$

While we are treating $f(x,y)=y^2-p_2(x)$, we denote $p_2(x)$ by $p(x)$ in this proof, for a less cumbersome notation.

Since $\mathbb{Q}[\mu][x,y]/(f_1(x,y),f_2(x,y))=\mathbb{Q}[\mu][x,y]/(p'(x),2y)$, which is isomorphic to $\mathbb{Q}[\mu][x]/(p'(x))$, it suffice to know explicitly the residue $V^{(i)}=V^{(i)}(x)$ of degree less than $q-1$ of the division $p(x)\,M^{(i)}(x)$ by $p'(x)$ for $1\le i\le q-1$. The key to this proof is that we actually know the quotient $Q^{(i)}=Q^{(i)}(x)\in\mathbb{Q}[\mu][x]$, as well as $V^{(i)}$, of this division! Namely, we will show that, if we define functions

$$Q^{(i)}(x)=\sum_{m=1}^{i} 2\mu_{2(i-m)}\,x^m+\frac{2i}{q}\,\mu_{2i},$$

then the expression

$$V^{(i)}(x)=p(x)\,M^{(i)}(x)-p'(x)\,Q^{(i)}(x) \tag{4.2}$$

is of degree less than $q-1$. Moreover, we can calculate all the terms of $V^{(i)}$ explicitly, which are no other than the V_{ij}s.

Let us start to calculate each term of x^k of the right hand side of (4.2) for any $k\ge 0$. We divide the calculation into four cases.
(i) The case $k\ge q$. In this case, $M^{(i)}$ has terms only up to x^{i-1} ($i-1\le q-2<q\le k$), and $p(x)$ has terms up to x^q ($q\le k$). Therefore, we find that the coefficient C_k of x^k in $M^{(i)}(x)\,p(x)$ is given by

$$\begin{aligned}C_k &= \sum_{m=k-q}^{i-1} 2(q+1-i+m)\,\mu_{2(i-1-m)}\,\mu_{2(q-k+m)}\\ &= \sum_{m'=q-1-k}^{i} 2(k-m'+1)\,\mu_{2(q-1-k+m')}\,\mu_{2(i-m')},\end{aligned}$$

where we have changed the summation index by $q-k+m=i-m'$. On the other hand, $Q^{(i)}$ has terms up to x^i ($i\le q-1<q\le k$), and $p'(x)$ has terms up to x^{q-1}

$(q-1<k)$, so we see

$$\text{“Coeff. of } x^k \text{ in } Q^{(i)}(x)\,p'(x)\text{”} = \sum_{m=k-q+1}^{i} 2\,\mu_{2(i-m)}\,\mu_{2(q-1-k+m)}(k-m+1).$$

So the right hand side of (4.2) has no term in x^k for $k \geq q$.
(ii) The case $k = q-1$. Since $M^{(i)}$ has terms only up to x^{i-1} $(i-1 \leq q-2 < q-1 = k)$, we see that

$$\text{“Coeff. of } x^k \text{ in } M^{(i)}(x)\,p(x)\text{”} = \sum_{m=0}^{i-1} 2(q+1-i+m)\,\mu_{2(i-1-m)}\,\mu_{2(1+m)}$$
$$= \sum_{m'=0}^{i-1} 2(q-m')\,\mu_{2m'}\,\mu_{2(i-m')} = \sum_{m'=1}^{i-1} 2(q-m')\,\mu_{2m'}\,\mu_{2(i-m')} + 2q\,\mu_0\mu_{2i},$$

where we have changed the index of summation by $m+1 = i-m'$. In this case $Q^{(i)}$ has terms up to x^i $(i \leq q-1 = k)$, and $p'(x)$ has terms up to x^{q-1} $(q = k)$, we have that the coefficient $C_k\ x^k$ in $Q^{(i)}(x)\,p'(x)$ is given by

$$C_k = \sum_{m=1}^{i} 2\,\mu_{2(i-m)}\,\mu_{2m}(q-m) + \frac{2i}{q}\mu_{2i}\,\mu_0\,q$$
$$= \sum_{m=1}^{i-1} 2\,\mu_{2(i-m)}\,\mu_{2m}(q-m) + 2(q-i)\,\mu_0\,\mu_{2i} + 2i\,\mu_{2i}\,\mu_0.$$

So the right hand side of (4.2) has no term in x^{q-1}.
(iii) The case $i-1 < k < q-1$.
Since $M^{(i)}$ has terms only up to x^{i-1}, we see that the coefficient D_k of x^k in $M^{(i)}(x)\,p(x)$ is given by

$$C_k = \sum_{m=0}^{i-1} 2(q+1+m-i)\,\mu_{2(i-m-1)}\,\mu_{2(q-k+m)}$$
$$= \sum_{m=1}^{i} 2(q+m-i)\,\mu_{2(i-m)}\,\mu_{2(q-1-k+m)}$$

by rewriting m as $m-1$. On the other hand, the coefficient C_k of x^k in $Q^{(i)}(x)\,p'(x)$ is

$$C_k = \frac{2i}{q}\mu_{2i}\,(k+1)\mu_{q-1-k} + \sum_{m=1}^{i} 2\,\mu_{2(i-m)}\,(k-m-1)\mu_{2(q-1-k+m)}.$$

So the coefficient of x^k in the right hand side of (4.2) is

$$\sum_{m=1}^{i} 2(q-1-k+2m-i)\,\mu_{2(i-m)}\,\mu_{2(q-1-k+m)} + \frac{2i}{q}(k+1)\mu_{2i}\,\mu_{2(q-1-k)}$$

and V_{ij}, which is no other than the value of this at $k = q-1-j$, is given by

$$\sum_{m=1}^{i} 2(j+2m-i)\,\mu_{2(i-m)}\,\mu_{2(j+m)} + \frac{2i}{q}(q-j)\mu_{2i}\,\mu_{2j}$$

as desired since $i < k+1 = q-j$.

(iv) The case $k < i-1$. Since $M^{(i)}$ has terms up to x^{i-1}, of higher degree than x^k, we see that the coefficient D_k of x^k in $M^{(i)}(x)\,p(x)$ is given by

$$D_k = \sum_{m=0}^{k} 2(q+1+m-i)\,\mu_{2(i-m-1)}\,\mu_{2(q-k+m)}$$
$$= \sum_{m=1}^{k+1} 2(q+m-i)\,\mu_{2(i-m)}\,\mu_{2(q-1-k+m)}$$

on replacing the summation index m by $m+1$. Similarly, $Q^{(i)}$ has terms up to x^i exceeding x^k again, and

$$\text{“Coeff. of } x^k \text{ in } p'(x)\,Q^{(i)}(x)\text{”}$$
$$= \frac{2i}{q}\mu_{2i}\,\mu_{q-1-k} + \sum_{m=1}^{k+1} 2\,\mu_{2(i-m)}\,(k-m-1)\mu_{2(q-1-k+m)}$$

with an extra term for $m = k+1$ which is zero. So the coefficient of x^k in the right hand side of (4.2) is

$$\sum_{m=1}^{k+1} 2(q-1-k+2m-i)\,\mu_{2(i-m)}\,\mu_{2(q-1-k+m)} + \frac{2i}{q}(k+1)\mu_{2i}\,\mu_{2(q-1-k)}$$

and then V_{ij}, which is no other than the value of this at $k = q-1-j$, is given by

$$V_{ij} = \sum_{m=1}^{q-j} 2(j+2m-i)\,\mu_{2(i-m)}\,\mu_{2(j+m)} + \frac{2i}{q}(q-j)\mu_{2i}\,\mu_{2j}$$

as desired since $q-j = k+1 \le i$. □

Lemma 4.3. *We have* $L_{v_j}(\mathit{\Delta}) = -2(q-j)(q+1-j)\mu_{2j}\,\mathit{\Delta}$ *for* $j = q-1,\ q-2,\ \cdots,\ 2,\ 1$.

Proof. Since the Hessian of $f(x,y) = y^2 - p(x)$ is

$$\begin{vmatrix} -p_2''(x) & 0 \\ 0 & 2 \end{vmatrix} = -2\,p_2''(x)$$
$$= -2\big(q(q-1)x^{q-2} + (q-2)(q-3)\mu_4 x^{q-4} + \cdots + 2\cdot 1\,\mu_{q-3}\big),$$

this lemma follows from (3.63). □

4.3 Heat equations for the $(2,3)$-curve

In this section we recall Weierstrass' result which gives a recursive relation for the coefficients of the power series expansion of his sigma function at the origin. We refer the reader to (12) and (13) in p. 314 of [12] also. Here we derive Weierstrass' result by following the method of [7], namely, following the theory described in previous sections, but without using the general results 4.1 and 4.3, in order to demonstrate the ideas of the theory.

Weierstrass' original method is explained in [26] and some explanation of it is available in [21]. It is easy to get L_0 and L_2:

$$V = \begin{bmatrix} 4\mu_4 & 6\mu_6 \\ 6\mu_6 & -\frac{4}{3}{\mu_4}^2 \end{bmatrix}, \quad \text{and} \quad \begin{cases} L_0 = 4\mu_4 \dfrac{\partial}{\partial \mu_4} + 6\mu_6 \dfrac{\partial}{\partial \mu_6} \\ L_2 = 6\mu_6 \dfrac{\partial}{\partial \mu_4} - \dfrac{4}{3}{\mu_4}^2 \dfrac{\partial}{\partial \mu_6}. \end{cases} \tag{4.4}$$

In this case, we see $V = T$ since

$$H\big((x,y),(z,w)\big) = -6x - 6z = [\,x \quad 1\,] \begin{bmatrix} & -6 \\ -6 & \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}.$$

Then $\Delta = 2^2 \cdot 3 \cdot \det(V)$. The differential forms

$$\omega_1 = \frac{dx}{2y}, \quad \eta_{-1} = \frac{xdx}{2y}$$

form a symplectic basis of $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\mu])$. We have $\boldsymbol{\omega} = (\,\omega_1,\ \eta_{-1}\,)$. Bearing in mind Lemma 3.18, we proceed by using $x^{-\frac{1}{2}}$ as the local parameter satisfying $\frac{\partial}{\partial \mu_j} x = 0$ for $j = 4, 6$, and we compute the matrix Γ as follows. Using $f(x,y) = 0$, we see $2y\frac{\partial}{\partial \mu_4} y = x$ and $2y \frac{\partial}{\partial \mu_6} y = 1$, so that

$$\frac{\partial}{\partial \mu_4} y = \frac{x}{2y}, \quad \frac{\partial}{\partial \mu_6} y = \frac{1}{2y}.$$

Therefore, we have

$$\frac{\partial}{\partial \mu_6}\omega_1 = -\frac{1}{4y^3}dx, \quad \frac{\partial}{\partial \mu_4}\omega_1 = \frac{\partial}{\partial \mu_6}\eta_{-1} = -\frac{x}{4y^3}dx, \quad \frac{\partial}{\partial \mu_4}\eta_{-1} = -\frac{x^2}{4y^3}dx, \tag{4.5}$$

and

$$d\Big(\frac{1}{y}\Big) = -\frac{1}{y^2}dy = -\frac{1}{y^2}\frac{dy}{dx}dx = -\frac{1}{y^2}\,\frac{3x^2+\mu_4}{2y}\,dx = 6\frac{\partial}{\partial \mu_4}\,\eta_{-1} + 2\mu_4 \frac{\partial}{\partial \mu_6}\,\omega_1,$$

$$\begin{aligned} d\Big(\frac{x}{y}\Big) &= \frac{ydx - xdy}{y^2} = \frac{y - x\frac{dy}{dx}}{y^2}\,dx = \frac{y - x\frac{3x^2+\mu_4}{2y}}{y^2}\,dx \\ &= \frac{y - \frac{3y^2 - 2\mu_4 x - 3\mu_6}{2y}}{y^2}\,dx = -\omega_1 - 4\mu_4 \frac{\partial}{\partial \mu_4}\omega_1 - 6\mu_6 \frac{\partial}{\partial \mu_6}\omega_1 = -\omega_1 - L_0\omega_1, \end{aligned} \tag{4.6}$$

$$\begin{aligned} d\Big(\frac{x^2}{y}\Big) &= \frac{2xydx - x^2dy}{y^2} = \frac{2xy - x^2\frac{dy}{dx}}{y^2}\,dx = \frac{2xy - x^2\frac{3x^2+\mu_4}{2y}}{y^2}\,dx \\ &= \frac{2xy - \frac{3x(y^2-\mu_4x-\mu_6)+\mu_4 x^2}{2y}}{y^2}\,dx = \frac{xdx}{2y} + \frac{\mu_4 x^2}{y^3}dx + \frac{3}{2}\frac{\mu_6 x}{y^3}dx \\ &= \eta_{-1} - L_0\eta_{-1} \\ &= \eta_{-1} + \frac{4}{3}{\mu_4}^2\frac{\partial}{\partial \mu_6}\omega_1 - 6\mu_6\frac{\partial}{\partial \mu_4}\omega_1 - \frac{2}{3}\mu_4 d\Big(\frac{1}{y}\Big) \qquad (\text{ by (4.6) }) \\ &= \eta_{-1} - L_2\omega_1 - \frac{2}{3}\mu_4 d\Big(\frac{1}{y}\Big). \end{aligned} \tag{4.7}$$

Accordingly, we see

$$L_2\eta_{-1} = 6\mu_6\frac{\partial}{\partial \mu_4}\eta_{-1} - \frac{4}{3}{\mu_4}^2\frac{\partial}{\partial \mu_6}\eta_{-1}$$

$$= -2\mu_6\mu_4\frac{\partial}{\partial\mu_6}\omega_1 - \frac{4}{3}{\mu_4}^2\frac{\partial}{\partial\mu_4}\omega_1 + \mu_6\, d\Big(\frac{1}{y}\Big) \qquad (\text{ by (4.5) and (4.6) })$$
$$= -\frac{\mu_4}{3}\Big(6\mu_6\frac{\partial}{\partial\mu_6}\omega_1 + 4\mu_4\frac{\partial}{\partial\mu_4}\omega_1\Big) + \mu_6\, d\Big(\frac{1}{y}\Big)$$
$$= -\frac{\mu_4}{3}L_0\omega_1 + \mu_6\, d\Big(\frac{1}{y}\Big) = \frac{\mu_4}{3}\omega_1 + \mu_6\, d\,\Big(\frac{1}{y}\Big).$$

Summarising these results, we have on $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\mu])$ that

$$L_0\,{}^t\boldsymbol{\omega} = \Gamma_0\,{}^t\boldsymbol{\omega},\quad L_2\,{}^t\boldsymbol{\omega} = \Gamma_2\,{}^t\boldsymbol{\omega},\quad \text{where}\ \ \Gamma_0 = \begin{bmatrix} -1 & \\ & 1\end{bmatrix},\ \ \Gamma_2 = \begin{bmatrix} & 1 \\ \frac{\mu_4}{3} & \end{bmatrix}.$$

Note that, by these equation, we have $L_j\varOmega = \Gamma_j\varOmega$ with $\varOmega = \begin{bmatrix}\omega' & \omega'' \\ \eta' & \eta''\end{bmatrix}$ as in (3.29). Since $L_0\det(T) = 12\det(T)$, $L_2\det(T) = 0$, $\varDelta = \det(T)$, and (3.65), we have arrived at

$$(4.8)\qquad \begin{aligned} (L_0 - H_0)\sigma(u) &= \Big(4\mu_4\frac{\partial}{\partial\mu_4} + 6\mu_6\frac{\partial}{\partial\mu_6} - u\frac{\partial}{\partial u} + 1\Big)\,\sigma(u) = 0,\\ (L_2 - H_2)\sigma(u) &= \Big(6\mu_6\frac{\partial}{\partial\mu_4} - \frac{4}{3}{\mu_4}^2\frac{\partial}{\partial\mu_6} - \frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{6}\mu_4 u^2\Big)\,\sigma(u) = 0,\end{aligned}$$

where $H_j = H^{L_j} + \frac{1}{8}L_j\log\varDelta$ for $j = 0$ and 2. From the first of (4.8) and the conditions **IC1**, **IC2**, the sigma function is of the form

$$\sigma(u) = u\sum_{n_4,n_6\geqq 0} b(n_4,n_6)\frac{(\mu_4u^4)^{n_4}(\mu_6u^6)^{n_6}}{(1+4n_4+6n_6)!}.$$

Using the second equation we then have a recurrence relation

$$(4.9)\qquad \begin{aligned} b(n_4,n_6) &= \tfrac{2}{3}(4n_4+6n_6-1)(2n_4+3n_6-1)b(n_4-1,n_6)\\ &\quad - \tfrac{8}{3}(n_6+1)b(n_4-2,n_6+1) + 12(n_4+1)b(n_4+1,n_6-1)\end{aligned}$$

with $b(n_4,n_6) = 0$ if $n_4 < 0$ or $n_6 < 0$. Since the term $b(n_4,n_6)$ on the left hand side has weight $4n_4+6n_6$, and the terms $b(i,j)$ on the right hand side have weight smaller than this, all terms may be found from (4.9). Therefore, any solution of (4.8) is a constant times the function

$$\sigma(u) = u + 2\mu_4\tfrac{u^5}{5!} + 24\mu_6\tfrac{u^7}{7!} - 36\mu_4^2\tfrac{u^9}{9!} - 288\mu_4\mu_6\tfrac{u^{11}}{11!} + \cdots.$$

4.4 Heat equations for the $(2,5)$-curve

In this section, we list the analogous results for the heat equations for the curve

$$\mathscr{C}_\mu\ :\ y^2 = x^5 + \mu_4x^3 + \mu_6x^2 + \mu_8x + \mu_{10}.$$

We note here that our results correct a sign in [6]; the overall constant $\frac{1}{80}$ at the 4th line from bottom in page 68 of [6] should be $-\frac{1}{80}$. Here we give the Hurwitz series version of the algorithm. Now, we take a usual symplectic basis of differentials

$$\omega_3 = \frac{1}{2y}dx,\quad \omega_1 = \frac{x}{2y}dx,\quad \eta_{-3} = \frac{3x^3+\mu_4x}{2y}dx,\quad \eta_{-1} = \frac{x^2}{2y}dx$$

of $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\mu])$. The matrix V for this case is given by

$$V=\begin{bmatrix} 4\mu_4 & 6\mu_6 & 8\mu_8 & 10\mu_{10} \\ 6\mu_6 & -\tfrac{12}{5}\mu_4^2+8\mu_8 & -\tfrac{8}{5}\mu_4\mu_6+10\mu_{10} & -\tfrac{4}{5}\mu_4\mu_8 \\ 8\mu_8 & -\tfrac{8}{5}\mu_4\mu_6+10\mu_{10} & -\tfrac{12}{5}\mu_6^2+4\mu_4\mu_8 & 6\mu_4\mu_{10}-\tfrac{6}{5}\mu_6\mu_8 \\ 10\mu_{10} & -\tfrac{4}{5}\mu_4\mu_8 & 6\mu_4\mu_{10}-\tfrac{6}{5}\mu_6\mu_8 & 4\mu_{10}\mu_6-\tfrac{8}{5}\mu_8^2 \end{bmatrix}.$$

Then $\varDelta=2^4\cdot 5\cdot\det(V)$. The operators L_j are given by

$${}^t[L_0\ L_2\ L_4\ L_6]=V\ {}^t\big[\tfrac{\partial}{\partial\mu_4}\ \ \tfrac{\partial}{\partial\mu_6}\ \ \tfrac{\partial}{\partial\mu_8}\ \ \tfrac{\partial}{\partial\mu_{10}}\big].$$

While the authors have the explicit commutation relations of these L_i, we shall not include these here because their explicit forms are not needed in this paper. However, these commutators are all in the span of the L_i. By (3.62), we see that these L_j's operate on the discriminant $\varDelta$ as follows:

$$[\,L_0\quad L_2\quad L_4\quad L_6\,]\varDelta=[\,40\quad 0\quad 12\,\mu_4\quad 4\,\mu_6\,]\varDelta.$$

The representation matrices Γ_j for the L_j acting on $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\mu])$ are

$$\Gamma_0=\begin{bmatrix} -3 & & & \\ & -1 & & \\ & & 3 & \\ & & & 1 \end{bmatrix},\quad \Gamma_2=\begin{bmatrix} & -1 & & \\ \tfrac{4}{5}\,\mu_4 & & & 1 \\ \tfrac{4}{5}\,{\mu_4}^2-3\mu_8 & & & -\tfrac{4}{5}\mu_4 \\ & \tfrac{3}{5}\,\mu_4 & 1 & \end{bmatrix},$$

$$\Gamma_4=\begin{bmatrix} -\mu_4 & & & 1 \\ \tfrac{6}{5}\,\mu_6 & & 1 & \\ \tfrac{6}{5}\,\mu_4\mu_6-6\mu_{10} & -\mu_8 & \mu_4 & -\tfrac{6}{5}\mu_6 \\ -\mu_8 & \tfrac{2}{5}\,\mu_6 & & \end{bmatrix},\ \Gamma_6=\begin{bmatrix} & & 1 & \\ \tfrac{3}{5}\,\mu_8 & & & \\ \tfrac{3}{5}\,\mu_4\mu_8 & -2\,\mu_{10} & & -\tfrac{3}{5}\,\mu_8 \\ -2\,\mu_{10} & \tfrac{1}{5}\,\mu_8 & & \end{bmatrix}.$$

Therefore, we find the following operators H_j:

$$\begin{aligned}
H_0&=3u_3\frac{\partial}{\partial u_3}+u_1\frac{\partial}{\partial u_1}-3,\\
H_2&=\frac{1}{2}\frac{\partial^2}{\partial {u_1}^2}+u_1\frac{\partial}{\partial u_3}-\frac{4}{5}\mu_4u_3\frac{\partial}{\partial u_1}-\frac{3}{10}\mu_4{u_1}^2-\Big(\frac{3}{2}\mu_8-\frac{2}{5}{\mu_4}^2\Big){u_3}^2,\\
H_4&=\frac{\partial^2}{\partial u_1\partial u_3}-\frac{6}{5}\mu_6u_3\frac{\partial}{\partial u_1}+\mu_4u_3\frac{\partial}{\partial u_3}-\frac{1}{5}\mu_6{u_1}^2+\mu_8u_1u_3+\Big(3\mu_{10}-\frac{3}{5}\mu_4\mu_6\Big){u_3}^2-\mu_4,\\
H_6&=\frac{1}{2}\frac{\partial^2}{\partial {u_3}^2}-\frac{3}{5}\mu_8u_3\frac{\partial}{\partial u_1}-\frac{1}{10}\mu_8{u_1}^2+2\mu_{10}u_3u_1-\frac{3}{10}\mu_8\mu_4{u_3}^2-\frac{1}{2}\mu_6.
\end{aligned}$$

By the equation $(L_0-H_0)\,\sigma(u)=0$ and the conditions **IC1** and **IC2**, the sigma function must be of the form

$$\sigma(u_3,u_1)=\sum_{\substack{m,n_4,n_6,n_8,n_{10}\geqq 0\\ 3-3m+4n_4+6n_6+8n_8+10n_{10}\geqq 0}}\Bigg[b(m,n_4,n_6,n_8,n_{10})\cdot\frac{{u_1}^3\Big(\dfrac{u_3}{{u_1}^3}\Big)^m\Big(\mu_4{u_1}^4\Big)^{n_4}\Big(\mu_6{u_1}^6\Big)^{n_6}\Big(\mu_8{u_1}^8\Big)^{n_8}\Big(\mu_{10}{u_1}^{10}\Big)^{n_{10}}}{m!\,(3-3m+4n_4+6n_6+8n_8+10n_{10})!}\Bigg].$$

Let $k = 3 - 3m + 4n_4 + 6n_6 + 8n_8 + 10n_{10}$. Then the other heat equations $(L_j - H_j)\,\sigma(u) = 0$ imply the following recursion scheme:

$$b(m, n_4, n_6, n_8, n_{10}) = \begin{cases} B_2 & (\text{ if } k > 1 \text{ and } m \geqq 0\,) \\ B_1 & (\text{ if } k = 1 \text{ and } m > 0\,) \\ B_0 & (\text{ if } k = 0 \text{ and } m > 1\,), \end{cases}$$

where the B_i are given by

$$\begin{aligned}
B_2 = {} & 20(n_8+1)\,b(m, n_4, n_6, n_8+1, n_{10}-1) \\
& +16(n_6+1)\,b(m, n_4, n_6+1, n_8-1, n_{10}) \\
& +12(n_4+1)\,b(m, n_4+1, n_6-1, n_8, n_{10}) \\
& -\tfrac{24}{5}(n_6+1)\,b(m, n_4-2, n_6+1, n_8, n_{10}) \\
& +\tfrac{3}{5}(k-3)(k-2)\,b(m, n_4-1, n_6, n_8, n_{10}) \\
& -\tfrac{8}{5}(n_{10}+1)\,b(m, n_4-1, n_6, n_8-1, n_{10}+1) \\
& -\tfrac{16}{5}(n_8+1)\,b(m, n_4-1, n_6-1, n_8+1, n_{10}) \\
& -2(k-2)\,b(m+1, n_4, n_6, n_8, n_{10}) \\
& -3m(m-1)\,b(m-2, n_4, n_6, n_8-1, n_{10}) \\
& +\tfrac{4}{5}m(m-1)\,b(m-2, n_4-2, n_6, n_8, n_{10}) \\
& +\tfrac{8}{5}m\,b(m-1, n_4-1, n_6, n_8, n_{10}), \\
B_1 = {} & +10(n_6+1)\,b(m-1, n_4, n_6+1, n_8, n_{10}-1) \\
& -\tfrac{12}{5}(n_8+1)\,b(m-1, n_4, n_6-2, n_8+1, n_{10}) \\
& -\tfrac{6}{5}(n_{10}+1)\,b(m-1, n_4, n_6-1, n_8-1, n_{10}+1) \\
& +8(n_4+1)\,b(m-1, n_4+1, n_6, n_8-1, n_{10}) \\
& -\tfrac{1}{5}(5m-10+8n_6-20n_8-30n_{10})\,b(m-1, n_4-1, n_6, n_8, n_{10}) \\
& -3(m-1)(m-2)\,b(m-3, n_4, n_6, n_8, n_{10}-1) \\
& +\tfrac{3}{5}(m-1)(m-2)\,b(m-3, n_4-1, n_6-1, n_8, n_{10}) \\
& +\tfrac{6}{5}(m-1)\,b(m-2, n_4, n_6-1, n_8, n_{10}), \\
B_0 = {} & -\tfrac{16}{5}(1+n_{10})\,b(m-2, n_4, n_6, n_8-2, n_{10}+1) \\
& -\tfrac{1}{5}(12n_8 - 40n_{10} - 5)\,b(m-2, n_4, n_6-1, n_8, n_{10}) \\
& +20(n_4+1)\,b(m-2, n_4+1, n_6, n_8, n_{10}-1) \\
& +12(n_8+1)\,b(m-2, n_4-1, n_6, n_8+1, n_{10}-1) \\
& -\tfrac{8}{5}(n_6+1)\,b(m-2, n_4-1, n_6+1, n_8-1, n_{10}) \\
& +\tfrac{3}{5}(m-2)(m-3)\,b(m-4, n_4-1, n_6, n_8-1, n_{10}) \\
& +\tfrac{6}{5}(m-2)\,b(m-3, n_4, n_6, n_8-1, n_{10}).
\end{aligned}$$

From these, we see that the expansion of $\sigma(u)$ is Hurwitz integral over $\mathbb{Z}[\frac{1}{5}]$.

Remark 4.10. Actually the above recurrence scheme is one of several possible recurrence relations. However, we see any such system gives the same solution space by the following argument. Here, of course, we suppose that $b(m, n_4, \cdots, n_{10}) = 0$ if k or any of the explicit arguments is negative. For any finite subset $S \subset \{(m, n_4, \cdots, n_{10}) \,|\, k, n_4, \cdots, n_{10} \geq 0\}$, we take the set E_S of relations h between $\{b(m, n_4, \cdots, n_{10})\}$ such that any $b(m, n_4, \cdots, n_{10})$ appears as a term in h provided that $(m, n_4, \cdots, n_{10}) \in S$. For instance, if we consider

$$S = \{(1,0,0,0,0),\ (0,0,0,0,0),\ (0,1,0,0,0),\ (1,1,0,0,0),\ (2,1,0,0,0)\},$$

then E_S consists of the following 4 equations:

$$\begin{aligned}
&b(0,0,0,0,0) = -2b(1,0,0,0,0),\\
&b(0,1,0,0,0) = 12b(0,0,0,0,0) - 10b(1,1,0,0,0),\\
&b(1,1,0,0,0) = \tfrac{6}{5}b(1,0,0,0,0) - 4b(2,1,0,0,0) - \tfrac{16}{5}b(0,0,0,0,0),\\
&b(2,1,0,0,0) = \tfrac{1}{5}\cdot 0 \cdot b(1,0,0,0,0).
\end{aligned}$$

The solution space of such a system of linear equations E_S is of dimension 1 or larger because we have at least one iteration system as above whose solution space is of dimension 1. Since E_S is independent of the choice of recursion system, any recursion system must include the same solution space of dimension 1.

The first few terms of the sigma expansion are given as follows (up to a constant multiple):

$$\begin{aligned}
\sigma(u_3, u_1) = u_3 &- 2\frac{{u_1}^3}{3!} - 4\mu_4\frac{{u_1}^7}{7!} - 2\mu_4\frac{u_3{u_1}^4}{4!} + 64\mu_6\frac{{u_1}^9}{9!} - 8\mu_6\frac{u_3{u_1}^6}{6!}\\
&- 2\mu_6\frac{{u_3}^2{u_1}^3}{2!3!} + \mu_6\frac{{u_3}^3}{3!} + (1600\mu_8 - 408{\mu_4}^2)\frac{{u_1}^{11}}{11!}\\
&- (4{\mu_4}^2 + 32\mu_8)\frac{u_3{u_1}^8}{8!} - 8\mu_8\frac{{u_3}^2{u_1}^5}{2!5!} - 2\mu_8\frac{{u_3}^3{u_1}^2}{3!2!} + \cdots.
\end{aligned}$$

4.5 Heat equations for the $(2,7)$-curve

We take the hyperelliptic genus three curve $\mathscr{C}$ in “Weierstrass” form

$$y^2 = f(x) = x^7 + \mu_4x^5 + \mu_6x^4 + \mu_8x^3 + \mu_{10}x^2 + \mu_{12}x + \mu_{14}.$$

The discriminant Δ of $\mathscr{C}$ is the resultant of f and f_1. It has 320 terms and is of weight 84. The matrix V is given by

$$V = \begin{bmatrix}
4\,\mu_4 & 6\,\mu_6 & 8\,\mu_8\\
6\,\mu_6 & -\frac{4}{7}\left(5\,{\mu_4}^2 - 14\,\mu_8\right) & -\frac{2}{7}\left(8\,\mu_6\mu_4 - 35\,\mu_{10}\right)\\
8\,\mu_8 & -\frac{2}{7}\left(8\,\mu_6\mu_4 - 35\,\mu_{10}\right) & \frac{4}{7}\left(21\,\mu_{12} - 6\,{\mu_6}^2 + 7\,\mu_4\mu_8\right)\\
10\,\mu_{10} & -\frac{12}{7}\left(\mu_4\mu_8 - 7\,\mu_{12}\right) & \frac{2}{7}\left(49\,\mu_{14} - 9\,\mu_6\mu_8 + 21\,\mu_4\mu_{10}\right)\\
12\,\mu_{12} & -\frac{2}{7}\left(4\,\mu_4\mu_{10} - 49\,\mu_{14}\right) & \frac{4}{7}\left(14\,\mu_4\mu_{12} - 3\,\mu_6\mu_{10}\right)\\
14\,\mu_{14} & -\frac{4}{7}\,\mu_4\mu_{12} & \frac{2}{7}\left(35\,\mu_4\mu_{14} - 3\,\mu_6\mu_{12}\right)
\end{bmatrix}$$

$$
\begin{matrix}
10\,\mu_{10} \\
-\frac{12}{7}\,(\mu_4\mu_8 - 7\,\mu_{12}) \\
\frac{2}{7}\,(21\,\mu_4\mu_{10} - 9\,\mu_6\mu_8 + 49\,\mu_{14}) \\
\frac{4}{7}\,(7\,\mu_6\mu_{10} - 6\,\mu_8{}^2 + 14\,\mu_4\mu_{12}) \\
\frac{2}{7}\,(21\,\mu_6\mu_{12} - 8\,\mu_{10}\mu_8 + 35\,\mu_4\mu_{14}) \\
\frac{8}{7}\,(7\,\mu_6\mu_{14} - \mu_{12}\mu_8)
\end{matrix}
\quad
\begin{matrix}
12\,\mu_{12} \\
-\frac{2}{7}\,(4\,\mu_4\mu_{10} - 49\,\mu_{14}) \\
\frac{4}{7}\,(14\,\mu_4\mu_{12} - 3\,\mu_6\mu_{10}) \\
\frac{2}{7}\,(21\,\mu_6\mu_{12} - 8\,\mu_{10}\mu_8 + 35\,\mu_4\mu_{14}) \\
\frac{4}{7}\,(7\,\mu_{12}\mu_8 - 5\,\mu_{10}{}^2 + 14\,\mu_6\mu_{14}) \\
\frac{2}{7}\,(21\,\mu_{14}\mu_8 - 5\,\mu_{12}\mu_{10})
\end{matrix}
$$

$$
\left.\begin{matrix}
14\,\mu_{14} \\
-\frac{4}{7}\,\mu_4\mu_{12} \\
\frac{2}{7}\,(35\,\mu_4\mu_{14} - 3\,\mu_6\mu_{12}) \\
\frac{8}{7}\,(7\,\mu_6\mu_{14} - \mu_{12}\mu_8) \\
\frac{2}{7}\,(21\,\mu_{14}\mu_8 - 5\,\mu_{12}\mu_{10}) \\
\frac{4}{7}\,(7\,\mu_{14}\mu_{10} - 3\,\mu_{12}{}^2)
\end{matrix}\right] .
$$

Here a calculation by Maple shows that $\Delta = 2^6 \cdot 3 \cdot \det(V)$. Then we have

$$
{}^t[L_0\ L_2\ L_4\ L_6\ L_8\ L_{10}] = V\ {}^t\!\left[\frac{\partial}{\partial\mu_4}\ \ \frac{\partial}{\partial\mu_6}\ \ \frac{\partial}{\partial\mu_8}\ \ \frac{\partial}{\partial\mu_{10}}\ \ \frac{\partial}{\partial\mu_{12}}\ \ \frac{\partial}{\partial\mu_{14}}\right].
$$

Using (3.62), their operation on Δ are given by

$$
[L_0\ L_2\ L_4\ L_6\ L_8\ L_{10}](\Delta) = [84\ \ 0\ \ 40\mu_4\ \ 24\mu_6\ \ 12\mu_8\ \ 4\mu_{10}]\Delta.
$$

As for the $(2,5)$-case, we have fundamental relations for these L_i as a set of generators of certain Lie algebra, which we do not include here. The symplectic basis of $H^1_{\mathrm{dR}}(\mathscr{C}/\mathbb{Q}[\mu])$ is

$$
\omega_5 = \frac{dx}{2y}, \quad \omega_3 = \frac{xdx}{2y}, \quad \omega_1 = \frac{x^2dx}{2y}, \quad \eta_{-5} = \frac{(5x^5 + 3\mu_4x^3 + 2\mu_6x^3 + \mu_8x^2)dx}{2y},
$$
$$
\eta_{-3} = \frac{(3x^4 + \mu_4x^2)dx}{2y}, \quad \eta_{-1} = \frac{x^3dx}{2y}.
$$

With respect to these, the matrices $\Gamma_j = \begin{bmatrix} -\beta_j & \alpha_j \\ -\gamma_j & {}^t\beta_j \end{bmatrix}$ are given as follows[4]:

$$
\alpha_0 = O, \quad \beta_0 = \begin{bmatrix} 5 & & \\ & 3 & \\ & & 1 \end{bmatrix}, \quad \gamma_0 = O,
$$

$$
\alpha_2 = \begin{bmatrix} & & \\ & & \\ & & 1 \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} & 3 & \\ -\frac{4}{7}\mu_4 & & 1 \\ & -\frac{8}{7}\mu_4 & \end{bmatrix},
$$

$$
\gamma_2 = \begin{bmatrix} -\frac{1}{7}(4\mu_4\mu_8 - 35\mu_{12}) & & \\ & -\frac{1}{7}(8\mu_4{}^2 - 21\mu_8) & \\ & & -\frac{5}{7}\mu_4 \end{bmatrix},
$$

$$
\alpha_4 = \begin{bmatrix} & & \\ & & 1 \\ & 1 & \end{bmatrix}, \quad \beta_4 = \begin{bmatrix} 3\mu_4 & & 1 \\ -\frac{6}{7}\mu_6 & \mu_4 & \\ & -\frac{12}{7}\mu_6 & \end{bmatrix},
$$

$$
\gamma_4 = \begin{bmatrix} -\frac{2}{7}(3\mu_6\mu_8 - 35\mu_{14}) & 3\mu_{12} & \\ 3\mu_{12} & -\frac{6}{7}(2\mu_4\mu_6 - 7\mu_{10}) & \mu_8 \\ & \mu_8 & -\frac{4}{7}\mu_6 \end{bmatrix},
$$

[4]These should not be confused with the symplectic basis of cycles α_j and β_j in (3.2)

$$\alpha_6 = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \quad \beta_6 = \begin{bmatrix} 2\mu_6 & \mu_4 & \\ -\frac{8}{7}\,\mu_8 & & \\ & -\frac{9}{7}\mu_8 & \end{bmatrix},$$

$$\gamma_6 = \begin{bmatrix} 3\mu_4\mu_{12} - \frac{8}{7}{\mu_8}^2 & 6\mu_{14} & \mu_{12} \\ 6\mu_{14} & -\frac{9}{7}\mu_4\mu_8 + 9\mu_{12} & 2\mu_{10} \\ \mu_{12} & 2\mu_{10} & -\frac{3}{7}\mu_8 \end{bmatrix},$$

$$\alpha_8 = \begin{bmatrix} & 1 & \\ 1 & & \\ & & \end{bmatrix}, \quad \beta_8 = \begin{bmatrix} \mu_8 & & \\ -\frac{10}{7}\mu_{10} & & \\ \mu_{12} & -\frac{6}{7}\mu_{10} & \end{bmatrix},$$

$$\gamma_8 = \begin{bmatrix} 6\mu_4\mu_{14} + 2\mu_6\mu_{12} - \frac{10}{7}\mu_8\mu_{10} & \mu_4\mu_{12} & 2\mu_{14} \\ \mu_4\mu_{12} & -\frac{6}{7}\mu_4\mu_{10} + 12\mu_{14} & 3\mu_{12} \\ 2\mu_{14} & 3\mu_{12} & -\frac{2}{7}\mu_{10} \end{bmatrix},$$

$$\alpha_{10} = \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix}, \quad \beta_{10} = \begin{bmatrix} & & \\ -\frac{5}{7}\,\mu_{12} & & \\ 2\,\mu_{14} & -\frac{3}{7}\,\mu_{12} & \end{bmatrix},$$

$$\gamma_{10} = \begin{bmatrix} 4\,\mu_6\mu_{14} - \frac{5}{7}\,\mu_8\mu_{12} & 2\,\mu_4\mu_{14} & \\ 2\,\mu_4\mu_{14} & -\frac{3}{7}\,\mu_4\mu_{12} & 4\,\mu_{14} \\ & 4\,\mu_{14} & -\frac{1}{7}\,\mu_{12} \end{bmatrix}.$$

These give a set of heat equations $(L_j - H_j)\,\sigma(u) = 0$ as before.

4.6 The sigma function for the $(2,7)$-curve

We now solve (3.73) in the $(2,7)$ case. The initial conditions **IC1**, **IC2** of (3.73) in this case are as follows:

$$\varphi(u) \in \mathbb{Q}[\mu][[u_5,\ u_3,\ u_1]], \text{ and } \varphi(u) \text{ is of homogeneous weight } 6.$$

Following [6] but in the Hurwitz series form as [26], for the (2,7) case we write any solution $\varphi(u)$ as

$$\begin{aligned} \varphi(u_5, u_3, u_1) = \sum_{\substack{\ell, m, n_4, n_6, n_8, \\ n_{10}, n_{12}, n_{14}}} \Bigg[& b(\ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ & \cdot \left(\mu_4 {u_1}^4\right)^{n_4} \left(\mu_6 {u_1}^6\right)^{n_6} \left(\mu_8 {u_1}^8\right)^{n_8} \left(\mu_{10} {u_1}^{10}\right)^{n_{10}} \left(\mu_{12} {u_1}^{12}\right)^{n_{12}} \left(\mu_{14} {u_1}^{14}\right)^{n_{14}} \\ & \cdot \frac{{u_1}^6 \left(\frac{u_5}{{u_1}^5}\right)^{\ell} \left(\frac{u_3}{{u_1}^3}\right)^{m}}{(6 - 5\ell - 3m + 4n_4 + 6n_6 + 8n_8 + 10n_{10} + 12n_{12} + 14n_{14})!\,\ell!\,m!} \Bigg], \end{aligned}$$

giving a solution of $(L_0 - H_0)\,\varphi(u) = 0$. If we define

$$k = 6 - 5\ell - 3m + 4n_4 + 6n_6 + 8n_8 + 10n_{10} + 12n_{12} + 14n_{14},$$

the above expression is rewritten as

$$\begin{aligned} \varphi(u_5, u_2, u_1) = \sum & b(\ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ & \cdot {\mu_4}^{n_4}\, {\mu_6}^{n_6}\, {\mu_8}^{n_8}\, {\mu_{10}}^{n_{10}}\, {\mu_{12}}^{n_{12}}\, {\mu_{14}}^{n_{14}}\, \frac{{u_5}^{\ell}}{\ell!}\, \frac{{u_3}^{m}}{m!}\, \frac{{u_1}^{k}}{k!}, \end{aligned} \tag{4.11}$$

where we require all the integer indices k, ℓ, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14} to be non-negative.

Note that the u-weight of this expression is $k_0 = 6+4n_4+6n_6+8n_8+10n_{10}+12n_{12}+14n_{14}$, which does not depend on ℓ or m. (Note also that $k = k_0-5\ell-3m$). For fixed n_4, n_6, n_8, n_{10}, n_{12}, $n_{14} \geq 0$, $k_0 \geq 0$ is fixed, and for non-negative k, we require $\ell = 0, \ldots, \lfloor (k_0+6)/5 \rfloor$, $m = 0, \ldots, \lfloor (6+k_0-5\ell)/3 \rfloor$. As noted above, if we insert this ansatz into the equation for $(L_0 - H_0)\varphi = 0$, we get an expression which is identically zero, for any set of $b(\ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14})$.

If we insert this ansatz into the expression for $(L_2 - H_2)\varphi = 0$, we get (after some algebra, and providing $k > 0$) the recurrence relation shown below, involving 20 terms (compare the equations on p.68 of [6] for the genus 2 case). We can structure this relation by the *weight* of each b coefficient of (4.11) (more precisely by the weight of the corresponding term in the expansion). We will call this P_2 :

$$\begin{aligned}
&-7\,b(\ell,m,n_4,n_6,n_8,n_{10},n_{12},n_{14})\\
&+14(2-k)\,b(\ell,m{+}1,n_4,n_6,n_8,n_{10},n_{12},n_{14})\\
&-42m\,b(\ell{+}1,m{-}1,n_4,n_6,n_8,n_{10},n_{12},n_{14})\\
&+196(n_{12}+1)\,b(\ell,m,n_4,n_6,n_8,n_{10},n_{12}{+}1,n_{14}{-}1)\\
&+168(n_{10}+1)\,b(\ell,m,n_4,n_6,n_8,n_{10}{+}1,n_{12}{-}1,n_{14})\\
&+140(n_8+1)\,b(\ell,m,n_4,n_6,n_8{+}1,n_{10}{-}1,n_{12},n_{14})\\
&+112(n_6+1)\,b(\ell,m,n_4,n_6{+}1,n_8{-}1,n_{10},n_{12},n_{14})\\
&-40(n_6+1)\,b(\ell,m,n_4{-}2,n_6{+}1,n_8,n_{10},n_{12},n_{14})\\
&+5(3-k)(2-k)\,b(\ell,m,n_4{-}1,n_6,n_8,n_{10},n_{12},n_{14})\\
&-8(n_{14}+1)\,b(\ell,m,n_4{-}1,n_6,n_8,n_{10},n_{12}-1,n_{14}+1)\\
&-16(n_{12}+1)\,b(\ell,m,n_4{-}1,n_6,n_8,n_{10}{-}1,n_{12}{+}1,n_{14})\\
&-24(n_{10}+1)\,b(\ell,m,n_4{-}1,n_6,n_8{-}1,n_{10}{+}1,n_{12},n_{14})\\
&-32(n_8+1)\,b(\ell,m,n_4{-}1,n_6{-}1,n_8{+}1,n_{10},n_{12},n_{14})\\
&+84(n_4+1)\,b(\ell,m,n_4{+}1,n_6{-}1,n_8,n_{10},n_{12},n_{14})\\
&-21m(m-1)\,b(\ell,m{-}2,n_4,n_6,n_8{-}1,n_{10},n_{12},n_{14})\\
&+8m(m-1)\,b(\ell,m{-}2,n_4{-}2,n_6,n_8,n_{10},n_{12},n_{14})\\
&+16m\,b(\ell,m{-}1,n_4{-}1,n_6,n_8,n_{10},n_{12},n_{14})\\
&-35\ell(\ell{-}1)\,b(\ell{-}2,m,n_4,n_6,n_8,n_{10},n_{12}{-}1,n_{14})\\
&+4\ell(\ell{-}1)\,b(\ell{-}2,m,n_4{-}1,n_6,n_8{-}1,n_{10},n_{12},n_{14})\\
&+8\ell\,b(\ell{-}1,m{+}1,n_4{-}1,n_6,n_8,n_{10},n_{12},n_{14}) = 0.
\end{aligned}$$

This relation applies only for $k > 1$, and can be written as

$$\begin{aligned}
P_2 : b(\ell,m,n_4,n_6,n_8,n_{10},n_{12},n_{14})
&= 2\,(2-k)\,b(\ell,1{+}m,n_4,n_6,n_8,n_{10},n_{12},n_{14})\\
&\quad -6m\,b(1{+}\ell,m{-}1,n_4,n_6,n_8,n_{10},n_{12},n_{14})\\
&\quad + \text{“lower weight terms”},
\end{aligned}$$

where the lower weight terms have coefficients which are quadratic or linear in ℓ, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}, times integers or rational numbers with denominators 7. Here the number $4n_4 + 6n_6 + 8n_8 + 10n_{10} + 12n_{12} + 14n_{14}$ for

$$b(\ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14})$$

is the μ-*weight* of the term. For P_2, the left hand side and the first two terms on the right hand side all have the μ-weight $W = 4n_4+6n_6+8n_8+10n_{10}+12n_{12}+14n_{14}$. The next highest μ-weight terms of the "lower weight terms" are of μ-weight $W-2$, and the lowest weight terms are of μ-weight $W - 12$.

Putting the same ansatz into $(L_4 - H_4)\varphi = 0$ we get another recurrence P_4 with 20 terms, providing $m > 0$ and $k > 0$. We can write this as

$$\begin{aligned} P_4 : &b(\ell, 1+m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ &= -7(k-1)\, b(1+\ell, m-1, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) + \text{"lower weight terms"}. \end{aligned}$$

Here the lower weight terms have the same property as P_2. We have another relation from the equation $(L_6 - H_6)\varphi = 0$

$$\begin{aligned} P_6 :\ &b(\ell, m+2, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ &\quad + 2\, b(1+\ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) = \text{"lower weight terms"}. \end{aligned}$$

We can write this in two different ways which will each come in useful

$$\begin{aligned} P_{6\text{a}} :\ &b(l, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ &= -2\, b(l+1, m-2, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) + \text{"lower weight terms"}, \\ P_{6\text{b}} :\ &2\, b(l, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ &= -b(l-1, m+2, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) + \text{"lower weight terms"}. \end{aligned}$$

Continuing, we have two further relations, from the equations $(L_8 - H_8)\varphi = 0$ and $(L_{10} - H_{10})\varphi = 0$,

$$\begin{aligned} P_8 &: b(\ell+1, m+1, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) = \text{"lower } \mu\text{-weight terms"}, \\ P_{10} &: b(\ell+2, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) = \text{"lower } \mu\text{-weight terms"}, \end{aligned}$$

where the lower μ-weight terms have the same properties as P_2 and P_4. The relations P_6, P_8, P_{10} have a total of 24, 24, 19 terms respectively. As before, we need to normalise the expansion, so we choose $b(1, 0, 0, 0, 0, 0, 0, 0) = 1$. We need to find relations which either express coefficients in terms of ones with lower or equal μ-weight.

Clearly we must take care with our recurrence relation to avoid infinite looping. We find that the following choice of recurrence scheme results in a sequence which decreases the μ-weight after no more than one extra step at any point in the recurrence :

$$b(\ell,m,n_4,n_6,n_8,n_{10},n_{12},n_{14})$$
$$= \begin{cases} 0 & \text{if } \min\{k,\ell,m,n_4,n_6,n_8,n_{10},n_{12},n_{14}\} < 0, \\ 1 & \text{if } \ell = 1,\ m = n_4 = n_6 = n_8 = n_{10} = n_{12} = n_{14} = 0, \\ \text{rhs}(P_2) & \text{if } k > 1, \\ \text{rhs}(P_4) & \text{if } k = 1,\ m > 0, \\ \text{rhs}(P_{6\text{a}}) & \text{if } k = 1,\ m = 0 \ \text{ (and } \ell > 0), \\ \text{rhs}(P_{6\text{b}}) & \text{if } k = 0,\ m > 1, \\ \text{rhs}(P_8) & \text{if } k = 0,\ m = 1 \text{ and } \ell > 0, \\ \text{rhs}(P_{10}) & \text{if } k = 0,\ m = 0 \text{ and } \ell > 1. \end{cases}$$

Note that the structure of this complicated linear recurrence relation does *not* depend on the moduli μ_i. We have used this to calculate the terms on the Hurwitz series for the solution up to weight 40 in u_i (weight 34 in the μ_i). As for the $(2,5)$-curve, there is another possible recursion scheme:

$$b(\ell,m,n_4,n_6,n_8,n_{10},n_{12},n_{14})$$
$$= \begin{cases} 0 & \text{if } \min\{k,\ell,m,n_4,n_8,n_{10},n_{12},n_{14}\} < 0, \\ 1 & \text{if } \ell = 1,\ m = n_4 = n_8 = n_{10} = n_{12} = n_{14} = 0, \\ \text{rhs}(P_{10}) & \text{if } \ell > 1, \\ \text{rhs}(P_8) & \text{if } \ell = 1,\ m > 0, \\ \text{rhs}(P_{6\text{a}}) & \text{if } \ell = 1,\ m = 0 \text{ and } k > 0, \\ \text{rhs}(P_4) & \text{if } \ell = 0,\ m > 0 \text{ and } k > 0, \\ \text{rhs}(P_{6\text{b}}) & \text{if } \ell = 0,\ m > 0 \text{ and } k = 0, \\ \text{rhs}(P_2) & \text{if } \ell = 0, \text{ and } m = 0. \end{cases}$$

We have used this to calculate the terms in the series up to weight 40 in $\{u_j\}$, or equivalently, weight 35 in the $\{\mu_i\}$. The first few terms of the expansion are given as follows (up to a constant multiple):

$$\begin{aligned} \varphi(u) = \sigma(u_5, u_3, u_1) = {} & 16\frac{{u_1}^6}{6!} - 2\frac{{u_1}^3 u_3}{3!} - 2\frac{{u_3}^2}{2!} + u_5 u_1 + 64\mu_4 \frac{{u_1}^{10}}{10!} + 36\mu_4 \frac{{u_1}^7 u_3}{7!} \\ & - 4\mu_4 \frac{{u_1}^4 {u_3}^2}{4!2!} - 2\mu_4 \frac{u_1 {u_3}^3}{3!} + 2\mu_4 \frac{u_5 {u_1}^5}{5!} - 512\mu_6 \frac{{u_1}^{12}}{12!} + 64\mu_6 \frac{{u_1}^9 u_3}{9!} \\ & + 16\mu_6 \frac{{u_1}^6 {u_3}^2}{6!2!} - 8\mu_6 \frac{{u_1}^3 {u_3}^3}{3!^2} - 8\mu_6 \frac{{u_3}^4}{4!} + 24\mu_6 \frac{u_5 {u_1}^7}{7!} + \cdots . \end{aligned}$$

Further studies are required to establish whether there are other recursion schemes which can be used to generate the series, and which recursions could be considered the most efficient in some sense.

4.7 Heat equations for the $(3,4)$-curve

We take the trigonal genus three curve $\mathscr{C} = \mathscr{C}_\mu^{3,4}$ in the Weierstrass form

$$y^3 = (\mu_8 + \mu_5 x + \mu_2 x^2)y + x^4 + \mu_6 x^2 + \mu_9 x + \mu_{12}.$$

The matrix V is given by

$$V = \begin{bmatrix} V_{1..3,\,1..3} & V_{1..3,\,4..6} \\ {}^tV_{1..3,\,4..6} & V_{4..6,\,4..6} \end{bmatrix},$$

where

$$V_{1..3,\,1..3} = \begin{bmatrix} 2\,\mu_2 & 5\,\mu_5 & 6\,\mu_6 \\ 5\,\mu_5 & \frac{1}{6}\,{\mu_2}^4 - 4\,\mu_2\mu_6 + 8\,\mu_8 & -\frac{1}{2}\,{\mu_2}^2\mu_5 + 9\,\mu_9 \\ 6\,\mu_6 & -\frac{1}{2}\,{\mu_2}^2\mu_5 + 9\,\mu_9 & \frac{2}{3}\,{\mu_2}^2\mu_6 + \frac{10}{3}\,\mu_2\mu_8 + \frac{5}{3}\,{\mu_5}^2 \end{bmatrix}$$

and

$$V_{1..3,\,4..6} = \begin{bmatrix} 8\,\mu_8 & 9\,\mu_9 & 12\,\mu_{12} \\ \frac{1}{12}\,{\mu_2}^3\mu_5 - \frac{1}{2}\,\mu_2\mu_9 - \frac{3}{2}\,\mu_5\mu_6 & \frac{1}{6}\,{\mu_2}^3\mu_6 - \frac{1}{3}\,{\mu_2}^2\mu_8 - \frac{1}{6}\,\mu_2{\mu_5}^2 - 3\,{\mu_6}^2 + 12\,\mu_{12} & \frac{1}{12}\,{\mu_2}^3\mu_9 - \frac{1}{6}\,\mu_2\mu_5\mu_8 - \frac{3}{2}\,\mu_6\mu_9 \\ \frac{4}{3}\,{\mu_2}^2\mu_8 - \frac{7}{12}\,\mu_2{\mu_5}^2 + 12\,\mu_{12} & \frac{4}{3}\,{\mu_2}^2\mu_9 - \frac{7}{6}\,\mu_2\mu_5\mu_6 + \frac{13}{3}\,\mu_5\mu_8 & 2\,{\mu_2}^2\mu_{12} - \frac{7}{12}\,\mu_2\mu_5\mu_9 + \frac{8}{3}\,{\mu_8}^2 \end{bmatrix},$$

and the remaining elements are given by

$$\begin{aligned}
V_{4,4} &= \frac{1}{24}\,{\mu_2}^2{\mu_5}^2 + 6\,\mu_2\mu_{12} - \frac{7}{2}\,\mu_5\mu_9 + 4\,\mu_6\mu_8, \\
V_{4,5} = V_{5,4} &= \frac{1}{12}\,{\mu_2}^2\mu_5\mu_6 + \frac{7}{6}\,\mu_2\mu_5\mu_8 - \frac{5}{12}\,{\mu_5}^3 - \frac{3}{2}\,\mu_6\mu_9, \\
V_{4,6} = V_{6,4} &= \frac{1}{24}\,{\mu_2}^2\mu_5\mu_9 + \frac{4}{3}\,\mu_2{\mu_8}^2 - \frac{5}{12}\,{\mu_5}^2\mu_8 + 6\,\mu_6\mu_{12} - \frac{9}{4}\,{\mu_9}^2, \\
V_{5,5} &= \frac{1}{6}\,{\mu_2}^2{\mu_6}^2 + 2\,{\mu_2}^2\mu_{12} + \frac{5}{3}\,\mu_2\mu_5\mu_9 - \frac{8}{3}\,\mu_2\mu_6\mu_8 - \frac{4}{3}\,{\mu_5}^2\mu_6 + \frac{8}{3}\,{\mu_8}^2, \\
V_{5,6} = V_{6,5} &= \frac{1}{12}\,{\mu_2}^2\mu_6\mu_9 + 3\,\mu_2\mu_5\mu_{12} - \frac{5}{6}\,\mu_2\mu_8\mu_9 - \frac{5}{12}\,{\mu_5}^2\mu_9 - \frac{1}{2}\,\mu_5\mu_6\mu_8, \\
V_{6,6} &= \frac{1}{24}\,{\mu_2}^2{\mu_9}^2 + 2\,\mu_2\mu_8\mu_{12} + {\mu_5}^2\mu_{12} - \frac{11}{6}\,\mu_5\mu_8\mu_9 + \frac{4}{3}\,\mu_6{\mu_8}^2.
\end{aligned}$$

The discriminant $\varDelta$ of $\mathscr{C}$ is calculated by an algorithm provided by Sylvester, has 670 terms and is of weight 72. A calculation by Maple shows $\varDelta = 3\cdot 4^2\cdot \det(V)$. Then we have

$$[\,L_0\;\;L_3\;\;L_4\;\;L_6\;\;L_7\;\;L_{10}]\,\varDelta = [\,72\;\;0\;\;-8{\mu_2}^2\;\;12\mu_6\;\;-8\mu_2\mu_5\;\;-{\mu_5}^2 - 4\mu_2\mu_8\,]\varDelta.$$

As in the $(2,7)$-case, we have fundamental relations for these L_i as a set of generators of certain Lie algebra. The symplectic basis of $H^1_{\rm dR}(\mathscr{C}/\mathbb{Q}[\mu])$ in this case is given by

$$\omega_5 = \frac{dx}{3y^2},\ \omega_2 = \frac{xdx}{3y^2},\ \omega_1 = \frac{ydx}{3y^2},\ \eta_{-5} = \frac{5x^2ydx}{3y^2},\ \eta_{-2} = \frac{2xydx}{3y^2},\ \eta_{-1} = \frac{x^2dx}{3y^2}.$$

The matrices $\Gamma_j = \begin{bmatrix} -\beta_j & \alpha_j \\ -\gamma_j & {}^t\beta_j \end{bmatrix}$ are given as follows[5] :

$$\alpha_0 = O,\quad \beta_0 = \begin{bmatrix} 5 & & \\ & 2 & \\ & & 1 \end{bmatrix},\quad \gamma_0 = O,$$

$$\alpha_3 = \begin{bmatrix} & & \\ & & 1 \\ & 1 & \end{bmatrix},\quad \beta_3 = \begin{bmatrix} & 2 & \\ \frac{1}{12}\,{\mu_2}^3 - \frac{3}{2}\,\mu_6 & & -\frac{1}{2}\,\mu_2 \\ -\frac{1}{6}\,\mu_5\mu_2 & -\frac{1}{3}\,{\mu_2}^2 & \end{bmatrix},$$

$$\gamma_3 = \begin{bmatrix} 5\,\mu_5\mu_8 - \frac{1}{3}\,{\mu_2}^2\mu_9 + \frac{1}{12}\,{\mu_2}^4\mu_5 - \frac{5}{3}\,\mu_2\mu_5\mu_6 & -\mu_2\mu_8 & \mu_9 \\ -\mu_2\mu_8 & \frac{2}{3}\,\mu_5\mu_2 & \frac{1}{6}\,{\mu_2}^3 - \mu_6 \\ \mu_9 & \frac{1}{6}\,{\mu_2}^3 - \mu_6 & \mu_5 \end{bmatrix},$$

$$\alpha_4 = \begin{bmatrix} & & \\ & 1 & \\ & & \frac{1}{3}\,\mu_2 \end{bmatrix},\quad \beta_4 = \begin{bmatrix} \frac{2}{3}\,{\mu_2}^2 & & 1 \\ -\frac{7}{12}\,\mu_5\mu_2 & & \\ \frac{5}{3}\,\mu_8 & \frac{2}{3}\,\mu_5 & \end{bmatrix},$$

$$\gamma_4 = \begin{bmatrix} -\frac{7}{12}\,{\mu_2}^2{\mu_5}^2 + 9\,\mu_2\mu_{12} + \frac{5}{3}\,\mu_5\mu_9 + \frac{11}{3}\,\mu_6\mu_8 & \frac{2}{3}\,\mu_2\mu_9 & -\frac{1}{3}\,\mu_2\mu_8 \\ \frac{2}{3}\,\mu_2\mu_9 & \frac{4}{3}\,\mu_2\mu_6 + \frac{4}{3}\,\mu_8 & -\frac{1}{2}\,\mu_5\mu_2 \\ -\frac{1}{3}\,\mu_2\mu_8 & -\frac{1}{2}\,\mu_5\mu_2 & \mu_6 \end{bmatrix},$$

$$\alpha_6 = \begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix},\quad \beta_6 = \begin{bmatrix} \mu_6 & & \\ \frac{1}{24}\,{\mu_2}^2\mu_5 - \frac{5}{4}\,\mu_9 & & -\frac{1}{4}\,\mu_5 \\ \frac{1}{3}\,\mu_2\mu_8 - \frac{5}{12}\,{\mu_5}^2 & -\frac{1}{6}\,\mu_5\mu_2 & \end{bmatrix},$$

$$\gamma_6 = \begin{bmatrix} \left\{ \begin{matrix} \frac{1}{24}\,{\mu_2}^3{\mu_5}^2 + 4\,{\mu_2}^2\mu_{12} - \frac{29}{12}\,\mu_2\mu_5\mu_9 \\ +\frac{4}{3}\,\mu_2\mu_6\mu_8 - \frac{5}{12}\,{\mu_5}^2\mu_6 + 6\,{\mu_8}^2 \end{matrix} \right\} & \frac{1}{2}\,\mu_5\mu_8 & 3\,\mu_{12} \\ \frac{1}{2}\,\mu_5\mu_8 & \frac{8}{3}\,\mu_2\mu_8 - \frac{1}{3}\,{\mu_5}^2 & \frac{1}{12}\,{\mu_2}^2\mu_5 - \frac{1}{2}\,\mu_9 \\ 3\,\mu_{12} & \frac{1}{12}\,{\mu_2}^2\mu_5 - \frac{1}{2}\,\mu_9 & 2\,\mu_8 \end{bmatrix},$$

$$\alpha_7 = \begin{bmatrix} & 1 & \\ 1 & & \\ & & \frac{1}{3}\,\mu_5 \end{bmatrix},\quad \beta_7 = \begin{bmatrix} \frac{2}{3}\,\mu_5\mu_2 & & \\ \frac{1}{12}\,{\mu_2}^2\mu_6 - \frac{7}{6}\,\mu_2\mu_8 - \frac{5\,{\mu_5}^2}{12} & & -\frac{1}{2}\,\mu_6 \\ \frac{1}{3}\,\mu_2\mu_9 - \frac{1}{2}\,\mu_5\mu_6 & -\frac{1}{3}\,\mu_2\mu_6 + \frac{2}{3}\,\mu_8 & \end{bmatrix},$$

$$\gamma_7 = \begin{bmatrix} \gamma_7^{[1,1]} & 2\,\mu_2\mu_{12} + \frac{2}{3}\,\mu_5\mu_9 - \mu_6\mu_8 & \frac{2}{3}\,\mu_5\mu_8 \\ 2\,\mu_2\mu_{12} + \frac{2}{3}\,\mu_5\mu_9 - \mu_6\mu_8 & \frac{8}{3}\,\mu_2\mu_9 - \frac{2}{3}\,\mu_5\mu_6 & -\frac{1}{6}\,{\mu_5}^2 + \frac{1}{6}\,{\mu_2}^2\mu_6 - \frac{1}{3}\,\mu_2\mu_8 \\ \frac{2}{3}\,\mu_5\mu_8 & -\frac{1}{6}\,{\mu_5}^2 + \frac{1}{6}\,{\mu_2}^2\mu_6 - \frac{1}{3}\,\mu_2\mu_8 & 2\,\mu_9 \end{bmatrix},$$

$$\big(\gamma_7^{[1,1]} = -\frac{5}{12}\,\mu_2{\mu_5}^3 - \frac{1}{2}\,\mu_5{\mu_6}^2 + 5\,\mu_{12}\mu_5 + \frac{14}{3}\,\mu_8\mu_9 + \frac{1}{12}\,{\mu_2}^3\mu_5\mu_6 - \frac{1}{6}\,{\mu_2}^2\mu_5\mu_8 - \mu_2\mu_6\mu_9\big),$$

$$\alpha_{10} = \begin{bmatrix} 1 & & \\ & & \\ & & \frac{1}{3}\,\mu_8 \end{bmatrix},\quad \beta_{10} = \begin{bmatrix} -\frac{1}{3}\,\mu_2\mu_8 & & \\ \frac{1}{24}\,{\mu_2}^2\mu_9 - \frac{5}{12}\,\mu_5\mu_8 & & -\frac{1}{4}\,\mu_9 \\ \mu_2\mu_{12} - \frac{5}{12}\,\mu_5\mu_9 + \frac{1}{3}\,\mu_6\mu_8 & -\frac{1}{6}\,\mu_2\mu_9 & \end{bmatrix},$$

$$\gamma_{10} = \begin{bmatrix} \gamma_{10}^{[1,1]} & 2\,\mu_{12}\mu_5 - \frac{5}{6}\,\mu_8\mu_9 & \frac{2}{3}\,{\mu_8}^2 \\ 2\,\mu_{12}\mu_5 - \frac{5}{6}\,\mu_8\mu_9 & 4\,\mu_2\mu_{12} - \frac{1}{3}\,\mu_5\mu_9 & -\frac{1}{6}\,\mu_5\mu_8 + \frac{1}{12}\,{\mu_2}^2\mu_9 \\ \frac{2}{3}\,{\mu_8}^2 & -\frac{1}{6}\,\mu_5\mu_8 + \frac{1}{12}\,{\mu_2}^2\mu_9 & 3\,\mu_{12} \end{bmatrix},$$

[5]These should not be confused with the symplectic basis of cycles α_j and β_j in (3.2)

$$\left(\gamma_{10}^{[1,1]} = \tfrac{1}{24}\,\mu_2{}^3\mu_5\mu_9 + \mu_2{}^2\mu_8{}^2 - \tfrac{5}{12}\,\mu_2\mu_5{}^2\mu_8 + 3\,\mu_2\mu_6\mu_{12} - \tfrac{7}{6}\,\mu_2\mu_9{}^2 \right.$$
$$\left. - \tfrac{5}{12}\,\mu_5\mu_6\mu_9 + \tfrac{1}{3}\,\mu_6{}^2\mu_8 + 9\,\mu_8\mu_{12} \right).$$

4.8 The sigma function for the $(3,4)$-curve

Following [6] and from the conditions **IC1**, **IC2** of (3.73), but using the Hurwitz series form, the sigma function is of the form

$$\sigma(u_5,u_2,u_1) = \sum_{\ell,m,n_2,n_5,n_6,n_8,n_9,n_{12}} \Bigg[b(\ell,m,n_2,n_5,n_6,n_8,n_9,n_{12})\,u_1{}^5\left(\frac{u_5}{u_1{}^5}\right)^{\ell}$$
$$\cdot \left(\frac{u_2}{u_1{}^2}\right)^{m} \left(\mu_2u_1{}^2\right)^{n_2}\left(\mu_5u_1{}^5\right)^{n_5}\left(\mu_6u_1{}^6\right)^{n_6}\left(\mu_8u_1{}^8\right)^{n_8}\left(\mu_9u_1{}^9\right)^{n_9}\left(\mu_{12}u_1{}^{12}\right)^{n_{12}}$$
$$\Big/\Big(\,\ell!\,m!\,(5-5\ell-2m+2n_2+5n_5+6n_6+8n_8+9n_9+12n_{12})!\,\Big)\Bigg].$$

for the $(3,4)$-curve. If we define

$$k = 5-5\ell-2m+2n_2+5n_5+6n_6+8n_8+9n_9+12n_{12},$$

we can rewrite the above expression as

$$\sigma(u_5,u_2,u_1) = \sum b(\ell,m,n_2,n_5,n_6,n_8,n_9,n_{12})$$
$$\cdot\,\frac{\mu_2{}^{n_2}\mu_5{}^{n_5}\mu_6{}^{n_6}\mu_8{}^{n_8}\mu_9{}^{n_9}\mu_{12}{}^{n_{12}}\,u_5{}^{\ell}u_2{}^{m}u_1{}^{k}}{\ell!\,m!\,k!},$$

where we require all the integer indices k, ℓ, m, n_2, n_5, n_6, n_8, n_9, n_{12} to be non-negative. Note that the u-weight of this expression is $k_0 = 5+2n_2+5n_5+6n_6+8n_8+9n_9+10n_{12}$, which does not depend on ℓ or m. (Note also that $k = k_0 - 5\ell - 2m$.) For fixed n_2, n_5, n_8, n_6, n_9, $n_{12} \geq 0$, $k_0 \geq 0$ is fixed, and for non-negative k, we require $\ell = 0, \ldots, \lfloor k_0/5 \rfloor$, $m = 0, \ldots, \lfloor (k_0-5\ell)/2 \rfloor$. In addition, we can use the condition that σ is an odd function, $\sigma(-u) = -\sigma(u)$; this tells us that if k_0 is even(odd) then we should restrict ourselves to m even(odd) respectively.

If we insert this ansatz into the equation for $(L_0 - H_0)\sigma = 0$, we get an expression which is identically zero, whatever the values for the $b(\ell,m,n_2,n_5,n_6,n_8,n_9,n_{12})$. If we insert the ansatz into the equation for $(L_3 - H_3)\sigma = 0$, we get (after some algebra) the recurrence relation shown below, involving 34 terms (compare the equations on p.68 of [6] for the genus 2 case). We can structure the relation by the *weight* of each b coefficient (more precisely by the weight of the corresponding term in the sigma expansion).

Contrarily to the $(2,3)$-, $(2,5)$-, $(2,7)$-curves, we could not find any approach for the $(3,4)$-curve to prove Hurwitz integrality of the expansion of $\sigma(u)$.

We call the recurrence relation, generated from $(L_3 - H_3)\sigma = 0$, R_3:

$$24\,b(\ell,m+1,n_2,n_5,n_6,n_8,n_9,n_{12})$$
$$+\,48m\,b(\ell+1,m-1,n_2,n_5,n_6,n_8,n_9,n_{12})$$

$$
\begin{aligned}
= &-12(4-k)(3-k)\, b(\ell, m, n_2, n_5-1, n_6, n_8, n_9, n_{12}) \\
&-36(n_8+1)\, b(\ell, m, n_2, n_5-1, n_6-1, n_8+1, n_9, n_{12}) \\
&+4(n_9+1)\, b(\ell, m, n_2-3, n_5, n_6-1, n_8, n_9+1, n_{12}) \\
&+2(n_{12}+1)\, b(\ell, m, n_2-3, n_5, n_6, n_8, n_9-1, n_{12}+1) \\
&+4(n_5+1)\, b(\ell, m, n_2-4, n_5+1, n_6, n_8, n_9, n_{12}) \\
&-8(n_9+1)\, b(\ell, m, n_2-2, n_5, n_6, n_8-1, n_9+1, n_{12}) \\
&+2(n_8+1)\, b(\ell, m, n_2-3, n_5-1, n_6, n_8+1, n_9, n_{12}) \\
&-96(n_5+1)\, b(\ell, m, n_2-1, n_5+1, n_6-1, n_8, n_9, n_{12}) \\
&-12(n_8+1)\, b(\ell, m, n_2-1, n_5, n_6, n_8+1, n_9-1, n_{12}) \\
&-12(n_6+1)\, b(\ell, m, n_2-2, n_5-1, n_6+1, n_8, n_9, n_{12}) \\
&-4(n_{12}+1)\, b(\ell, m, n_2-1, n_5-1, n_6, n_8-1, n_9, n_{12}+1) \\
&-4(n_9+1)\, b(\ell, m, n_2-1, n_5-2, n_6, n_8, n_9+1, n_{12}) \\
&+120(n_2+1)\, b(\ell, m, n_2+1, n_5-1, n_6, n_8, n_9, n_{12}) \\
&-12(3-k)\, \underline{b(\ell, m+1, n_2-1, n_5, n_6, n_8, n_9, n_{12})} \\
&-24m(3-k)\, b(\ell, m-1, n_2, n_5, n_6-1, n_8, n_9, n_{12}) \\
&-8m(m-1)\, b(\ell, m-2, n_2-1, n_5-1, n_6, n_8, n_9, n_{12}) \\
&+24\ell(3-k)\, b(\ell-1, m, n_2, n_5, n_6, n_8, n_9-1, n_{12}) \\
&-60\ell(\ell-1)\, b(\ell-2, m, n_2, n_5-1, n_6, n_8-1, n_9, n_{12}) \\
&+4m(3-k)\, b(\ell, m-1, n_2-3, n_5, n_6, n_8, n_9, n_{12}) \\
&-\ell(\ell-1)\, b(\ell-2, m, n_2-4, n_5-1, n_6, n_8, n_9, n_{12}) \\
&+4\ell(\ell-1)\, b(\ell-2, m, n_2-2, n_5, n_6, n_8, n_9-1, n_{12}) \\
&+20\ell(\ell-1)\, b(\ell-2, m, n_2-1, n_5-1, n_6-1, n_8, n_9, n_{12}) \\
&+24\ell m\, b(\ell-1, m-1, n_2-1, n_5, n_6, n_8-1, n_9, n_{12}) \\
&+8m\, b(\ell, m-1, n_2-2, n_5, n_6, n_8, n_9, n_{12}) \\
&+4\ell\, b(\ell-1, m, n_2-1, n_5-1, n_6, n_8, n_9, n_{12}) \\
&-2\ell\, b(\ell-1, m+1, n_2-3, n_5, n_6, n_8, n_9, n_{12}) \\
&+36\ell\, b(\ell-1, m+1, n_2, n_5, n_6-1, n_8, n_9, n_{12}) \\
&-36(n_{12}+1)\, b(\ell, m, n_2, n_5, n_6-1, n_8, n_9-1, n_{12}+1) \\
&-72(n_9+1)\, b(\ell, m, n_2, n_5, n_6-2, n_8, n_9+1, n_{12}) \\
&+288(n_9+1)\, b(\ell, m, n_2, n_5, n_6, n_8, n_9+1, n_{12}-1) \\
&+192(n_5+1)\, b(\ell, m, n_2, n_5+1, n_6, n_8-1, n_9, n_{12}) \\
&+216(n_6+1)\, b(\ell, m, n_2, n_5, n_6+1, n_8, n_9-1, n_{12}).
\end{aligned}
$$

Note the two expressions on the left hand side, which are the highest weight terms, at weight $W = 2n_2 + 5n_5 + 6n_6 + 8n_8 + 9n_9 + 12n_{12}$. The next highest weight term (underlined) is of weight $W-2$, and the lowest weight terms are of weight

$W-13$. Putting the ansatz into the equation for $(L_4 - H_4)\sigma$ we get another recurrence with 27 terms, which we call R_4:

$$
\begin{aligned}
-12\, b(\ell, m+2, n_2, & n_5, n_8, n_6, n_9, n_{12}) \\
& + 24(4-k)\, b(\ell+1, m, n_2, n_5, n_6, n_8, n_9, n_{12}) \\
= 4\, & \underline{b(\ell, m, n_2-1, n_5, n_6, n_8, n_9, n_{12})} \\
& +16\ell m\, b(\ell-1, m-1, n_2-1, n_5, n_6, n_8, n_9-1, n_{12}) \\
& -14\ell\, b(\ell-1, m+1, n_2-1, n_5-1, n_6, n_8, n_9, n_{12}) \\
& +40\ell\, b(\ell-1, m, n_2, n_5, n_6, n_8-1, n_9, n_{12}) \\
& +16m\, b(\ell, m-1, n_2, n_5-1, n_6, n_8, n_9, n_{12}) \\
& -7\ell(\ell-1)\, b(\ell-2, m, n_2-2, n_5-2, n_6, n_8, n_9, n_{12}) \\
& +108\ell(\ell-1)\, b(\ell-2, m, n_2-1, n_5, n_6, n_8, n_9, n_{12}-1) \\
& +44\ell(\ell-1)\, b(\ell-2, m, n_2, n_5, n_8-1, n_6-1, n_9, n_{12}) \\
+12(5-k)(4-k)\, & b(\ell, m, n_2, n_5, n_6-1, n_8, n_9, n_{12}) \\
& +8\ell(4-k)\, b(\ell-1, m, n_2-1, n_5, n_6, n_8-1, n_9, n_{12}) \\
& +20\ell(\ell-1)\, b(\ell-2, m, n_2, n_5-1, n_6, n_8, n_9-1, n_{12}) \\
& -288(n_8+1)\, b(\ell, m, n_2, n_5, n_6, n_8+1, n_9, n_{12}-1) \\
& -64(n_{12}+1)\, b(\ell, m, n_2, n_5, n_6, n_8-2, n_9, n_{12}+1) \\
& -40(n_6+1)\, b(\ell, m, n_2, n_5-2, n_6+1, n_8, n_9, n_{12}) \\
& -216(1+n_5)\, b(\ell, m, n_2, n_5+1, n_6, n_8, n_9-1, n_{12}) \\
& -80(n_6+1)\, b(\ell, m, n_2-1, n_5, n_6+1, n_8-1, n_9, n_{12}) \\
-4(\ell-1-n_9 & -8n_5-2n_6+k+2m-2n_2) \\
& \cdot b(\ell, m, n_2-2, n_5, n_6, n_8, n_9, n_{12}) \\
& -104(n_9+1)\, b(\ell, m, n_2, n_5-1, n_6, n_8-1, n_9+1, n_{12}) \\
& +14(n_8+1)\, b(\ell, m, n_2-1, n_5-2, n_6, n_8+1, n_9, n_{12}) \\
& +14(n_{12}+1)\, b(\ell, m, n_2-1, n_5-1, n_6, n_8, n_9-1, n_{12}+1) \\
& +28(n_9+1)\, b(\ell, m, n_2-1, n_5-1, n_6-1, n_8, n_9+1, n_{12}) \\
& -144(n_2+1)\, b(\ell, m, n_2+1, n_5, n_6-1, n_8, n_9, n_{12}) \\
& +16m(m-1)\, b(\ell, m-2, n_2, n_5, n_6, n_8-1, n_9, n_{12}) \\
& +16m(m-1)\, b(\ell, m-2, n_2-1, n_5, n_6-1, n_8, n_9, n_{12}) \\
& +12m(4-k)\, b(\ell, m-1, n_2-1, n_5-1, n_6, n_8, n_9, n_{12}).
\end{aligned}
$$

As for R_3, the two expressions on the left hand side, are the highest weight terms, at weight $W = 2n_2 + 5n_5 + 6n_6 + 8n_8 + 9n_9 + 12n_{12}$. The next highest weight terms are of weight $W-2$, and the lowest weight terms are of weight $W-14$.

We see that the two recurrence relations have the same terms in b. Hence we can take linear combinations to get two relations, each with only one leading term at weight W

$$S_{3,4}: b(\ell, m, n_2, n_5, n_8, n_6, n_9, n_{12}) = \frac{1}{k-m+1}\,\text{(lower weight terms)} \quad (m \neq 0),$$
$$T_{3,4}: b(\ell, m, n_2, n_5, n_8, n_6, n_9, n_{12}) = \frac{1}{k-m}\,\text{(lower weight terms)} \quad (l \neq 0).$$

These $S_{3,4}$ and $T_{3,4}$ connect the left hand side with terms of relative weight -2 and lower, down to -14. In addition we have other relations from the equations $(L_6 - H_6)\sigma = 0$, $(L_7 - H_7)\sigma = 0$, and $(L_{10} - H_{10})\sigma = 0$ that

$$R_6: b(\ell, m, n_2, n_5, n_8, n_6, n_9, n_{12}) = \text{“lower weight terms”},$$
$$R_7: b(\ell, m, n_2, n_5, n_8, n_6, n_9, n_{12}) = \text{“lower weight terms”},$$
$$R_{10}: b(\ell, m, n_2, n_5, n_8, n_6, n_9, n_{12}) = \text{“lower weight terms”},$$

respectively. Here the right hand sides are linear in the coefficients b with coefficients at most quadratic in k, ℓ, m, n_2, n_5, n_8, n_6, n_9, n_{12} over the rationals but each denominator is a divisor of 24.

R_6, R_7, R_{10} have a total of 37, 47, 42 terms respectively and connect the left hand side with terms of relative weight -5, -5, -8 and lower, down to -16, -17, -20 respectively.

Ideally we would like to proceed as follows. Suppose we have already calculated the b coefficients at weight $W-2$. Then we would like to use one of the above to calculate each coefficient at weight W. We could proceed in this manner to calculate coefficients at successive weight levels to the required number of terms. However this approach needs some modification. Recall that the weight does not depend on ℓ or m. Clearly if $\ell > 1$ we can use R_6, and if $\ell = 1$, $m > 0$, we can use R_7. Similarly if $\ell = 1$, $k > 0$, we can use R_6. A short calculation shows that if $\ell = 1$, one of these two possibilities holds except in the special case $\ell = 1$, $m = n_2 = n_5 = n_8 = n_6 = n_9 = n_{12} = 0$ which is covered later. For the case $\ell = 0$ we cannot use R_6, R_7, R_{12}. If $m \neq 0$ and $m \neq (k+1)$ we can use $S_{3,4}$. All the possibilities considered so far will reduce the weight by 2. There remain the cases $m = 0$ and $m = (k+1)$ to deal with.

The case $\ell = 0$, $m = 0$ is handled as follows. Take $48(k-m)\,T_{3,4}$:

$$\begin{aligned}
&48(m-k)\, b(\ell, m, n_2, n_5, n_6, n_8, n_9, n_{12}) \\
&\quad = 8\, b(\ell-1, m, n_2-1, n_5, n_6, n_8, n_9, n_{12}) \\
&\qquad + 12k\, b(\ell-1, m+2, n_2, n_5, n_6, n_8, n_9, n_{12}) + \cdots .
\end{aligned}$$

Shifting by $n_2 \to n_2 + 1$, the first term in the right hand side is expressed as

$$\begin{aligned}
&b(\ell-1, m, n_2, n_5, n_6, n_8, n_9, n_{12}) \\
&\quad = -6(m-k-2)\, b(\ell, m, n_2+1, n_5, n_6, n_8, n_9, n_{12}) \\
&\qquad - \tfrac{3}{2}(k+2)\, b(\ell-1, m+2, n_2, n_5, n_6, n_8, n_9, n_{12}) + \cdots .
\end{aligned}$$

On the right hand side we now have two terms of non-negative relative weight as above of relative weight $+2$ which comes from the underlined term in R_3 and of relative weight 0 which comes from the underlined term in R_4. Putting $\ell = 1$, $m = 0$, we have, say $T^{(0)}_{3,4}$, that

$$\begin{aligned} & b(0,0,n_2,n_5,n_6,n_8,n_9,n_{12}) \\ & \quad = -6\,(k-3+5\,\ell+2\,m)\,b(1,0,n_2+1,n_5,n_6,n_8,n_9,n_{12}) \\ & \qquad - \tfrac{3}{2}\,(k-3+5\,\ell+2\,m)\,b(0,2,n_2,n_5,n_6,n_8,n_9,n_{12}) + \cdots . \end{aligned}$$

The first term in the right hand side has $\ell = 1$, $m = 0$, and $k = 5+2n_2+\cdots > 0$. Hence we can apply R_7 to this term to give a term with maximum relative weight $+2-5 = -3$. The second term has $\ell = 0$, $m = 2$, and $k = 1+2n_2+5n_5+\cdots$ so $k+1 > 2$ and hence $k+1 \neq m$. For this term we can apply $S_{3,4}$ to produce a term of maximum relative weight $0-2 = -2$. Hence both terms of weight ≥ 0 can be expressed as terms of relative weight ≤ -2, so our chain eventually decreases in weight. The case $\ell = 0$, $m = (k+1)$ is treated as follows. Take R_3, shift by $m \to m-1$, and set $\ell = 0$ to get

$$\begin{aligned} R^{(0)}_3 :\ & b(0,m,n_2,n_5,n_6,n_8,n_9,n_{12}) \\ & = -2(m-1)\,b(1,m-2,n_2,n_5,n_6,n_8,n_9,n_{12}) + \text{“lower weight terms”}. \end{aligned}$$

Now the first term on the right, $b(1,m-2,n_2,n_5,n_6,n_8,n_9,n_{12})$, is of the same weight as the term on the left. Write this as

$$b(1,m',n_2,n_5,n_6,n_8,n_9,n_{12}),$$

with corresponding k-value k'. If $\min(k',m') < 0$ then this term is zero as discussed above. If $k' = 0$, $m' = 0$, it is easy to show that $m' = n_2 = n_5 = n_8 = n_6 = n_9 = n_{12} = 0$, and this term $b(1,0,0,0,0,0,0,0)$ cannot be reduced further. Otherwise one or both of $k' = 0$, m' is positive, so we can apply R_7 or R_6 to reduce the term to terms of relative weight ≤ -2, so our chain terminates or decreases in weight. These choices, plus the requirement discussed above that $b(\ell,m,n_2,n_5,n_6,n_8,n_9,n_{12}) = 0$ if any of the $\{k,\ell,m,n_2,n_5,n_6,n_8,n_9,n_{12}\}$ are negative, define all the $b(\ell,m,n_2,n_5,n_6,n_8,n_9,n_{12})$ in terms of the so-far undefined $b(1,0,0,0,0,0,0,0)$. Therefore the solution of the system

$$(L_j - H_j)\sigma(u) = 0 \quad (j = 0,\ 3,\ 4,\ 6,\ 7,\ 10)$$

is of dimension one. Choosing $b(1,0,0,0,0,0,0,0) = 1$, we summarise with $k =$

$5-5\ell-2m+2n_2+5n_5+8n_8+6n_6+9n_9+12n_{12}$ as defined as above as follows:

$$b(\ell, m, n_2, n_5, n_8, n_6, n_9, n_{12}) =$$

$$= \begin{cases} 0 & \text{if } \min(k,\ell,m,n_2,n_5,n_6,n_8,n_9,n_{12})<0, \\ 1 & \text{if } \ell=1,\ m=n_2=n_5=n_8=n_6=n_9=n_{12}=0, \\ \mathrm{rhs}(R_{10}) & \text{if } \ell>1, \\ \mathrm{rhs}(R_7) & \text{if } \ell>0,\ m>0, \\ \mathrm{rhs}(R_6) & \text{if } \ell>0,\ k>0, \\ \mathrm{rhs}(S_{3,4}) & \text{if } \ell=0,\ m\neq 0 \text{ and } m\neq(k+1), \\ \mathrm{rhs}(T^{(0)}_{3,4}) & \text{if } l=m=0, \\ \mathrm{rhs}(R^{(0)}_3) & \text{if } \ell=0 \text{ and } m=(k+1). \end{cases}$$

We have used this to calculate the terms in the sigma series up to weight 40 in $\{u_j\}$, or equivalently, weight 35 in the $\{\mu_i\}$. The first few terms of the sigma expansion are given as follows (up to a constant multiple):

$$\begin{aligned}\sigma(u_5,u_2,u_1) &= u_5+\frac{6\,u_1^5}{5!}-2\frac{u_1{u_2}^2}{2!}-2\mu_2\frac{u_1^3{u_2}^2}{2!\,3!}+30\mu_2\frac{{u_1}^7}{7!}-2{\mu_2}^2\frac{u_1{u_2}^4}{4!}\\ &-2{\mu_2}^2\frac{{u_1}^5{u_2}^2}{2!\,5!}+126\,{\mu_2}^2\frac{{u_1}^9}{9!}+24\mu_5\frac{{u_1}^8u_2}{8!}-\mu_5\frac{u_5u_2{u_1}^3}{3!}+8\mu_5\frac{{u_2}^5}{5!}\\ &-2\mu_6\frac{2\,u_5{u_2}^2{u_1}^2}{2!\,2!}+6\mu_6\frac{u_5{u_1}^6}{6!}+24\mu_6\frac{{u_1}^7{u_2}^2}{2!\,7!}-2{\mu_3}^3\frac{{u_1}^7{u_2}^2}{2!\,7!}\\ &-2{\mu_2}^3\frac{{u_1}^3{u_2}^4}{4!\,3!}+432\mu_6\frac{{u_1}^{11}}{11!}+510{\mu_2}^3\frac{{u_1}^{11}}{11!}\\ &-\mu_2\mu_5{u_1}^5\frac{u_5u_2}{5!}-\mu_2\mu_5\frac{u_5{u_2}^3u_1}{3}+288\mu_2\mu_5\frac{{u_1}^{10}u_2}{10!}+\cdots.\end{aligned}$$

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