The Addition Law Attached to a Stratification of a Hyperelliptic Jacobian Variety

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Abstract. This article shows explicit relation between fractional expressions of Schottky-Klein type for hyperelliptic σ-function and a product of differences of the algebraic coordinates on each stratum of natural stratification in a hyperelliptic Jacobian.

1. Introduction

In this paper we shall consider the addition law on the Jacobian for any hyperelliptic curve. In the theory of Abelian functions, addition laws are definitive for a class of corresponding functions.

To describe our investigation, we start from genus one case, and recall that Weierstrass showed that principal relations in the theory of elliptic functions can be derived from the well-known addition formula

$$\frac{\sigma(v+u)\sigma(v-u)}{\sigma(v)^2\sigma(u)^2} = \wp(v) - \wp(u), \tag{1.1}$$

where $\wp(u) = -(d^2/du^2)\log\sigma(u)$.

If we consider an elliptic curve C_1 defined by $y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$ with unique point ∞ at infinity. Let $(x(u), y(u))$ be the inverse function of

$$(x, y) \mapsto u = \int_\infty^{(x,y)} \frac{dx}{2y} \quad \text{modulo the periods.}$$

Then $x(u)$ equals to the function $\wp(u)$ attached to C_1 up to an additive constant. Hence we have

$$\frac{\sigma(v+u)\sigma(v-u)}{\sigma(v)^2\sigma(u)^2} = x(v) - x(u). \tag{1.2}$$

We recall two or three kinds of generalizations of (1.1) or (1.2). While for the sake of simplicity we restrict here materials only about genus two curve C_2 defined by $y^2 = x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x^1 + \lambda_0$, in many cases such generalizations are proved for any hyperelliptic curve. Before describing our generalizations, we

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note that there is a nice generalization of the Weierstrass elliptic σ-function (see 3.1 below for the definition), which is a theta function on $\mathbb{C}^2$ (now we are assuming the genus is two). Indeed if we define $\wp_{ij}(u) = -(\partial^2/\partial u_i \partial u_j)\log\sigma(u)$, then we have a classical formula[1]

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp_{11}(v) - \wp_{11}(u) - \wp_{12}(u)\wp_{22}(v) + \wp_{12}(v)\wp_{22}(u), \tag{1.3}$$

which is a natural generalization of (1.1) based on the idea of Jacobi. This formula is expressed in terms of the algebraic coordinates as

$$\begin{aligned}\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} &= \frac{f(x_1,x_2)-2y_1y_2}{(x_1-x_2)^2} - \frac{f(z_1,z_2)-2w_1w_2}{(w_1-w_2)^2} \\ &\quad + x_1x_2(z_1+z_2) - z_1z_2(x_1+x_2),\end{aligned} \tag{1.4}$$

where $u = \big(\int_\infty^{(x_1,y_1)} + \int_\infty^{(x_2,y_2)}\big)(\frac{dx}{2y}, \frac{xdx}{2y})$, $v = \big(\int_\infty^{(z_1,w_1)} + \int_\infty^{(z_2,w_2)}\big)(\frac{dx}{2y}, \frac{xdx}{2y})$, and $f(x,z)$ is a rather complicated polynomial of x and z such that $f(x,z) = f(z,x)$ (see [**B1**, p.211] for this).

On the other hand, the following is another generalization of (1.2) given by the third author. Let u and v vary on the canonical universal Abelian covering of the curve C_2 presented in $\mathbb{C}^2$ (this is no other than $\kappa^{-1}(\Theta^{[1]})$ by notation in Section 2 below). and $x(u)$ be a function such that $u = \int_\infty^{(x(u),y(u))}(\frac{dx}{2y}, \frac{xdx}{2y})$. Then we have

$$-\frac{\sigma(v+u)\sigma(v-u)}{\sigma_2(v)^2\sigma_2(u)^2} = x(v) - x(u), \tag{1.5}$$

where $\sigma_2(u) = \partial\sigma(u)/\partial u_2$

Now, it is natural to want a unified understanding of whole the formulae (1.3), (1.4), and (1.5) above. There are several hints. The first hint would be the fact that the right hand side of (1.3) has a determinant expression of [**EEP**], which is given by using the genus two case of results in [**Ô**]. The second hint is the following formula:

$$\begin{aligned}&-\frac{\sigma(u^{(1)}+u^{(2)}+v)\sigma(u^{(1)}+u^{(2)}-v)}{\sigma(u^{(1)}+u^{(2)})^2\sigma_2(v)^2} \\ &\qquad = x(v)^2 - \wp_{22}(u^{(1)}+u^{(2)})x(v) - \wp_{12}(u^{(1)}+u^{(2)})\end{aligned} \tag{1.6}$$

for $u^{(1)}$, $u^{(2)}$, and v varying on the canonical universal Abelian covering of C_2 in $\mathbb{C}^2$. This appeared in [**G**] at the first time, and a generalization of this for any hyperellitpic curve was reported in [**BES**, (3.21)], without proof.

The main result (Theorem 4.2 below) of this paper seems to be a unification of (1.5) and (1.6). There should exist a unification of all the formulae above, and such a formulation might be given in near future.

Notations. The symbol $\big(d^\circ(z) \geq n\big)$ denotes terms of total degree at least n with respect to a variable z.

[1] Generalization of this formula to the case $g = 3$ was given by Baker [**B2**]. Recently in [**BEL**] the addition law for all genera $g > 1$ of hyperelliptic curve was written in terms of pffafian built in Kleinian $\wp$-functions.

2. Staratification of the Jacobian of a Hyperelliptic Curve

Throughout this article we deal with a hyperelliptic (or elliptic) curve C_g of genus $g > 0$ given by the affine equation

$$y^2 = f(x),$$

where we are assuming that $f(x)$ is of the form

$$f(x) = x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_2 x^2 + \lambda_1 x + \lambda_0$$

with λ_j's being complex numbers. Then a canonical basis of the space of the differentials of the first kind on C_g is given by

$$\omega_1 := \frac{dx}{2y}, \quad \omega_2 := \frac{xdx}{2y}, \quad \cdots, \quad \omega_g := \frac{x^{g-1}dx}{2y}$$

with the algebraic coordinate (x, y) of C_g, Let α_j and β_j $(j = 1, \cdots, g)$ be a standard homology basis on C_g. Namely, they give

$$\mathrm{H}_1(C_g, \mathbb{Z}) = \bigoplus_{j=1}^{g} \mathbb{Z}\alpha_j \oplus \bigoplus_{j=1}^{g} \mathbb{Z}\beta_j,$$

and their intersection products are given by $[\alpha_i, \alpha_j] = 0$, $[\beta_i, \beta_j] = 0$, $[\alpha_i, \beta_j] = -[\beta_i, \alpha_j]\delta_{i,j}$. We denote matrices of the half-periods with respect to the differentials ω_i and the homology basis α_j, β_j by

$$\omega' := \frac{1}{2}\left[\int_{\alpha_j} \omega_i\right], \quad \omega'' := \frac{1}{2}\left[\int_{\beta_j} \omega_i\right].$$

We introduce differentials of the second kind

$$dr_j := \frac{1}{2y}\sum_{k=j}^{2g-j}(k+1-j)\lambda_{k+1+j}x^k dx, \quad (j = 1, \ldots, g)$$

and the half-periods matrix

$$\eta' := \frac{1}{2}\left[\int_{\alpha_j} dr_i\right], \quad \eta'' := \frac{1}{2}\left[\int_{\beta_j} dr_i\right]$$

of this differentials with respect to α_j and β_j. These $2g$ meromorphic differentials u_i and r_i $(i = 1, \cdots, g)$ are chosen in such the way that the half-periods matrices ω', ω'', η', η'' satisfy the generalized Legendre relation

$$\mathfrak{M}\begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix} {}^t\mathfrak{M} = \frac{\sqrt{-1}\pi}{2}\begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix},$$

where $\mathfrak{M} = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}$. Let $\Lambda = 2\big(\mathbb{Z}^g\omega' \oplus \mathbb{Z}^g\omega''\big)$. Then Λ is a lattice in $\mathbb{C}^g$ and the Jacobi variety $\mathcal{J}(C_g)$ of C_g is given by

$$\mathcal{J}(C_g) := \mathbb{C}^g/\Lambda.$$

We use the modulus $\mathbb{T} := {\omega'}^{-1}\omega''$ to define the σ-function of C_g later.

For $k = 0, \cdots, g$, the *Abel map* ϕ_k from the k-th symmetric product $\mathrm{Sym}^k(C_g)$ of the curve C_g to $\mathcal{J}(C_g)$ is the map

$$\phi_k : \mathrm{Sym}^k(C_g) \to \mathcal{J}(C_g) \text{ given by } \ (Q_1, \ldots, Q_k) \mapsto \sum_{i=1}^{k} \int_{\infty}^{Q_i} (\omega_1, \cdots, \omega_g) \mod \Lambda.$$

We denote the modulo Λ map by κ:

$$\kappa : \mathbf{C}^g \to \mathcal{J}(C_g) = \mathbb{C}^g/\Lambda.$$

We denote $\Theta^{[k]}$ the image $\phi_k(\mathrm{Sym}^k(C_g))$ of the Abel map ϕ_k above. Now we have the following stratification:

$$\{O\} = \Theta^{[0]} \subset \Theta^{[1]} \subset \Theta^{[2]} \subset \cdots \subset \Theta^{[g-1]} \subset \Theta^{[g]} = \mathcal{J}(C_g),$$

where O is the origin of $\mathcal{J}(C_g)$. It is known that each $\Theta^{[k]}$ is a subvariety of $\mathcal{J}(C_g)$. We shall refer each subvariety $\Theta^{[k]}$ by k-th stratum of $\mathcal{J}(C_g)$.

The following Lemma is shown by a straightforward calculation of the Abelian integral

$$u_i = \int_{\infty}^{(x,y)} \frac{x^{i-1}dx}{2y}$$

by using power series expansion and integral by its terms.

Lemma 2.1. *Let $u = (u_1, \cdots, u_g) \in \kappa^{-1}(\Theta^{[1]})$. We denote by $\big(x(u), y(u)\big) \in C_g$ the algebraic coordinate such that whose image by the Abel map ϕ_k is u modulo Λ. Then we have the follwoing properties.*
(1) *The variable u_g is a local parameter around $(0, \cdots, 0)$ on $\kappa^{-1}(\Theta^{[1]})$, and so that $u_1, \cdots, u_g$ are functions of u_g on $\kappa^{-1}(\Theta^{[1]})$ near $(0, \cdots, 0)$.*
(2) *The function $x(u)$ has the following Laurent expansions around $(0, 0, \cdots, 0)$ on $\kappa^{-1}(\Theta^{[1]})$:*

$$\begin{aligned} x(u) &= {u_g}^{-2} + \big(d^\circ(u_g) \geq 0\big), \\ y(u) &= {u_g}^{-2g-1} + \big(d^\circ(u_g) \geq -2g+1\big). \end{aligned}$$

For a proof of the above, we refer the reader to Lemmas 3.8 and 3.9 in [**Ô**], for instance. $\square$

3. The Sigma Function and Its Derivatives

In this section, we will introduce the hyperelliptic θ-function and σ-function. The later one is a natural generalization of the Weierstrass σ-function.

Let a and b be two vectors in $\mathbb{R}^g$. We recall the theta function with respect to the lattice of periods generated by 1_g and $\mathbb{T} = {\omega'}^{-1}\omega''$ with characteristic ${}^t[a\ b]$, which is a function of $z \in \mathbb{C}^g$ defined by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z) = \theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\mathbb{T}) = \sum_{n\in\mathbb{Z}^g} \exp\left[2\pi\sqrt{-1}\left\{\frac{1}{2}\,{}^t(n+a)\mathbb{T}(n+a) + {}^t(n+a)(z+b)\right\}\right],$$

as usual. Let $\delta' = {}^t\left[\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right]$ and $\delta'' = {}^t\left[\frac{g}{2}, \frac{g-1}{2}, \cdots, \frac{1}{2}\right]$. Then the half-period vector $\delta'\omega' + \delta''\omega''$ is no other than so-called Riemann constant for $\mathcal{J}(C_g)$.

Definition 3.1. The σ-function (see, for example, [**B1**, p.336]) is given by

$$\sigma(u) = \gamma_0 \exp\left(-\frac{1}{2}\,{}^t u\eta'{\omega'}^{-1}u\right)\vartheta\begin{bmatrix} \delta'' \\ \delta' \end{bmatrix}\left(\frac{1}{2}{\omega'}^{-1}u;\mathbb{T}\right),$$

where γ_0 is a certain non-zero constant depending of C_g, which is explained in [**BEL** p.32] and [**Ô**, Lemma 4.2]. We regards the domain $\mathbb{C}^g$ where this function is defined to be $\kappa^{-1}(\mathcal{J}(C_g))$.

Following the paper [**Ô**], we introduce multi-indices $\natural^n$ and their associated derivatives $\sigma_{\natural^n}(u)$ of $\sigma(u)$ as follows:

Definition 3.2. We define that

$$\natural^n = \begin{cases} \{n+1, n+3, \cdots, g-1\} & \text{if } g-n \equiv 0 \mod 2, \\ \{n+1, n+3, \cdots, g\} & \text{if } g-n \equiv 1 \mod 2. \end{cases}$$

By using this notation we have partial derivatives of $\sigma(u)$ associated these multi-indeces, namely,

$$\sigma_{\natural^n}(u) = \left(\prod_{i\in\natural^n}\frac{\partial}{\partial u_i}\right)\sigma(u).$$

Moreover we denote $\sharp := \natural^1$ and $\flat := \natural^2$, so that $\sigma_\sharp(u) = \sigma_{\natural^1}(u)$ and $\sigma_\flat(u) = \sigma_{\natural^2}(u)$.

Several examples of $\sigma_{\natural^n}(u)$ are given in the following table[2].

genus	$\sigma_\sharp \equiv \sigma_{\natural^1}$	$\sigma_\flat \equiv \sigma_{\natural^2}$	$\sigma_{\natural^3}$	$\sigma_{\natural^4}$	$\sigma_{\natural^5}$	$\sigma_{\natural^6}$	$\sigma_{\natural^7}$	$\sigma_{\natural^8}$	$\cdots$
1	σ	σ	σ	σ	σ	σ	σ	σ	$\cdots$
2	σ_2	σ	σ	σ	σ	σ	σ	σ	$\cdots$
3	σ_2	σ_3	σ	σ	σ	σ	σ	σ	$\cdots$
4	σ_{24}	σ_3	σ_4	σ	σ	σ	σ	σ	$\cdots$
5	σ_{24}	σ_{35}	σ_4	σ_5	σ	σ	σ	σ	$\cdots$
6	σ_{246}	σ_{35}	σ_{46}	σ_5	σ_6	σ	σ	σ	$\cdots$
7	σ_{246}	σ_{357}	σ_{46}	σ_{57}	σ_6	σ_7	σ	σ	$\cdots$
8	σ_{2468}	σ_{357}	σ_{468}	σ_{57}	σ_{68}	σ_7	σ_8	σ	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

[2] One can see that numbers appearing in those multy-index $\natural^n$ are naturally related to the Weierstrass gap sequence 1, 3, 5, $\cdots$, $2g-1$ at the Weierstrass point ∞ at infinity.

For $u \in \kappa^{-1}\mathcal{J}(C_g)$, we denote u' and u'' the unique elements in $\mathbb{R}^g$ such that $u = 2(u'\omega' + u''\omega'')$. We introduce a $\mathbb{C}$-valued $\mathbb{R}$-bilinear form $L(\ ,\ )$ defined by

$$L(u, v) = {}^t u(\eta' u' + \eta'' u'')$$

for u, $v \in \kappa^{-1}\mathcal{J}(C_g)(= \mathbb{C}^g)$. Let

$$\chi(\ell) = \exp\left[2\pi\sqrt{-1}({}^t\ell'\delta'' - {}^t\ell''\delta') - \pi\sqrt{-1}\,{}^t\ell'\ell''\right].$$

The following facts are essential for our main result.

Proposition 3.3. (1) *For $u \in \kappa^{-1}(\Theta^{[n]})$ and $\ell \in \Lambda$, we have*

$$\sigma_{\natural^n}(u + \ell) = \chi(\ell)\sigma_{\natural^n}(u) \exp L(u + \tfrac{1}{2}\ell, \ell).$$

(2) *Let n be a positive integer $n \le g$. Let v, $u^{(1)}$, $u^{(2)}$, $\cdots$, $u^{(n)}$ be elements in $\kappa^{-1}(\Theta^{[1]})$. If $u^{(1)} + \cdots + u^{(n)} \notin \kappa^{-1}(\Theta^{[n-1]})$, then the function $v \mapsto \sigma_{\natural^{n+1}}(u^{(1)} + \cdots + u^{(n)} + v)$ has only zeroes at $v = (0, \cdots, 0)$ modulo Λ of order $g - n$ and at $-u^{(1)}$ modulo Λ of order 1. Around $(0, 0, \cdots, 0)$ we have the following expansion with respect to v_g :*

$$\sigma_{\natural^{n+1}}(u^{(1)} + \cdots + u^{(n)} + v) = (-1)^{(g-n)(g-n-1)/2}\sigma_{\natural^n}(u){v_g}^{g-n} + (d^\circ(v_g) \ge g - n + 1).$$

(3) *For $v \in \kappa^{-1}(\Theta^{[1]})$,*

$$\sigma_{\natural^1}(v) = -(-1)^{g(g-1)/2}{v_g}^g + (d^\circ(v_g) \ge g + 2).$$

Proof. The assertion (1) is proved by Lemma 7.3 in [**Ô**]. The assertions (2) and (3) are proved by Proposition 7.5 in [**Ô**]. $\square$

Remark 3.4. (1) If $n = g$, the assertion 3.3(1) is no other than the classical relation of $\sigma(u)$ with respect to the translation by any period $\ell \in \Lambda$ (see [**B1**, p.286]). Namely, we have the same relations for translation by any period $\ell \in \Lambda$ for the special partial-derivatives $\sigma_{\natural^n}(u)$ on $\kappa^{-1}(\Theta^{[n]})$ as that of σ itself. The essence of the proof in [**Ô**] of this fact is that the derivative $\sigma_{\breve{\natural}^n}(u)$ for any proper subset $\breve{\natural}^n$ of $\natural^n$ vanishes on the strutum $\Theta^{[n]}$, which is proved by investigating Riemann singularity theorem explicitly.
(2) We see by considering the unification of 3.3(2) and 3.3(3) above that it would be natural to define $\sigma_{\natural^0}(u) = -1$ (a constant function).
(3) The statements 3.3(2) and (3) complement the Riemann singularity theorem ([**R**]) (see also [**ACGH**, p.226-227]) with pointing the orders of vanishing on each stratum $\Theta^{[n]}$ in terms of σ function.

4. Main Result

In this Section we start to recall the follwoing formula (Lemma 8.1 in [**Ô**]) without proof.

Lemma 4.1. *Suppose u and v be variables on $\kappa^{-1}(\Theta^{[1]})$. Then we have*

$$(-1)^g \frac{\sigma_\flat(u+v)\sigma_\flat(u-v)}{\sigma_\sharp(u)^2\sigma_\sharp(v)^2} = x(u) - x(v).$$

The following relation is our main theorem and is an extension of both of (3.21) in [**BES**] and the formula above.

Theorem 4.2. *Let m and n be positive integers such that $m+n \le g+1$. Let*

$$u = \sum_{i=1}^{m} \int_{\infty}^{(x_i,y_i)} (\omega_1, \cdots, \omega_g) \ \in \kappa^{-1}(\Theta^{[m]}), \quad v = \sum_{j=1}^{n} \int_{\infty}^{(x'_j,y'_j)} (\omega_1, \cdots, \omega_g) \ \in \kappa^{-1}(\Theta^{[n]})$$

Then the following relation holds :

$$\frac{\sigma_{\natural^{m+n}}(u+v)\sigma_{\natural^{m+n}}(u-v)}{\sigma_{\natural^m}(u)^2\sigma_{\natural^n}(v)^2} = (-1)^{\delta(g,n)} \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i - x'_j),$$

where $\delta(g,n) = \frac{1}{2}n(n-1) + gn$.

Proof. We prove the desired formula by induction with respect to m and n. First of all we suppose that $2g$ points $u^{(1)}, \cdots, u^{(g)}$ and $v^{(1)}, \cdots, v^{(g)}$ are given. Then by 3.3 we see that both sides of the desired formula are function on $\Theta^{[1]}$ with respect to each variable in $u^{(i)}$s and $v^{(j)}$s. We let $u^{[i]} = u^{(1)} + \cdots + u^{(i)}$ and $v^{[j]} = v^{(1)} + \cdots + v^{(j)}$ for $0 \le i \le g$ and $0 \le j \le g$. If $m = n = 1$, the assertion is just the Lemma 4.1. Therefore, the assertion is proved by reducing

$$\frac{\sigma_{\natural^{m+n+1}}(u^{[m]} + v^{[n+1]})\sigma_{\natural^{m+n+1}}(u^{[m]} - v^{[n+1]})}{\sigma_{\natural^m}(u^{[m]})^2\sigma_{\natural^{n+1}}(u^{[n+1]})^2} = (-1)^{\delta(g,n+1)} \prod_{i=1}^{m}\prod_{j=1}^{n+1}(x_i - x'_j) \tag{4.3a}$$

and

$$\frac{\sigma_{\natural^{m+n+1}}(u^{[m+1]} + v^{[n]})\sigma_{\natural^{m+n+1}}(u^{[m+1]} - v^{[n]})}{\sigma_{\natural^{m+1}}(u^{[m+1]})^2\sigma_{\natural^n}(u^{[n]})^2} = (-1)^{\delta(g,n)} \prod_{i=1}^{m+1}\prod_{j=1}^{n}(x_i - x'_j), \tag{4.3b}$$

to the formula

$$\frac{\sigma_{\natural^{m+n}}(u^{[m]} + v^{[n]})\sigma_{\natural^{m+n}}(u^{[m]} - v^{[n]})}{\sigma_{\natural^m}(u^{[m]})^2\sigma_{\natural^n}(v^{[n]})^2} = (-1)^{\delta(g,n)} \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i - x'_j). \tag{4.4}$$

We denote the index of sign in 3.3(2) by $\epsilon(g,n)$, that is $\epsilon(g,n)=(g-n)(g-n-1)/2$. Then the left hand side of (4.3a) is

$$\begin{aligned}
&\frac{\sigma_{\natural^{m+n+1}}(u^{[m]}+v^{[n+1]})\sigma_{\natural^{m+n+1}}(u^{[m]}-v^{[n+1]})}{\sigma_{\natural^m}(u^{[m]})^2\sigma_{\natural^{n+1}}(v^{[n+1]})^2}\\
&=\Big[\sigma_{\natural^{m+n}}(u^{[m]}+v^{[n]})\big\{(-1)^{\epsilon(g,m+n)}\big(v_g^{(n+1)}\big)^{g-m-n}+\cdots\big\}\\
&\qquad\times\sigma_{\natural^{m+n}}(u^{[m]}-v^{[n]})\big\{(-1)^{\epsilon(g,m+n)}\big(-v_g^{(n+1)}\big)^{g-m-n}+\cdots\big\}\Big]\\
&\qquad\Big/\sigma_{\natural^m}(u^{[m]})^2\sigma_{\natural^n}(v^{[n]})^2\big\{(-1)^{\epsilon(g,n)}\big(v_g^{(n+1)}\big)^{g-n}+\cdots\big\}^2\\
&=\frac{\sigma_{\natural^{m+n}}(u^{[m]}+v^{[n]})\sigma_{\natural^{m+n}}(u^{[m]}-v^{[n]})}{\sigma_{\natural^m}(u^{[m]})^2\sigma_{\natural^n}(u^{[n]})^2}\left\{(-1)^{g-m-n}\frac{1}{\big(v_g^{(n+1)}\big)^{2m}}+\cdots\right\}
\end{aligned}\tag{4.3a$'$}$$

by using the Lemma 2.1. The right hand side of (4.3a) is

$$\begin{aligned}
&(-1)^{\delta(g,n+1)}\prod_{i=1}^{m}\prod_{j=1}^{n+1}(x_i-x_j')\\
&=(-1)^{\delta(g,n+1)}(-x_{n+1}')^m\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i-x_j')+\big(d^\circ(x_{n+1})\le m-1\big)\\
&=\frac{(-1)^{\delta(g,n+1)+m}}{\big(v^{(n+1)}\big)^{2m}}\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i-x_j')+\big(d^\circ(v^{(n+1)})\ge -2m+1\big)
\end{aligned}\tag{4.3a$''$}$$

The index of sign of the last in (4.3a$''$) is

$$\begin{aligned}
\delta(g,n+1)+m&=\tfrac{1}{2}(n+1)n+g(n+1)-1+m\\
&=\tfrac{1}{2}n(n-1)+n+gn+g-1+m\\
&\equiv\tfrac{1}{2}n(n-1)+gn-1+(g-n-m)\mod 2\\
&=\delta(g,n)+(g-n-m).
\end{aligned}$$

This is equal to the sum of indeces of sign in (4.4) and one in the last factor in (4.3a$'$). Thus, the leading terms of expansions with respect to $v_g^{(n+1)}$ of the two sides completely coincide. Till here, the assumption $m+n\le g+1$ is not so essential. Now we are going to check the divisors of the two sides regarding as functions of $v^{(n+1)}$ modulo Λ. Using the assumption $m+n\le g+1$, we can determine the divisors of two sides exactly by Proposition 3.4. For the left hand side of (4.3a), the numerator has zeroes at $v^{(n+1)}=(0,0,\cdots,0)$ of order $2(g-m-n)$ modulo Λ, at $\pm u^{(1)}$, $\cdots$, $\pm u^{(m)}$, $-v^{(1)}$, $\cdots$, $-v^{(n)}$ of order 1 modulo Λ. The denominator has zeroes at $v^{(n+1)}=(0,0,\cdots,0)$ of order $2(g-n)$ modulo Λ, at $-v^{(1)}$, $\cdots$, $-v^{(n)}$ of order 1 modulo Λ. Therefore the left hand side of (4.3a) has only pole at $v^{(n+1)}=(0,0,\cdots,0)$ of order $2m$ modulo Λ, and has zeroes at $\pm u^{(1)}$, $\cdots$, $\pm u^{(m)}$ of order 1 modulo Λ. These pole and zeroes coincide with those of the left hand side including their order, because, for u and $v\in\kappa^{-1}(\Theta^{[1]})$, we have $x(u)-x(v)$

if and only if $u = \pm v$. Thus, we have reduced the formula (4.3a) to the equality (4.4). The formula (4.3b) is similarly reduced to (4.4). Hence, we have proved the assertion. $\square$

Baker ([**B1**] and [**B2**]) defined

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u)$$

for $0 \leq i \leq g$, $0 \leq j \leq g$, and $u = (u_1, \cdots, u_g) \in \mathbb{C}^g$, which is natutal generalizations of Weierstrass $\wp$ function. As we mentioned before 4.2 the following special case of $(m, n) = (g, 1)$ in 4.2 appeared in [**BES**, (3.21)], which was the motivation of this paper.

Corollary 4.5. ([**BES**, (3.21)]) *We suppose that (x, y), (x_1, y_1), $\cdots$, (x_g, y_g) are $g+1$ points on C_g. Let*

$$\begin{aligned} F_g(x) &= (x - x_1)(x - x_2) \cdots (x - x_g) \\ u &= \sum_{i=1}^{g} \int_{\infty}^{(x_i, y_i)} (\omega_1, \cdots, \omega_g) \in \kappa^{-1}(\Theta^{[g]})(= \mathbb{C}^g), \\ v &= \int_{\infty}^{(x,y)} (\omega_1, \cdots, \omega_g) \in \kappa^{-1}(\Theta^{[1]}). \end{aligned}$$

Then we have the relation

$$\begin{aligned} \frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2 \sigma_\sharp(v)^2} &= (-1)^{\frac{1}{2}g(g+1)} F_g(x) \\ &= x^g - \wp_{gg}(u) x^{g-1} - \wp_{g,g-1}(u) x^{g-2} - \cdots - \wp_{g1}(u). \end{aligned}$$

Proof. The first equality is obvious from 4.2. The second equality follow from the fact

$$\wp_{g,g-k+1}(u) = (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq g} x_{i_1} x_{i_2} \cdots x_{i_k}$$

for $1 \leq k \leq g$, which was given by Baker(see [**B3**], for example). $\square$

5. Some Remarks

We finally remark on related works and future of our result. The polynomial F_g plays the role of the master polynomial in the theory of hyperelliptic functions. Bolza enabled to express the polynomial F_g in terms of Kleinian $\wp$-functions; in this context we shall call master polynomial F_g as Bolza polynomial. Its zeroes give solution to the Jacobi inversion problem. In the 2×2 Lax representation of dynamic systems associated with a hyperelliptic curve it plays the role of U polynomial among Jacobi's U, V, W-triple ([**Mu**]). Vanhaecke studied the properties of $\Theta^{[k]}$ using U, V, W-polynomials which are constructed on the basis of the master polynomial F_g ([**V2**]). In [**BES**], the authors applied aforementioned particular case of the addition theorem with Bolza polynomial to compute the norm of wave function to the Schrödinger equation with finite-gap potential.

A similar polynomial $F_k(z) \equiv U(z) := (z-x_1)\cdots(z-x_k)$ over $\Theta^{[k]}$ which plays essential roles in the studies of structures of the subvarieties ([**AF**] and [**Ma**]), namely,

$$\frac{\sigma_{\natural^{k+1}}(u-v)\sigma_{\natural^{k+1}}(u+v)}{\sigma_{\natural^1}(v)^2\sigma_{\natural^k}(u)^2} = (-1)^{g-k}F_k(x).$$

For the case of $k=1$, it appeared in [**Ô**] as Frobenius-Stickelberger type relation of higher genus, which determines an algebraic structure of the curve.

We emphasize that study of θ-divisor attract now more attention. In particular inversion of higher genera hyperelliptic integrals with respect to the restriction to θ-divisor was recently used in the problem of analytic description by means of reduction of the Benney system of conformal map of a domain in half-plane to its complement [**BG1**], [**BG2**]. The same method of inversion of an ultraelliptic integral was used in [**EPR**] to describe motion of double pendulum.

In was shown in series of publications of Vanhaecke [**V1**], [**V2**], Abenda and Fedorov [**AF**] and one from authors [**Ma**] and others that θ-divisor can serve as a carrier of integrability. Grant [**G**], Cantor [**C**] found algebraic structures on $\Theta^{[1]}$ which is related to division polynomials whose zeros determines n-time points. Recently on from the authors and Eilbeck and Previato used Grant-Jorgenson formula ([**G**] and [**J**]) to derive an analogue of the Frobenius-Stickelberger addition formula for three variables in the case of genus two hyperelliptic curve ([**EEP**]).

Thus we expect that our main theorem has some effects on these studies based on the Riemann singularity theorem.

References

[**AF**] S. Abenda, Yu. Fedorov, *On the weak Kowalevski-Painlevé Property for hyperelliptically separable systems*, Acta Appl. Math. **60** (2000), 137-178.

[**ACGH**] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris,, *Geometry of algebraic curves I*, Springer, Berlin, 1985.

[**B1**] H. F. Baker, *Abelian functions – Abel's theorem and the allied theory including the theory of the theta functions –*, Cambridge Univ. Press, 1897; reprinted in 1995..

[**B2**] H. F. Baker, *On the hyperelliptic sigma functions*, Amer. J. Math. **20** (1898), 301-384..

[**B3**] H.F. Baker, *On a system of differential equations leading to periodic functions*, Acta Math. 27 (1903), 135-156.

[**BEL**] V. M. Buchstaber, V. Z. Enolskii, and D. V. Leykin, *Reviews in Mathematics and Mathematical Physics (London)* (Novikov, S. P. and Krichever, I. M., eds.), Gordon and Breach, pp. 1-125..

[**BES**] E. D. Belokolos, V. Z. Enolskii, M. Salerno, *Wannier functions for quasi-periodic potentials*, Teor.Mat.Fiz. (2005).

[**BG1**] S. Baldwin and J. Gibbons, *Hyperelliptic reduction of the Benney moment equations*, J.Phys.A:Math.Gen. **36** (2003), 8393-8413.

[**BG2**] S. Baldwin and J. Gibbons, *Higher genus hyperelliptic reductions of the Benney equations*, J.Phys.A:Math.Gen. **37** (2004), 5341-5354.

[**C**] D. G. Cantor, *On the analogue of the division polynomials for hyperelliptic curves*, J. reine angew. Math. **447** (1994), 91-145.

[**EEP**] J. C. Eilbeck, V. Z. Enolskii, and E. Previato, *On a generalized Frobenius–Stickelberger addition formula*, Lett. Math. Phys **65** (2003), 5-17.

[**EPR**] V. Z. Enolskii, M. Pronine and P. Richter, *Double pendulum and θ-divisor* , J.Nonlin.Sci. **13:2** (2003), 157-174.

[**G**] D. Grant, *A generalization of a formula of Eisenstein*, Proc. London Math. Soc. **62** (1991), 121-132.

[**J**] J. Jorgenson, *On directional derivative of the theta function along its theta-divisor*, Israel J.Math. **77** (1992), 274-284.

[**Ma**] S. Matsutani, *Relations of al functions over subvarieties in a hyperelliptic Jacobian*, Cubo Journal (2005) (to appear).

[**Mu**] D. Mumford, *Tata lectures on theta, vol II*, Birkhäuser, 1984.

[**Ô**] Y. Ônishi, *Determinant expressions for hyperelliptic functions (with an Appendix by Shigeki Matsutani)*, Proc. Edinburgh Math. Soc., (2005), 1-38.

[**R**] G. F. B. Riemman, *Ueber das Verschwinden der Theta-Functionen*, J. reine angew. Math. **65** (1866).

[**PV**] M. Pedroni, P. Vanhaecke, *A Lie algebraic generalization of the Mumford system, its symmetries and its multi-Hamiltonian structure*, Regular and Chaotic dynamics **3** (1998), 132-160.

[**V1**] P. Vanhaecke, *Stratifications of hyperelliptic Jacobians and the Sato Grassmannian*, Acta. Appl. Math. **40** (1995), 143-172.

[**V2**] P. Vanhaecke, *Integrable systems and symmetric products of curves*, Math. Z. **227** (1998), 93-127.

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