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Determinant Expressions in Abelian Functions for Purely Pentagonal Curves of Degree Six

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In this paper, we describe precise conjecture (see 9.1) of Frobenius-Stickelberger-type formulae for purely d-gonal curves with unique point at infinity, namely, for a curve defined by

$$y^d = x^s + \lambda_1 x^{s-1} + \lambda_2 x^{s-2} + \cdots + \lambda_{s-1} x + \lambda_s \quad (\lambda_j\text{s are constants})$$

with $\gcd(d, s) = 1$ and $d < s$. In this paper we mainly proves the case $(d, s) = (5, 6)$. Although our proof is applicable for more general d, it seems for us that we need a slight progress to prove the conjecture for *all* d. However, this case would be a strong evidence supporting the conjecture. The expected general proof might make the proof of this paper shorter. Expecting for the reader to find such the good proof, this paper contains many materials which do not need for our proof of the example. Simply saying, it seems that we need a improved Riemann singularity theorem in order to find one of the best proofs.

Proposition 12.1 is the key of our proof. Our proof is quite different from that for the case of hyperelliptic curves [**Ô3**]. It was impossible to apply the proof in the case of trigonal curves [**Ô4**]. The proof of this paper starts from investigating fundamental properties of certain derivative $\sigma_\flat$ (see 12.1) of generalized sigma function, which properties show directly the 2 variable case of the conjecture (see 12.14). In the next time we prove the conjecture when the number of variables are exactly the genus of the curve (see 13.1). Then, we prove the conjecture by decreasing the number of variables inductively. Finally, the cases whose number of variables are bigger than genus are proved inductively. This final step is very similar with those of [**Ô1**], [**Ô2**], [**Ô3**], [**Ô4**].

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Conventions. We denote by $\mathbf{Z}$ the ring of rational integers, by $\mathbf{Q}$ the field of rational numbers, by $\mathbf{R}$ the field of real numbers, and by $\mathbf{C}$ the field of complex numbers. The transpose of a vector u is denoted by ${}^t u$. The symbol $d^\circ(z_1, \cdots, z_m) \geqq N$ means a power series whose all terms are of total degree bigger than or equal to N with respect to variables $z_1, \cdots, z_m$. This symbol does *not* mean that the variables z_1, $\cdots$, z_m are all the variables contained in this power series.

Contents

1. Schur-Weierstrass polynomial.

In this section, we recall fundamentals on Schur-Weierstrass polynomials. The main reference is [**BEL2**] and [**Mac**]. We assume that (d,s) is coprime pair of two integers such that $s > d \geqq 2$.

Definition 1.1 The decreasing sequence of positive integers that is not expressed of the form $ad + bs$ with positive integers a and b is denoted by $\mathbf{Weier}_{d,s}$, and is called the *Weierstrass sequence* for (d,s). We denote its elements as

$$\mathbf{Weier}_{d,s} = \{w_g, w_{g-1}, \cdots, w_1\}.$$

The number g of this sequence is called its *length*.

Lemma 1.2 (Lemma 1.3 of [**BEL2**]) *We have the followings on* $\mathbf{Weier}_{d,s}$:
(1) *The length g of $\mathbf{Weier}_{d,s}$ is $(d-1)(s-1)/2$;*
(2) *The maximum element w_g in $\mathbf{Weier}_{d,s}$ is $2g-1$;*
(3) *If $w \in \mathbf{Weier}_{d,s}$, then $2g-1-w \notin \mathbf{Weier}_{d,s}$;*
(4) *If $w > \tilde{w}$ and $w \in \mathbf{Weier}_{d,s}$, then $w - \tilde{w} \in \mathbf{Weier}_{d,s}$;*
(5) *For $i = 1, \cdots, g$, we have $i \leqq w_i \leqq 2i-1$; especially, $w_1 = 1$.*

We prepare g variables

$$(u^{(1)}_{\langle w_1 \rangle}, \cdots, u^{(g)}_{\langle w_1 \rangle}) = (u^{(1)}_{\langle 1 \rangle}, \cdots, u^{(g)}_{\langle 1 \rangle}). \tag{1.3}$$

For $1 \leqq k \leqq 2g-1$, let

$$u^{(i)}_{\langle k \rangle} := \tfrac{1}{k}(u^{(i)}_{\langle 1 \rangle})^k, \qquad u_{\langle k \rangle} := u^{(1)}_{\langle k \rangle} + u^{(2)}_{\langle k \rangle} + \cdots + u^{(g)}_{\langle k \rangle}, \tag{1.4}$$

and

$$\begin{aligned} u^{(i)} &:= (u^{(i)}_{\langle w_g \rangle}, u^{(i)}_{\langle w_{g-1} \rangle}, \cdots, u^{(i)}_{\langle w_1 \rangle}), \\ u &:= u^{(1)} + u^{(2)} + \cdots + u^{(g)} \\ &= (u_{\langle w_g \rangle}, u_{\langle w_{g-1} \rangle}, \cdots, u_{\langle w_1 \rangle}). \end{aligned} \tag{1.5}$$

For each $k \geqq 0$, we denote by

$$(-1)^k U_k(u_{\langle w_1 \rangle}, \cdots, u_{\langle w_g \rangle}) \tag{1.6}$$

the k-th *complete symmetric polynomial* of degree k with respect to $u^{(1)}_{\langle 1 \rangle}, \cdots, u^{(n)}_{\langle 1 \rangle}$. If there is no afraid of confusing, we simply denote as

$$U_k = U_k(u_{\langle w_1 \rangle}, \cdots, u_{\langle w_g \rangle}). \tag{1.7}$$

If $u^{(j+1)}_{\langle 1 \rangle} = \cdots = u^{(g)}_{\langle 1 \rangle} = 0$, we denote the $U_k(u_{\langle w_1 \rangle}, \cdots, u_{\langle w_g \rangle})$ as

$$U^{[j]}_k = U^{[j]}_k(u_{\langle w_1 \rangle}, \cdots, u_{\langle w_g \rangle}) := U_k(u^{(1)} + u^{(2)} + \cdots + u^{(j)}). \tag{1.8}$$

Definition 1.9 The *Schur-Weierstrass polynomial* of (d, s) is defined by

$$\begin{aligned} S_{d,s} &= S_{d,s}(u_{\langle 1\rangle} u_{\langle 2\rangle}, \cdots, u_{\langle 2g-2\rangle}, u_{\langle 2g-1\rangle}) \\ &:= \left| U_{w_i - g + j} \right|_{1 \leqq i \leqq g, 1 \leqq j \leqq g}. \end{aligned}$$

To show that the determinant in 1.9 is no other than $S_{d,s}$ in [**BEL2**], p.86, we let

$$p_j := j\, u_{\langle j\rangle}. \tag{1.10}$$

We introduce new variables $s_{\langle w_1\rangle}$, $s_{\langle w_2\rangle}$, $\cdots$, $s_{\langle w_g\rangle}$ which satisfy

$$p_j = -{s_{\langle w_1\rangle}}^j - {s_{\langle w_2\rangle}}^j - \cdots - {s_{\langle w_g\rangle}}^j, \quad (1 \leqq j \leqq 2g-1). \tag{1.11}$$

We check that there exist such variables as follows: Let $\varepsilon_k(\boldsymbol{s})$ with $\boldsymbol{s} = (s_{\langle w_1\rangle}, s_{\langle w_2\rangle}, \cdots, s_{\langle w_g\rangle})$ be the fundamental symmetric polynomial of degree k with respect to s_js. Then, as is well-known, that $s_{\langle w_1\rangle}$, $s_{\langle w_2\rangle}$, $\cdots$, $s_{\langle w_g\rangle}$ satisfy

$$\varepsilon_k(\boldsymbol{s}) = \frac{1}{k!} \begin{vmatrix} -p_1 & 1 & & & \\ -p_2 & -p_1 & 2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ -p_{k-1} & -p_{k-2} & -p_{k-3} & \cdots & k-1 \\ -p_k & -p_{k-1} & -p_{k-2} & \cdots & -p_1 \end{vmatrix} \quad (k = 1, 2, \cdots, 2g-1). \tag{1.12}$$

For the details about this fact, we refer the reader to [**Mac**, p.29, $\ell. -4$ and p.28, $\ell.13$]. By the fundamental theorem of algebraic equations, we see that for given value of $\varepsilon_k(\boldsymbol{s})$ there is the set $\boldsymbol{s}$ which satisfies (1.11). Therefore, the determinant in 1.9 coincides with the Schur-Weierstrass polynomial $S_{d,s}(-p_1, -p_2, \cdots, -p_{2g-1})$ just above Theorem 4.1 of [**BEL2**].

Remark 1.13 The u_j here is different from z_j in [**BEL2**]. The relation is $z_{w_k} = -ku_{\langle k\rangle}$. Moreover, we remark here that the direction of the integrals in (3.10) is opposite from those of (5.3) in [**BEL2**]. The integrals their-selves are of constant multiples each another.

While the determinant in 1.9 is obviously a function of $2g-1$ variables $u_{\langle 1\rangle}$, $u_{\langle 2\rangle}$, $\cdots$, $u_{\langle 2g-2\rangle}$, $u_{\langle 2g-1\rangle}$, it is in fact independent of some of these variables as follows:

Proposition-Definition 1.14 The determinant (1.4) is a polynomial of only g variables $u_{\langle w_1\rangle}$, $u_{\langle w_2\rangle}$, $\cdots$, $u_{\langle w_g\rangle}$. So that, we denote it as

$$S(u_{\langle w_g\rangle}, u_{\langle w_{g-1}\rangle}, \cdots, u_{\langle w_1\rangle}) := S_{d,s}(u_{\langle 1\rangle} u_{\langle 2\rangle}, \cdots, u_{\langle 2g-2\rangle}, u_{\langle 2g-1\rangle}).$$

We call this also the *Schur-Weierstrass polynomial* of (d, s).

Proof. This is just Theorem 4.1 of [**BEL2**, p.86]. $\square$

We introduce special derivatives of the Schur-Weierstrass polynomial. We start by the following definition:

Definition 1.15 The *attached Sato weight* for the k-th stratum is defined by

$$\mathrm{aw}(k) = \sum_{j=1}^{g-k} (w_j - j + 1).$$

The following Lemma would be useful for computing $\mathrm{aw}(k)$ for a small k:

Lemma 1.16

$$\sum_{j=1}^{g} (w_j - j + 1) = \frac{(d^2-1)(s^2-1)}{24}.$$

Proof. Looking at the determinant expression (1.20) of $S(u)$ for the case $(d, s) = (5, 6)$, we see that the left hand side of the assertion is the sum of Sato weights of its diagonal entries, and is equal to right hand side as is computed at just before Lemma 4.4 in [**BEL2**]. $\square$

We denote by $\widetilde{\Theta}^{[1]}$ the subset of $\mathbf{C}^g$ where a vector $u^{(j)}$ in (1.15) varies, namely, we let

$$\widetilde{\Theta}^{[1]} = \{(\tfrac{1}{w_g}{u_{\langle 1\rangle}}^{w_g}, \tfrac{1}{w_{g-1}}{u_{\langle 1\rangle}}^{w_{g-1}}, \cdots, \tfrac{1}{w_2}{u_{\langle 1\rangle}}^{w_2}, u_{\langle 1\rangle}) \,\big|\, u_{\langle 1\rangle} \in \mathbf{C}\} \subset \mathbf{C}^g. \tag{1.17}$$

More generally, we define

$$\widetilde{\Theta}^{[n]} = \{u^{(1)} + \cdots + u^{(n)} \,\big|\, u^{(j)} \in \widetilde{\Theta}^{[1]},\ 1 \leqq j \leqq n\} \subset \mathbf{C}^g. \tag{1.18}$$

We define the following special derivatives of the Schur-Weierstrass polynomial attached to each the stratification $\widetilde{\Theta}^{[n]}$.

Definition 1.19 For $0 \leqq k \leqq g$, the symbol $\natural^k$ denotes the multi-index with respect to the elements of $\mathbf{Weier}_{d,s}$ that is the last one according to the lexicographic order such that its total weight is $\mathrm{aw}(k)$ and that

$$S_{\natural^k}(\widetilde{\Theta}^{[k]}) \tag{1.20}$$

does not vanish.

In this situation, we have the following vanishing properties. While we do not currently use these facts in our proof, they are interesting in the comparison with the vanishing properties in 7.6 of $\sigma_{\natural^k}(u)$, and are the natural degenerations of the corresponding properties of $\sigma_{\natural^k}(u)$.

Proposition 1.21 *Let I be a multi-index with respect to the elements of* $\mathbf{Weier}_{d,s}$*. Then we have the following properties:*
(1) *If* $\mathrm{sw}(I) < \mathrm{aw}(n)$*, then* $S_I(\widetilde{\Theta}^{[n]}) = 0$.
(2) *Suppose* $\mathrm{sw}(I) = \mathrm{aw}(n)$*. Then we have* $S_I(\widetilde{\Theta}^{[n-1]}) = 0$ *by* (1)*. In this situation, if* $S_I(u)$ *is not a zero function on* $\widetilde{\Theta}^{[n]}$*, then, for* $u \in \widetilde{\Theta}^{[n]}$*, we have*

$$S_I(u) = 0 \iff u \in \widetilde{\Theta}^{[n-1]}.$$

Especially, for $u \in \widetilde{\Theta}^{[n]}$*, we have*

$$S_{\natural^n}(u) = 0 \iff u \in \widetilde{\Theta}^{[n-1]}.$$

(3) *Suppose* $1 \leqq n \leqq g$, $u^{(j)} \in \widetilde{\Theta}^{[1]}$ $(1 \leqq j \leqq n)$*, and* $v \in \widetilde{\Theta}^{[1]}$*, then we have*

$$\begin{aligned} &S_{\natural^{n+1}}(u^{(1)} + \cdots + u^{(n)} + v) \\ &\quad = S_{\natural^n}(u^{(1)} + \cdots + u^{(n)}) {v_{\langle 1\rangle}}^{w_n+n-1} + \big(d^\circ(v_{\langle 1\rangle}) \geqq w_n + n\big). \end{aligned}$$

Proof. Because of the following three reason we do not prove this here.
(1) This Proposition is not used in our proof of the main results.
(2) This is proved by similar argument as in [**Ô3**].
(3) If we have presented the proof it would be quite long. □

We describe the case $(d,s) = (5,6)$ explicitly. In this case

$$\begin{aligned} g &= 10, \\ \mathbf{Weier}_{5,6} &= \{w_{10}, w_9, w_8, w_7, w_6, w_5, w_4, w_3, w_2, w_1\} \\ &= \{\ 19,\ 14,\ 13,\ \ 9,\ \ 8,\ \ 7,\ \ 4,\ \ 3,\ \ 2,\ \ 1\}, \end{aligned}$$

$$S(u_{\langle 19\rangle}, u_{\langle 14\rangle}, \cdots, u_{\langle 1\rangle}) \tag{1.22}$$
$$= \begin{vmatrix} U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} & U_{19} \\ U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\ U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} \\ U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & U_9 \\ & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 \\ & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 \\ & & & & & U_0 & U_1 & U_2 & U_3 & U_4 \\ & & & & & & U_0 & U_1 & U_2 & U_3 \\ & & & & & & & U_0 & U_1 & U_2 \\ & & & & & & & & U_0 & U_1 \end{vmatrix}.$$

Here the empty entries mean their entries are 0. The attached Sato weights $\mathrm{aw}(k)$ are given as follows:

(1.23)

k	1	2	3	4	5	6	7	8	9	10
$\mathrm{aw}(k)$	25	19	13	10	7	4	3	2	1	0

(1.24)

$S_\sharp$	$S_\flat$	$S_{\natural^3}$	$S_{\natural^4}$	$S_{\natural^5}$	$S_{\natural^6}$	$S_{\natural^7}$	$S_{\natural^8}$	$S_{\natural^9}$	$S_{\natural^n}$ $(n \geqq 10)$
$S_{\langle 4,8,13\rangle}$	$S_{\langle 379\rangle}$	$S_{\langle 148\rangle}$	$S_{\langle 37\rangle}$	$S_{\langle 7\rangle}$	$S_{\langle 4\rangle}$	$S_{\langle 3\rangle}$	$S_{\langle 2\rangle}$	$S_{\langle 1\rangle}$	S

2. Derivatives of Schur-Weierstrass polynomial of (5,6)

Any partial derivative of Schur-Weierstrass polynomial is given simply by a sum of shifts of suitable rows. Namely, we have

$$\frac{\partial}{\partial u_{\langle k\rangle}} = (-1)^{k+1}\sum_{r\geqq 0} U_r \frac{\partial}{\partial U_{k+r}}, \tag{2.1}$$

where we are writing simply $U_j = U_j^{[g]}(\boldsymbol{u}_{\langle 1\rangle})$. If we operate (2.1) to the determinant (1.9) that express the Schur-Weierstrass polynomial, we get the sum of 10 terms whose one of rows is shifted to the right by k entries. At the operation, we regard each row is successive 10 terms of two-sides sequence

$$\cdots,\ 0,\ 0,\ 0,\ U_0,\ U_1,\ U_2,\ U_3,\ U_4,\ U_5,\ \cdots. \tag{2.2}$$

If we understand this operation as above, we can obtain the derivatives of $S(\boldsymbol{u})$ easily. For instance,

$$\begin{aligned} S_{\langle 148\rangle}(\boldsymbol{u}) &= \frac{\partial^3}{\partial u_{\langle 1\rangle}\partial u_{\langle 4\rangle}\partial u_{\langle 8\rangle}} S(\boldsymbol{u}) \\ &= \begin{vmatrix} U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} & U_{19} \\ U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\ U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} \\ U_0 & U_1 & U_2 & U_3 & \boxed{U_4} & U_5 & U_6 & U_7 & U_8 & U_9 \\ & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & \boxed{U_8} \\ & & U_0 & \boxed{U_1} & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 \\ & & & & & U_0 & U_1 & U_2 & U_3 & U_4 \\ & & & & & & U_0 & U_1 & U_2 & U_3 \\ & & & & & & & U_0 & U_1 & U_2 \\ & & & & & & & & U_0 & U_1 \end{vmatrix}. \end{aligned} \tag{2.3}$$

Here, the boxes $\boxed{}$ in the determinant above mean that we are going to shifting to the right such as the boxed entries to be U_0. The reader should check that, except only the unique term above, other all terms vanish. This should be sophisticatedly explained in the future.

Continuing the calculation above, we have arrived that

$$
\begin{aligned}
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} & U_{19} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} \\
 & & & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 \\
 & & & & & & & & & U_0 \\
 & & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 \\
 & & & & & U_0 & U_1 & U_2 & U_3 & U_4 \\
 & & & & & & U_0 & U_1 & U_2 & U_3 \\
 & & & & & & & U_0 & U_1 & U_2 \\
 & & & & & & & & U_0 & U_1
\end{vmatrix} \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} & U_{19} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} \\
 & & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 \\
 & & & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 \\
 & & & & & U_0 & U_1 & U_2 & U_3 & U_4 \\
 & & & & & & U_0 & U_1 & U_2 & U_3 \\
 & & & & & & & U_0 & U_1 & U_2 \\
 & & & & & & & & U_0 & U_1 \\
 & & & & & & & & & U_0
\end{vmatrix} \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} \\
U_5 & U_6 & U_7 \\
U_4 & U_5 & U_6
\end{vmatrix}.
\end{aligned} \tag{2.4}
$$

We list below the important derivatives of $S(\boldsymbol{u})$ by using similar calculation as above. While only the results (2.5) and (2.6) below are important for our proof of main results, whole the list below would be useful for the reader.

$$
\begin{aligned}
S_{\langle 13,8,4\rangle}(\boldsymbol{u}) &= \frac{\partial^3}{\partial u_{\langle 13\rangle}\partial u_{\langle 8\rangle}\partial u_{\langle 4\rangle}} S(\boldsymbol{u}) \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} & U_{19} \\
U_5 & U_6 & U_7 & \boxed{U_8} & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & \boxed{U_{13}} \\
U_0 & U_1 & U_2 & U_3 & \boxed{U_4} & U_5 & U_6 & U_7 & U_8 & U_9 \\
 & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 \\
 & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 \\
 & & & & & U_0 & U_1 & U_2 & U_3 & U_4 \\
 & & & & & & U_0 & U_1 & U_2 & U_3 \\
 & & & & & & & U_0 & U_1 & U_2 \\
 & & & & & & & & U_0 & U_1
\end{vmatrix} \\
&= -U_{10}.
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
S_{\langle 973\rangle}(\boldsymbol{u}) &= \frac{\partial^3}{\partial u_{\langle 9\rangle}\partial u_{\langle 7\rangle}\partial u_{\langle 3\rangle}} S(\boldsymbol{u}) \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} & U_{19} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\
U_4 & U_5 & U_6 & \boxed{U_7} & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} \\
U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & \boxed{U_9} \\
 & U_0 & U_1 & U_2 & \boxed{U_3} & U_4 & U_5 & U_6 & U_7 & U_8 \\
 & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 \\
 & & & & & U_0 & U_1 & U_2 & U_3 & U_4 \\
 & & & & & & U_0 & U_1 & U_2 & U_3 \\
 & & & & & & & U_0 & U_1 & U_2 \\
 & & & & & & & & U_0 & U_1
\end{vmatrix} \\
&= \begin{vmatrix}
U_{10} & U_{11} \\
U_5 & U_6
\end{vmatrix}.
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
S_{\langle 841\rangle}(\boldsymbol{u}) &= \frac{\partial^3}{\partial u_{\langle 8\rangle}\partial u_{\langle 4\rangle}\partial u_{\langle 1\rangle}} S(\boldsymbol{u}) \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} & U_{19} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} \\
U_0 & U_1 & U_2 & U_3 & \boxed{U_4} & U_5 & U_6 & U_7 & U_8 & U_9 \\
 & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & \boxed{U_8} \\
 & & U_0 & \boxed{U_1} & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 \\
 & & & & & U_0 & U_1 & U_2 & U_3 & U_4 \\
 & & & & & & U_0 & U_1 & U_2 & U_3 \\
 & & & & & & & U_0 & U_1 & U_2 \\
 & & & & & & & & U_0 & U_1
\end{vmatrix} \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} \\
U_5 & U_6 & U_7 \\
U_4 & U_5 & U_6
\end{vmatrix}.
\end{aligned} \tag{2.7}
$$

$$
S_{\langle 73\rangle}(\boldsymbol{u}) = \frac{\partial^2}{\partial u_{\langle 3\rangle}\partial u_{\langle 7\rangle}} S(\boldsymbol{u})
$$

$$
\begin{aligned}
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} & U_{19} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} \\
U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & U_9 \\
 & U_0 & U_1 & U_2 & \boxed{U_3} & U_4 & U_5 & U_6 & U_7 & U_8 \\
 & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & \boxed{U_7} \\
 & & & & & U_0 & U_1 & U_2 & U_3 & U_4 \\
 & & & & & & U_0 & U_1 & U_2 & U_3 \\
 & & & & & & & U_0 & U_1 & U_2 \\
 & & & & & & & & U_0 & U_1
\end{vmatrix} \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} \\
U_5 & U_6 & U_7 & U_8 \\
U_4 & U_5 & U_6 & U_7 \\
U_0 & U_1 & U_2 & U_3
\end{vmatrix}.
\end{aligned} \tag{2.8}
$$

$$
S_{\langle 7\rangle}(\boldsymbol{u}) = \frac{\partial}{\partial u_{\langle 7\rangle}} S(\boldsymbol{u})
$$

$$
\begin{aligned}
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} & U_{19} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} \\
U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & U_9 \\
 & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 \\
 & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & \boxed{U_7} \\
 & & & & & U_0 & U_1 & U_2 & U_3 & U_4 \\
 & & & & & & U_0 & U_1 & U_2 & U_3 \\
 & & & & & & & U_0 & U_1 & U_2 \\
 & & & & & & & & U_0 & U_1
\end{vmatrix} \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\
U_5 & U_6 & U_7 & U_8 & U_9 \\
U_4 & U_5 & U_6 & U_7 & U_8 \\
U_0 & U_1 & U_2 & U_3 & U_4 \\
 & U_0 & U_1 & U_2 & U_3
\end{vmatrix}.
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
S_{\langle 4\rangle}(\boldsymbol{u}) &= \frac{\partial}{\partial u_{\langle 4\rangle}} S(\boldsymbol{u}) \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} & U_{19} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} \\
U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & U_9 \\
 & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 \\
 & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 \\
 & & & & & U_0 & U_1 & U_2 & U_3 & \boxed{U_4} \\
 & & & & & & U_0 & U_1 & U_2 & U_3 \\
 & & & & & & & U_0 & U_1 & U_2 \\
 & & & & & & & & U_0 & U_1
\end{vmatrix} \\
&= -\begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 \\
U_0 & U_1 & U_2 & U_3 & U_4 & U_5 \\
 & U_0 & U_1 & U_2 & U_3 & U_4 \\
 & & U_0 & U_1 & U_2 & U_3
\end{vmatrix}.
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
S_{\langle 3\rangle}(\boldsymbol{u}) &= \frac{\partial}{\partial u_{\langle 3\rangle}} S(\boldsymbol{u}) \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} & U_{19} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} \\
U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & U_9 \\
 & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 \\
 & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 \\
 & & & & & U_0 & U_1 & U_2 & U_3 & U_4 \\
 & & & & & & U_0 & U_1 & U_2 & \boxed{U_3} \\
 & & & & & & & U_0 & U_1 & U_2 \\
 & & & & & & & & U_0 & U_1
\end{vmatrix} \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} \\
U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 \\
 & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 \\
 & & U_0 & U_1 & U_2 & U_3 & U_4 \\
 & & & & & U_0 & U_1
\end{vmatrix}.
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
S_{\langle 2\rangle}(\boldsymbol{u}) &= \frac{\partial}{\partial u_{\langle 2\rangle}} S(\boldsymbol{u}) \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} & U_{19} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} \\
U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & U_9 \\
 & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 \\
 & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 \\
 & & & & & U_0 & U_1 & U_2 & U_3 & U_4 \\
 & & & & & & U_0 & U_1 & U_2 & U_3 \\
 & & & & & & & U_0 & U_1 & \boxed{U_2} \\
 & & & & & & & & U_0 & U_1
\end{vmatrix} \\
&= -\begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} \\
U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 \\
 & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 \\
 & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 \\
 & & & & & U_0 & U_1 & U_2 \\
 & & & & & & U_0 & U_1
\end{vmatrix}.
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
S_{\langle 1\rangle}(\boldsymbol{u}) &= \frac{\partial}{\partial u_{\langle 1\rangle}} S(\boldsymbol{u}) \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} & U_{19} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} \\
U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & U_9 \\
 & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 \\
 & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 \\
 & & & & & U_0 & U_1 & U_2 & U_3 & U_4 \\
 & & & & & & U_0 & U_1 & U_2 & U_3 \\
 & & & & & & & U_0 & U_1 & U_2 \\
 & & & & & & & & U_0 & \boxed{U_1}
\end{vmatrix} \\
&= \begin{vmatrix}
U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} \\
U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} \\
U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} \\
U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 \\
 & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 \\
 & & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 \\
 & & & & & U_0 & U_1 & U_2 & U_3 \\
 & & & & & & U_0 & U_1 & U_2 \\
 & & & & & & & U_0 & U_1
\end{vmatrix}.
\end{aligned}
\tag{2.13}
$$

3. Purely d-gonal curves with unique point at infinity.

Let (d, s) be the pair introduced in the beginning of Section 1, and let C be the algebraic curve defined by

$$y^d = x^s + \lambda_1 x^{s-1} + \lambda_2 x^{s-2} + \cdots + \lambda_{s-1} x + \lambda_s \quad (\lambda_j \text{ are constants}), \tag{3.1}$$

where we assume C is a complete projective curve having the unique point ∞ at infinity. The genus of C is $g = (d-1)(s-1)/2$ if it is non-singular. Let

$$\overline{\mathbf{Weier}}_{d,s} \tag{3.2}$$

be the sequence of increasing non-negative integers which are expressed as $ad + bs$ with non-negative integers a and b. We denote its elements by

$$\begin{aligned}\overline{\mathbf{Weier}}_{d,s} &= \{a_1 d + b_1 s, a_2 d + b_2 s, \cdots, a_g d + b_g s, \cdots\} \\ &= \{0, d, \cdots, (s-1)d, \cdots\}.\end{aligned} \tag{3.3}$$

Corresponding this sequence we consider monomials with respect to $x(u)$ and $y(u)$ $x^{a_j} y^{b_j}$ and the determinant

$$\begin{aligned}&\Big|(x^{a_j} y^{b_j})(u^{(i)})\Big|_{1 \leqq i,j \leqq n} \\ &= \begin{vmatrix} 1 & x(u^{(1)}) & x^{a_3} y^{b_3}(u^{(1)}) & \cdots & x^{a_j} y^{b_j}(u^{(1)}) & \cdots & x^{a_n} y^{b_n}(u^{(1)}) \\ 1 & x(u^{(2)}) & x^{a_3} y^{b_3}(u^{(2)}) & \cdots & x^{a_j} y^{b_j}(u^{(2)}) & \cdots & x^{a_n} y^{b_n}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & x(u^{(n)}) & x^{a_3} y^{b_3}(u^{(n)}) & \cdots & x^{a_j} y^{b_j}(u^{(n)}) & \cdots & x^{a_n} y^{b_n}(u^{(n)}) \end{vmatrix}.\end{aligned} \tag{3.4}$$

This is so important for our main result. As is well-known, the differentials

$$\begin{aligned}&\omega_{\langle w_g \rangle} = \frac{\mathrm{d}x}{dy^{d-1}}, \quad \omega_{\langle w_{g-1} \rangle} = \frac{x\mathrm{d}x}{dy^{d-1}}, \quad \omega_{\langle w_{g-2} \rangle} = \frac{x^{a_3} y^{b_3} \mathrm{d}x}{dy^{d-1}}, \\ &\qquad\qquad \cdots\cdots, \quad \omega_{\langle w_1 \rangle} = \frac{x^{a_g} y^{b_g} \mathrm{d}x}{dy^{d-1}}\end{aligned} \tag{3.5}$$

form a basis of the space of holomorphic 1-forms on C. For g variable points (x_1, y_1), (x_2, y_2), $\cdots$, (x_g, y_g) on C and the point ∞, the values of integrals

$$\begin{aligned}u &= (u_{\langle w_g \rangle}, u_{\langle w_{g-1} \rangle}, \cdots, u_{\langle w_1 \rangle}) \\ &= \Big(\int_\infty^{(x_1, y_1)} + \int_\infty^{(x_2, y_2)} + \cdots + \int_\infty^{(x_g, y_g)} \Big) (\omega_{\langle w_g \rangle}, \omega_{\langle w_{g-1} \rangle}, \cdots, \omega_{\langle w_1 \rangle})\end{aligned} \tag{3.6}$$

with any paths of integrals fill the whole space $\mathbf{C}^g$. Points in $\mathbf{C}^g$ is denoted by u, v, $u^{(j)}$ etcetera, and their natural coordinates are denoted as $(u_{\langle w_g \rangle}, u_{\langle w_{g-1} \rangle}, \cdots, u_{\langle w_1 \rangle})$, $(v_{\langle w_g \rangle}, v_{\langle w_{g-1} \rangle}, \cdots, v_{\langle w_1 \rangle})$, $(u^{(j)}_{\langle w_g \rangle}, u^{(j)}_{\langle w_{g-1} \rangle}, \cdots, u^{(j)}_{\langle w_1 \rangle})$, respectively. All the values of (3.6) with respect to closed paths of integrals are denoted by Λ, which is a lattice in

$\mathbf{C}^g$. The Jacobian variety of C such that whose closed points over $\mathbf{C}$ is just $\mathbf{C}^g/\Lambda$ is denoted by J. The map modulo Λ is denoted by

$$\kappa : \mathbf{C}^g \to \mathbf{C}^g/\Lambda = J(\mathbf{C}). \tag{3.7}$$

Needless to say, $\Lambda = \kappa^{-1}\big((0, 0, \cdots, 0)\big)$. We identify the point in J given by (3.6) with the coset in $\mathrm{Pic}^\circ(C)$ represented by the divisor $(x_1, y_1)+(x_2, y_2)+\cdots+(x_g, y_g)-g \cdot \infty$ modulo linear equivalence. We embed C into J by this identification as

$$\begin{aligned} \iota : C &\hookrightarrow \mathrm{Pic}^\circ(C) = J, \\ P &\mapsto P - \infty. \end{aligned} \tag{3.8}$$

More generally, for $0 \leqq k \leqq g$ we have a map ι from k-th symmetric product $\mathrm{Sym}^k(C)$ to J defined by

$$\begin{aligned} \iota : \mathrm{Sym}^k(C) &\to \mathrm{Pic}^\circ(C) = J \\ (P_1, \cdots, P_k) &\mapsto P_1 + \cdots + P_k - k \cdot \infty. \end{aligned} \tag{3.9}$$

We denote by $W^{[k]}$ the image of this map: $W^{[k]} = \iota(\mathrm{Sym}^k(C)$. Especially, we have $W^{[0]} = (0, 0, \cdots, 0)$, $W^{[1]} = \iota(C)$, $W^{[g]} = J$. We denote by $[-1]$ the (-1)-multiplication in J, namely

$$[-1](u_{\langle w_g\rangle}, \cdots, u_{\langle w_2\rangle}, u_{\langle w_1\rangle}) = (-u_{\langle w_g\rangle}, \cdots, -u_{\langle w_2\rangle}, -u_{\langle w_1\rangle}). \tag{3.10}$$

We let

$$\Theta^{[k]} = W^{[k]} \cup [-1]W^{[k]}, \tag{3.11}$$

and we call this the *standard theta subset* of dimension k. We note that $\kappa^{-1}\iota(C)$ is a universal Abelian covering of C.

Remark 3.12 If $d = 2$, we see $\Theta^{[k]} = W^{[k]}$. In general, we have $\Theta^{[k]} \neq W^{[k]}$ for $1 \leqq k \leqq g$.

Lemma 3.13 *If $u = (u_{\langle w_g\rangle}, u_{\langle w_{g-1}\rangle}, \cdots, u_{\langle w_1\rangle}) \in \kappa^{-1}\iota(C)$, each $u_{\langle w_j\rangle}$ $2 \leqq j \leqq g$ is expanded with respect to $u_{\langle 1\rangle}$ as*

$$u_{\langle w_j\rangle} = \frac{1}{w_j} {u_{\langle 1\rangle}}^{w_j} + \cdots .$$

Proof. We take $t = 1/\sqrt[d]{x}$ as a local parameter at ∞ on the curve C. Then we have that

$$\begin{aligned} u_{\langle w_j\rangle} &= \int_\infty^{(x,y)} \frac{x^{a_{g-j+1}} y^{b_{g-j+1}} \mathrm{d}x}{dy^{d-1}} \\ &= \int_0^t \frac{t^{-a_j d - b_j s} + \cdots}{dt^{s(d-1)} + \cdots} \cdot \frac{-d}{t^{d+1}}\, \mathrm{d}t \\ &= \int_0^t \big(- t^{w_j - 1} + \cdots \big) \mathrm{d}t \\ &= \frac{1}{w_j} t^{w_j} + \cdots , \end{aligned} \tag{3.14}$$

and we have proved the statement. $\square$

Lemma 3.15 *If $u = (u_{\langle w_g \rangle}, u_{\langle w_{g-1} \rangle}, \cdots, u_{\langle w_1 \rangle}) \in \kappa^{-1}\iota(C)$, then the functions $x(u)$ and $y(u)$ are expanded with respect to $u_{\langle w_1 \rangle} = u_{\langle 1 \rangle}$ as:*

$$x(u) = \frac{1}{{u_{\langle 1 \rangle}}^d} + \cdots, \qquad y(u) = \frac{1}{{u_{\langle 1 \rangle}}^s} + \cdots.$$

Proof. These are proved similarly as in 3.9. $\square$

Remark 3.16 We see that $u_{\langle 1 \rangle}$ is a local parameter on $\kappa^{-1}\iota(C)$ at the origin of $\mathbf{C}^g$.

The group of d-th roots of unity acts on C as follows. Let

$$\zeta = e^{2\pi\sqrt{-1}/d}. \tag{3.17}$$

For a point (x, y) on C, the action is given by

$$[\zeta](x, y) = (x, \zeta y). \tag{3.18}$$

If we take an integer μ such that $(2g-1)\mu$ ($\equiv s\mu$) $\equiv 1$ mod d, then we see

$$[\zeta]u_{\langle w_g \rangle} = u_{\langle w_g \rangle}, \qquad [\zeta]u_{\langle 1 \rangle} = \zeta^\mu u_{\langle 1 \rangle} \tag{3.16}$$

etcetera. Namely,

$$\begin{aligned}&[\zeta](u_{\langle w_g \rangle}, u_{\langle w_{g-1} \rangle}, \cdots, u_{\langle 2 \rangle}, u_{\langle 1 \rangle}) \\ &\qquad = (\zeta u_{\langle w_g \rangle}, \zeta^{\mu w_{g-1}} u_{\langle w_{g-1} \rangle}, \cdots, \zeta^{\mu w_2} u_{\langle w_2 \rangle}, \zeta^\mu u_{\langle 1 \rangle}).\end{aligned} \tag{3.19}$$

On $x(u)$ and $y(u)$, we have that

$$x([\zeta]u) = x(u), \qquad y([\zeta]u) = \zeta y(u). \tag{3.20}$$

4. The sigma function.

Now we are going to define the *sigma function*

$$
\begin{aligned}
\sigma(u) &= \sigma(u_{\langle w_g\rangle}, u_{\langle w_{g-1}\rangle}, \cdots, u_{\langle w_1\rangle}) \\
&= \sigma(u_{\langle 2g-1\rangle}, u_{\langle 2g-d-1\rangle}, \cdots, u_{\langle 1\rangle})
\end{aligned} \tag{4.1}
$$

associated to the curve C according to Chapter 1 of [**BEL1**]. The function $\sigma(u)$ is constructed by the date

—. the basis (3.5) of differential forms of 1st kind;
—. certain special g differential forms $\eta_1, \eta_2, \cdots, \eta_g$ of 2nd kind;
—. representatives $\alpha_1, \alpha_2, \cdots, \alpha_g, \beta_1, \beta_2, \cdots, \beta_g$ of generators of $H^0(C, \mathbf{Z})$ such that their intersections are $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = \delta_{ij}$, $\alpha_i \cdot \beta_j = 0$.

The below is according to p.68 and around the formula 1.2 in p.194 of [**Ba1**]. We consider the periods matrix

$$
[\,\omega' \;\; \omega''\,] = \left[\int_{\alpha_i} \omega_j \quad \int_{\beta_i} \omega_j\right]_{i,j=1,2,\cdots,g} \tag{4.2}
$$

determined by ω_j of (3.5). Let us introduce

$$
\Omega\big((x,y),(x',y')\big) = \frac{1}{(x-x')dy^{d-1}} \sum_{k=1}^{n} y^{d-k} {y'}^{k-1}, \tag{4.3}
$$

and define differential forms $\eta_j = \eta_j(x,y)$ of 2nd kind such that

$$
\begin{aligned}
&\frac{\partial\,\Omega\big((x',y'),(x,y)\big)}{\partial x} - \frac{\partial\,\Omega\big((x,y),(x',y')\big)}{\partial x'} \\
&\qquad = \sum_{i=1}^{g} \frac{\omega_i(x,y)}{\mathrm{d}x}\frac{\eta_i(x',y')}{\mathrm{d}x'} - \frac{\omega_i(x',y')}{\mathrm{d}x'}\frac{\eta_i(x,y)}{\mathrm{d}x}.
\end{aligned} \tag{4.4}
$$

We consider the matrix of their periods

$$
[\,\eta' \;\; \eta''\,] = \left[\int_{\alpha_i} \omega_j \quad \int_{\beta_i} \omega_j\right]_{i,j=1,2,\cdots,g}. \tag{4.5}
$$

We unify these matrices as $M = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}$. Since the sigma function that we want to define is a function of $u_{\langle w_g\rangle}$, $u_{\langle w_{g-1}\rangle}$, $\cdots$, $u_{\langle w_1\rangle}$, and M, we write

$$
\sigma(u) = \sigma(u; M) = \sigma(u_{\langle w_g\rangle}, u_{\langle w_{g-1}\rangle}, \cdots, u_{\langle w_1\rangle}; M). \tag{4.6}
$$

Here, M satisfies

$$
M \begin{bmatrix} & -1_g \\ 1_g & \end{bmatrix} {}^t\!M = 2\pi\sqrt{-1} \begin{bmatrix} & -1_g \\ 1_g & \end{bmatrix} \tag{4.7}
$$

(see (1.14) in p.11 of [**BEL1**]). This is the *generalized Legendre relation.* Let

$$\delta = \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \in \left(\tfrac{1}{2}\mathbf{Z}\right)^{2g} \tag{4.8}$$

be a theta characteristic that gives the Riemann constant with respect to $[\omega' \ \omega'']$ with base point of C to be ∞ (see (1.18) in p.15 of [**BEL1**] and pp.163–166 of [**MumTataI**]). We remark here that the canonical divisor class of C is $(2g-2)\infty$ by (3.5), so that any theta characteristic is given by a vector in $\left(\frac{1}{2}\mathbf{Z}\right)^{2g}$. Let

$$\begin{aligned} \widetilde{\sigma}(u) = {} & \exp(-\tfrac{1}{2}u\eta'{\omega'}^{-1}\ {}^t u)\vartheta[\delta]\left({\omega'}^{-1}\ {}^t u;\ {\omega'}^{-1}\omega''\right) \\ = {} & \exp(-\tfrac{1}{2}u\eta'{\omega'}^{-1}\ {}^t u) \\ & \times \sum_{n\in\mathbf{Z}^{10}} \exp\left[2\pi i\left\{\tfrac{1}{2}\ {}^t(n+\delta'){\omega'}^{-1}\omega''(n+\delta') + {}^t(n+\delta')(z+\delta'')\right\}\right]. \end{aligned} \tag{4.9}$$

Then, the sigma function is a non-zero constant multiple of $\widetilde{\sigma}(u)$. The ratio of $\sigma(u)$ by $\widetilde{\sigma}(u)$ is fixed in 5.1.

In this paper, for a given $u \in \mathbf{C}^g$, we denote by u' and u'' the elements in $\mathbf{R}^g$ determined by

$$u = u'\omega' + u''\omega''. \tag{4.10}$$

Then, for u, $v \in \mathbf{C}^g$, and ℓ $(= \ell'\omega' + \ell''\omega'' \in \Lambda)$, we define

$$\begin{aligned} L(u,v) &:= {}^t u(\eta' v' + \eta'' v''), \\ \xi(\ell) &:= \pi\sqrt{-1}\big(2({}^t\ell'\delta'' - {}^t\ell''\delta') + {}^t\ell'\ell''\big)\ (\in \pi\sqrt{-1}\,\mathbf{Z}). \end{aligned} \tag{4.11}$$

Under this situation, we describe important properties of the function $\sigma(u;M)$ as follows:

Lemma 4.12 *For any $u \in \mathbf{C}^g$, $\ell \in \Lambda$ and $\gamma \in \mathrm{Sp}(2g,\mathbf{Z})$, we have*
(1) $\sigma(u+\ell;M) = \sigma(u;M)\exp[L(u+\frac{1}{2}\ell,\ell)+\xi(\ell)]$,
(2) $\sigma(u;\gamma M) = \sigma(u;M)$,
(3) $\sigma(u;M) = 0 \iff u \in \Theta^{[g-1]}$.

Proof. This is a special case of [**Ba1**], p.286, l.22. See also [**BEL1**], p.12, Theorem 1.1 and p.15. $\square$

Lemma 4.13 *If we fix M, the space of solutions $\varphi(u;M)$, that are entire in u, of the functional equation*

$$\varphi(u+\ell;M) = \varphi(u;M)\exp[L(u+\tfrac{1}{2}\ell,\ell)+\xi(\ell)]$$

of type 4.12(1) *is* 1*-dimensional. In other words, for a fixed M,* 4.11(1) *characterizes the sigma function up to multiplicative non-zero constant.*

Proof. We prove the statement by using Frobenius theorem in [**F**], which are explained also in [**L**], p.93, Theorem 3.1. First of all, the Riemann form

$$E(u,v) = L(u,v) - L(v,u) \tag{4.14}$$

attached to $L(\ ,\ )$ has the following properties:

$$\begin{cases} E(\sqrt{-1}u, v) = E(\sqrt{-1}v, u) \text{ and this is positive definite;} \\ E(u,v) = 2\pi\sqrt{-1}({}^t u' v'' - {}^t u'' v'). \end{cases} \tag{4.15}$$

These are proved similarly as Lemma 3.1.2 of [**Ô98**]. Especially, the second in 4.15 shows that $E(\ ,\ )$ has values in $\sqrt{-1}\mathbf{R}$, and the set of the values of this on $\Lambda \times \Lambda$ is $2\pi\sqrt{-1}\mathbf{Z}$. So that, the Pfaffian of $E(\ ,\ )$ is 1. On the other hand, if we define

$$\xi_0(\ell) = \xi(\ell) + \tfrac{1}{2}L(\ell,\ell). \tag{4.16}$$

Then it is easy to show that our $\sigma(u)$ is belongs to

$$\mathrm{Th}\big(\tfrac{1}{2\pi\sqrt{-1}}L, \tfrac{1}{2\pi\sqrt{-1}}\xi_0\big) \tag{4.17}$$

where we are using the notation of [**L**], Chapter VI. Therefore, [**L**], p.93, Theorem 3.1 shows the statement. $\square$

5. The Schur-Weierstrass polynomial and the sigma function.

The Schur-Weierstrass polynomial coincide with the limit of the sigma function when all the λ_j in (3.1) bring close to 0. More concretely, this is stated as follows:

Proposition-Definition 5.1 *There is a non-zero constant c that does not depend on the coefficients λ_js in* (3.1) *such that*

$$\sigma(u) = cS(u) + (d^\circ(\lambda_1, \lambda_2, \cdots, \lambda_s) \geqq 1).$$

We define the sigma function $\sigma(u)$ to be the one whose c is 1. *This definition is slightly different from* [**Ô2**] *and* [**Ô3**].

Proof. This is the main result of [**BEL2**]. □

Lemma 5.2 *The function $\sigma(u)$ has power-series expansion with respect to $u_{\langle w_g\rangle}$, $\cdots$, $u_{\langle w_2\rangle}$, $u_{\langle w_1\rangle}$ and belongs to $\mathbf{Q}[\lambda_1, \cdots, \lambda_s][[u_{\langle w_g\rangle}, \cdots, u_{\langle w_1\rangle}]]$. Moreover, it is homogeneous of Sato weight*

$$\mathrm{sw}\big(\sigma(u)\big) = (d^2-1)(s^2-1)/24.$$

Remark 5.3 It is plausible that the power-series expansion of $\sigma(u)$ above is Hurwitz integral series with respect to $u_{\langle w_g\rangle}$, $\cdots$, $u_{\langle w_1\rangle}$ over $\mathbf{Z}[\lambda_1, \cdots, \lambda_s]$.

Lemma 5.4 *Let $\zeta = e^{2\pi\sqrt{-1}/d}$. Then we have*

$$\sigma(-[\zeta]u) = (-1)^b\zeta^\nu\sigma(u), \qquad S(-[\zeta]u) = (-1)^b\zeta^\nu S(u),$$

where b is an integer such that $b \equiv \frac{(d^2-1)(n^2-1)}{24} \bmod 2$, and ν is an integer such that $\nu \equiv \frac{(d^2-1)(n^2-1)}{24(2g-1)} \bmod d$.

Proof. We know that $-\Lambda = \Lambda$, $[\zeta]\Lambda = \Lambda$. The former is seen by considering the path of integral of opposite direction for each $\ell \in \Lambda$. The later is seen from that the action of $[\zeta]$ is coming from an automorphism of the curve C and by considering the path of integral given by the action $[\zeta]$. These facts and 4.13 show that there exists a constant K such that

$$\sigma(-[\zeta]u) = K\sigma(u). \tag{5.5}$$

On the other hand, 5.2, the invariance of Sato weight by the action $[\zeta]$, and (3.16) give that the constant K is of the claimed form. □

Lemma 5.6 *Let $\zeta = e^{2\pi\sqrt{-1}/5}$. If $(d, s) = (5, 6)$, then we have*

$$\sigma(-[\zeta]u) = -\sigma(u), \qquad S(-[\zeta]u) = -S(u).$$

6. Riemann singularity theorem.

In this Section we investigate the zeroes of $\sigma(u)$ and of its derivatives by using Riemann singularity theorem and Brill-Noether matrix. Riemann singularity theorem is stated as follows in our language.

Proposition 6.1 (Riemann singularity theorem) *For a given $u \in \kappa^{-1}(\Theta^{[g-1]})$, Let $P_1 + \cdots + P_{g-1} - (g-1)\cdot\infty$ be a divisor on C that represent u modulo Λ. Then we have*

$$\dim \Gamma(C, \mathcal{O}(P_1 + \cdots + P_{g-1})) = r + 1$$

if and only if the following two conditions hold:
(1) *For all $h \leqq r$ and for all $i_1, \cdots, i_h \in \{w_g, w_{g-1}, \cdots, w_2, w_1\}$,*

$$\sigma_{\langle i_1 i_2 \cdots i_h \rangle}(u) = 0;$$

(2) *There exists a multi-index of $(r+1)$ suffices $\langle i_1, i_2, \cdots, i_{r+1}\rangle$, such that*

$$\sigma_{\langle i_1 i_2 \cdots i_{r+1} \rangle}(u) \neq 0.$$

Proof. Using the relation (4.13) of a theta series and the sigma function, this statement reduces to the fact described in [**ACGH**], pp.226–227. $\square$

To compute $\dim \Gamma(C, \mathcal{O}(P_1 + \cdots + P_{g-1}))$, we define the Brill-Noether matrix as follows. For concreteness, we take local parameter t around each point P on C as

$$t = \begin{cases} y & \text{if } y(P) = 0, \\ x - x(P) & \text{if } y(P) \neq 0 \text{ and } P \neq \infty, \\ x^{-1/d} & \text{if } P = \infty. \end{cases} \tag{6.2}$$

We denote by $P(t)$ the point on the neighborhood of P on C that corresponds to the value t of the local parameter above. Let Ω^1 be the sheaf of the holomorphic differential forms. For each $\mu \in \Gamma(C, \Omega^1)$, let

$$I^\ell \mu(P) = \frac{d^{\ell+1}}{dt^{\ell+1}} \int_\infty^{P(t)} \mu \Big|_{t=0}. \tag{6.3}$$

Though this depends on the choice of t, it is not important for our discussion. So that, we do not write t on this symbol. Since μ is holomorphic, $I^\ell \mu(P)$ takes a finite value at each point P. Let P_js are pairwise different points. We consider effective divisor $D := \sum_{j=1}^k n_j P_j$ with $n_j > 0$ for all j. The matrix of size $\deg D := \sum n_j$ columns and g rows whose $(n_1 + \cdots + n_{j-1} + \ell, i)$-entry is $I^\ell \omega_i(P_j)$ is called the *Brill-Noether matrix* associated to D, and is denoted by

$$B(D), \tag{6.4}$$

where $1 \leqq \ell \leqq n_j$ and ω_i being those of (3.5). Our calculation is based on the following.

Proposition 6.5 *Let D be a effective divisor on C. Then we have*

$$\dim \Gamma(C, \mathcal{O}(D)) = \deg D + 1 - \mathrm{rank} B(D).$$

Proof. For each $\mu \in \Gamma(C, \Omega^1)$, there is unique set of constants $c_1, \cdots, c_g \in \mathbf{C}$ such that $\mu = c_1\omega_1 + \cdots + c_g\omega_g$. If we write $D = \sum_{j=1}^{k} n_j P_j$, then the following three conditions are equivalent:

(6.6a) $\mu \in \Gamma(C, \Omega^1(-D))$,
(6.6b) $\delta^\ell \mu(P_j) = 0$ for all j and ℓ with $1 \leqq j \leqq k$ and $1 \leqq \ell \leqq n_j$, and
(6.6c) $B(D) \begin{bmatrix} c_1 \\ \vdots \\ c_g \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

Hence

$$\dim \Gamma(C, \Omega^1(-D)) = g - \mathrm{rank} B(D). \tag{6.7}$$

The Riemann-Roch theorem shows that

$$\dim \Gamma(C, \mathcal{O}(D)) = \deg D - g + 1 + \dim \Gamma(C, \Omega^1(-D)), \tag{6.8}$$

and that $\dim \Gamma(C, \mathcal{O}(D)) = \deg D + 1 - \mathrm{rank} B(D)$. $\square$

This Proposition is used in 11.1 and 12.1 later.

7. Derivatives of the sigma function and the stratification.

Definition 7.1 We define the attached Sato weight for the k-th stratum by

$$\mathrm{aw}(k) = \sum_{j=1}^{g-k}(w_j - j + 1).$$

The reader should compare this with (1.15).

Example 7.2 If $(d, s) = (5, 6)$, then $\mathrm{aw}(k)$ is given as follows:

k	1	2	3	4	5	6	7	8	9	10
$\mathrm{aw}(k)$	25	19	13	10	7	4	3	2	1	0

Definition 7.3 Let I be a multi-index taken from the set $\mathbf{Weier}_{d,s} = \{w_g, w_{g-1}, \cdots, w_2, w_1\}$. We ignore its order. In this paper we call such a multi-index simply *multi-index*. In this paper we always indicate any multi-index by $\langle\ \ \rangle$. For instance, we denote as $I = \langle 4332111\rangle$. The derivative $\sigma_I(u)$ of $\sigma(u)$ corresponding a multi-index I is defined by

$$\sigma_I(u) = \left(\prod_{i\in I}\frac{\partial}{\partial u_{\langle i\rangle}}\right)\sigma(u).$$

Of course, multiplicated $i \in I$ we derivate appropriate higher partial derivative.

Definition 7.4 Let I be a multi-index. Then we denote by $\mathrm{sw}(I)$ the simple sum of the elements of I. Here sw means Sato weight.

Definition 7.5 We denote by $\natural^n$ a multi-index such that $\mathrm{sw}(\natural^n) = \mathrm{aw}(n)$ and that it is the last one according to lexicographic order of indices of Sato weight $\mathrm{aw}(n)$ with $S_{\natural^n}(u)$ being not a zero-function on $\kappa^{-1}(\Theta^{[n]})$. Since we use $\natural^1$ and $\natural^2$ often, we denote these as $\sharp = \natural^1$, $\flat = \natural^2$.

In the case $(d, s) = (5, 6)$, we have the following definition (see also 1.21).

Definition 7.6 We define the following special derivatives $\sigma(u)$:

$\sigma_\sharp$	$\sigma_\flat$	$\sigma_{\natural^3}$	$\sigma_{\natural^4}$	$\sigma_{\natural^5}$	$\sigma_{\natural^6}$	$\sigma_{\natural^7}$	$\sigma_{\natural^8}$	$\sigma_{\natural^9}$	$\sigma_{\natural^n}$ $(n \geqq 10)$
$\sigma_{\langle 4,8,13\rangle}$	$\sigma_{\langle 379\rangle}$	$\sigma_{\langle 148\rangle}$	$\sigma_{\langle 37\rangle}$	$\sigma_{\langle 7\rangle}$	$\sigma_{\langle 4\rangle}$	$\sigma_{\langle 3\rangle}$	$\sigma_{\langle 2\rangle}$	$\sigma_{\langle 1\rangle}$	σ
σ_{753}	σ_{864}	$\sigma_{10,7,5}$	σ_{86}	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ

It is very plausible that the following hypothesis is true.

Working hypothesis 7.7 *Let I be a multi-index. Then we have the following assertions.*
(1) *If* $\mathrm{sw}(I) < \mathrm{aw}(n)$*, then* $\sigma_I(\kappa^{-1}(\Theta^{[n]})) = 0$*.*
(2) *Assume* $\mathrm{sw}(I) = \mathrm{aw}(n)$*. Then* σ_I *satisfies the translational relation*

$$\sigma_I(u+\ell) = \sigma_I(u)\exp[L(u+\tfrac{1}{2},\ell)+\xi(\ell)]$$

for $u \in \Theta^{[n]}$ *and* $\ell \in \Lambda$*, where* $L(\cdot\,,\cdot)$ *and* $\xi(\cdot)$ *are those defined in* (4.11)*. Especially, for* $u \in \Theta^{[n]}$ *and* $\ell \in \Lambda$*, we have*

$$\sigma_{\natural^n}(u+\ell) = \sigma_{\natural^n}(u)\exp[L(u+\tfrac{1}{2},\ell)+\xi(\ell)].$$

In this situation, we have $\sigma_I(\kappa^{-1}(\Theta^{[n-1]})) = 0$ *by* (1)*, If* σ_I *is not a zero-function on* $\kappa^{-1}(\Theta^{[n]})$*, then for* $u \in \kappa^{-1}(\Theta^{[n]})$ *we have*

$$\sigma_I(u) = 0 \iff u \in \kappa^{-1}(\Theta^{[n-1]}).$$

Especially,

$$\sigma_{\natural^n}(u) = 0 \iff u \in \kappa^{-1}(\Theta^{[n-1]}).$$

(3) *Let* $1 \leqq n \leqq g$*. If* $u \in \kappa^{-1}(\Theta^{[n]})$ *and* $v \in \kappa^{-1}\iota(C)$*, then we have an expansion of the form*

$$\sigma_{\natural^{n+1}}(u+v) = \sigma_{\natural^n}(u){v_{\langle 1\rangle}}^{w_n+n-1} + \big(d^\circ(v_{\langle 1\rangle}) \geqq w_n+n\big).$$

Theorem 7.8 *The hypothesis* 7.7 *is true for the cases*

$$\begin{aligned}(d,s) = &\,(2,2g+1) \quad (g \geqq 1),\\ &\,(3,4),\ (3,5),\ (3,7), \text{etc.},\\ &\,(5,6), \text{etcetera}.\end{aligned}$$

Remark 7.9 For $(d,s) = (2,2g+1)$, 7.8 is proved by [**Ô3**], and for $(d,s) = (3,4)$ that is proved by [**Ô4**]. The similar method shows 7.8 for each case $(d,s) = (3,s)$. While this paper shows the case of $(d,s) = (5,6)$, we need more improvement of method in order to prove for any (d,s) once for all.

In this paper, we use a method which we call *peeling method* to prove 7.8. It seems that the most sophisticated way to prove 7.6 requires quite precise version of Riemann singular theorem. If we had have such theorem, then we can reduce many things about the sigma function to the Schur-Weierstrass polynomial.

Remark 7.10 (1) We have $\mathrm{sw}(I) = 0$ if and only if $I = \langle\rangle$. Therefore, 7.6 states that $\sigma(u)$ satisfies the translational relation on $\mathbf{C}^g = \kappa^{-1}(\Theta^{[g]})$, that $\sigma(\Theta^{[g-1]}) = 0$. These facts are no other than 4.12.
(2) We have $\mathrm{sw}(I) = 1$ only for $I = \{1\}$. The hypothesis 7.6 asserts that $\sigma_{\langle 1\rangle}(u)$ satisfies the translational relation on $\Theta^{[g-1]}$, and that $\sigma_{\langle 1\rangle}(\Theta^{[g-2]}) = 0$.
(3) For the cases $\mathrm{sw}(I) > 1$ we prove 7.8 in Sections later.

8. Table of derivatives of sigma functions

Here we present $\sigma_{\natural^n}(u)$ defined in 7.5 explicitly for several cases of (d, s) in the following table. If $d = 2$, namely, if C is a hyperelliptic curve, then we have very simple rule of $\natural^n$. If $d \geqq 3$, we have not know such a simple rule.

8.1 Table of $\natural^n$

(d,s)	g	$\sharp = \natural^1$	$\flat = \natural^2$	$\natural^3$	$\natural^4$	$\natural^5$	$\natural^6$	$\natural^7$	$\natural^8$	$\natural^9$	$\natural^{10}$	$\natural^{11}$	$\natural^{12}$	$\cdots$
$(2,3)$	1	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(2,5)$	2	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(2,7)$	3	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(2,9)$	4	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(2,11)$	5	$\langle 3,7\rangle$	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(2,13)$	6	$\langle 1,5,9\rangle$	$\langle 3,7\rangle$	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(2,15)$	7	$\langle 3,7,11\rangle$	$\langle 1,5,9\rangle$	$\langle 3,7\rangle$	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$(3,4)$	3	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(3,5)$	4	$\langle 4\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(3,7)$	6	$\langle 2,8\rangle$	$\langle 1,5\rangle$	$\langle 4\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(3,9)$	7	$\langle 4,10\rangle$	$\langle 2,7\rangle$	$\langle 1,5\rangle$	$\langle 4\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(3,10)$	9	$\langle 4,8,14\rangle$	$\langle 1,7,11\rangle$	$\langle 4,8\rangle$	$\langle 2,7\rangle$	$\langle 1,5\rangle$	$\langle 4\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$(5,6)$	10	$\langle 3,8,14\rangle$	$\langle 2,4,13\rangle$	$\langle 1,3,9\rangle$	$\langle 2,8\rangle$	$\langle 7\rangle$	$\langle 4\rangle$	$\langle 3\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	$\cdots$
$(5,7)$	12	$\langle 1,6,11,18\rangle$	$\langle 4,8,16\rangle$	$\langle 2,6,13\rangle$	$\langle 1,4,11\rangle$	$\langle 3,9\rangle$	$\langle 1,8\rangle$	$\langle 6\rangle$	$\langle 4\rangle$	$\langle 3\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\,\rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

9. Statement of The Main Result.

Conjecture 9.1 *Let $n \geqq 2$ be an integer. Then we have*

$$\frac{\sigma_{\natural^n}(u^{(1)}+u^{(2)}+\cdots+u^{(n)})\prod_{1\leqq i<j\leqq n}\prod_{\nu=1}^{d-1}\sigma_\flat(u^{(i)}+[\zeta^\nu]u^{(j)})}{\prod_{j=1}^{n}\sigma_\sharp(u^{(j)})^{(d-1)(n-1)+1}}$$
$$= \pm \left|(x^{a_j}y^{b_j})(u^{(i)})\right|_{1\leqq i,j\leqq n} \cdot \left|(x^{j-1})(u^{(i)})\right|_{1\leqq i,j\leqq n}^{d-2},$$

where the signature "$\pm$" coincides with the signature appeared in the top of the right hand side of

$$S_{\natural^n} = \pm \left| U_{w_{g+1-i}-\mathrm{aw}(n)+1-j} \right|_{1\leqq i,j\leqq n}.$$

Especially, if $n \geqq g$, then it is always to be "$+$".

If 9.1 is true, then we have the following.

Corollary 9.2 (Kiepert-type formula) *If* 9.1 *is true, then, for $n \geqq g$ and $u \in \kappa^{-1}\iota(C)$, we have*

$$\psi_n(u) := \frac{\sigma(nu)}{\sigma_\sharp(u)^{n(d-1)(n-1)+n}} = \pm\, y^{n(n-1)/2}(u)\cdot \left|(x^{a_j}y^{b_j})^{(i-1)}\right|_{2\leqq i,j\leqq n}(u),$$

where ${}^{(i)}$ means $(\mathrm{d}/\mathrm{d}u_{\langle 1\rangle})^i$, and the signature "$\pm$" coincide with the top signature in the right hand side of

$$S_{\natural^n} = \pm \left| U_{w_{g+1-i}-\mathrm{aw}(n)+1-j} \right|_{1\leqq i,j\leqq n}.$$

Especially, if $n \geqq g$, then it is always to be "$+$". Here, the determinant is of size $(n-1)\times(n-1)$.

While we prove this for the case $(d, s) = (5, 6)$ in Section 21, general case is proved similarly.

Theorem 9.3 *The Conjecture* 9.1 *and* 9.2*) are true for $(d, s) = (2, 2g + 1)$ for any $g \geqq 0$, $(3, 4)$, $(3, 5)$, $(5, 6)$, etcetera.*

References are [**Ô02**], [**Ô04**], [**Ô05**], and [**Ô-trig**]. For the case $(d, s) = (5, 6)$, they are proved in the following Sections.

10. The peeling method.

We explain how to read Tables form 22.3 to 22.13. Roughly speaking,

—. Each number in the 1st column indicates the Sato weight of the index of the corresponding function in the 2nd column.

—. Each $\langle ij\cdots\ell\,\rangle$ in the second column means the function $\sigma_{\langle ij\cdots\ell\rangle}$.

—. In each of the 3rd column, if we have only one number there, the number means the largest k such that the function in the 2nd column vanishes completely on $\Theta^{[k]}$. If we have three numbers k, $\langle jk\cdots m\rangle$, and c there in this order, this means the function in the 2nd column equals to c times of $\sigma_{\langle jk\cdots m\rangle}$ of $\Theta^{[k]}$.

For instance, the top of Table 22.4 is as follows:

Weight	Function	Stratum	$\langle 1111\rangle$	$\langle 211\rangle$	$\langle 22\rangle$	$\langle 31\rangle$	$\langle 4\rangle$
1	$\langle 1\rangle$	8	1/3!	2/2!/2	0	1/1!/3	0

The index $\langle 1\rangle$ in the second box means the function $\sigma_{\langle 1\rangle}$. While this function is derivation by $u_{\langle 1\rangle}$, we have its Sato weight 1 in the 1st box. The third box means

$$\sigma_{\langle 1\rangle}(u)=0 \quad (\text{for} \ \ u\in\Theta^{[8]}). \tag{10.1}$$

The following five boxes means that the terms of ${u_{\langle 1\rangle}}^3$ in the expansion of $\sigma_{\langle 1\rangle}(u^{[7]}+v)$ $(v\in\Theta^{[1]})$ with respect to $v_{\langle 1\rangle}$ is of the form

$$\sigma_{\langle 1\rangle}(u^{[7]}+v)=\cdots+\big(\tfrac{1}{3!}\sigma_{\langle 1111\rangle}+\tfrac{2}{3!\cdot 2}\sigma_{\langle 211\rangle}+\tfrac{1}{1!\cdot 3}\sigma_{\langle 3\rangle}\big)(u^{[7]}){u_{\langle 1\rangle}}^3+\cdots. \tag{10.2}$$

Another example is the sixth column of Table 22.4:

Weight	Function	Stratum	$\langle 1111\rangle$	$\langle 211\rangle$	$\langle 22\rangle$	$\langle 31\rangle$	$\langle 4\rangle$
3	$\langle 21\rangle$	$7,\langle 3\rangle,-1$	0	1/1!	0	0	0

The $\langle 21\rangle$ in the second box means the function $\sigma_{\langle 21\rangle}$. Whole this function is obtained by $\partial^2/\partial u_{\langle 2\rangle}\partial u_{\langle 1\rangle}$ from $\sigma(u)$, the number 3 in the 1st box is the sum of Sato weight of $u_{\langle 2\rangle}$ and of $u_{\langle 1\rangle}$. The 3rd box means that

$$\sigma_{\langle 21\rangle}(u)=(-1)\times\sigma_{\langle 3\rangle}(u) \quad (\text{for} \ \ u\in\Theta^{[7]}) \tag{10.3}$$

The following boxes mean the terms of ${u_{\langle 1\rangle}}^3$ in the expansion of $\sigma_{\langle 111\rangle}(u^{[6]}+v)$ $(v\in\Theta^{[1]})$ with respect to $v_{\langle 1\rangle}$ is of the form

$$\sigma_{\langle 21\rangle}(u^{[6]}+v)=\cdots+\tfrac{1}{1!}\sigma_{\langle 211\rangle}(u^{[6]}){v_{\langle 1\rangle}}^3+\cdots. \tag{10.4}$$

We shall be attentive mostly on the 3rd column and the last row. The other columns are written easily. Especially, the 5th column and its following columns are written only by knowledge of Taylor expansions

In each Table from 22.1 to 22.13, the last row means the linear relation(s) of the null space of the matrix given by the Table.

11. Derivative of the sigma attached to the first stratum.

Proposition 11.1 *For the curve C with $(d,s)=(5,6)$ (genus $g=10$), the hypothesis* 9.6 *is valid. Namely, for $\sharp=\langle 4,8,13\rangle$, we have the following statements*:

(1) *$\sigma_\sharp(u)$ is not a zero-function on $\kappa^{-1}(\Theta^{[1]})$.*

(2) *Suppose $v\in\kappa^{-1}(\Theta^{[1]})$. Then $\sigma_\sharp(v)=0 \iff v\in\Lambda$.*

(3) *If $u\in\kappa^{-1}(\Theta^{[1]})$, then*

$$\sigma_\sharp(u+\ell)=\sigma_\sharp(u)\exp[L(u+\tfrac{1}{2},\ell)+\xi(\ell)],$$

where $L(\cdot\,,\cdot)$ and $\xi(\cdot)$ are those of (4.11).

(4) *If $v\in\kappa^{-1}\iota(C)$, then*

$$\sigma(v)=-{v_{\langle 1\rangle}}^{10}+\big(d^\circ(v_{\langle 1\rangle})\geqq 11\big).$$

Proof. Let $Q\in C$ be the corresponding point to a generic $v\in\kappa^{-1}(\Theta^{[1]})$ modulo Λ. The Brill-Noether matrix $B(D)$ associated to $D=Q+8\infty$ is given by

$$\begin{bmatrix}
\varGamma^1\omega_1(v) & \varGamma^1\omega_2(v) & \varGamma^1\omega_3(v) & \varGamma^1\omega_4(v) & \varGamma^1\omega_5(v) & \varGamma^1\omega_6(v) & \varGamma^1\omega_7(v) & \varGamma^1\omega_8(v) & \varGamma^1\omega_9(v) & \varGamma^1\omega_{10}(v)\\
0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0
\end{bmatrix}, \tag{11.2}$$

and $\operatorname{rank}B(D)=7$. Thus, we have $\dim\varGamma(C,\mathcal{O}(D))=3$ by 6.5. Hence $\sigma_{\langle j_1 j_2\rangle}(v)=0$ for any j_1, j_2 by 6.1. Therefore, the translational relation holds for $\sigma_{\langle 13,8,4\rangle}(v)$ on $\kappa^{-1}(\Theta^{[1]})$, The principle of arguments, that is a calculation of the integral around a fundamental polygon obtained from the Riemann surface associated to C after taking logarithm for the translational relation, shows that there are exactly 10 zeroes with counting their multiplicities Since $S_{\langle 13,8,4\rangle}(v)=-{v_{\langle 1\rangle}}^{10}$, we see $\sigma_{\langle 13,8,4\rangle}(v)$ is not a zero-function and its zeroes are only at the points in Λ. $\square$

12. The formula for $n = 2$

Proposition 12.1 *The divisor of the entire function* $v \mapsto \sigma_\flat(u^{(1)} + v)$ *on* $\kappa^{-1}(\Theta^{[1]})$ *is well-defined modulo* Λ. *This function has zeroes of order* 6 *at each point in* Λ, *zeroes of order* 1 *at* $[\zeta^j]u^{(1)}$ $(j = 1, \cdots, 4)$ *modulo* Λ, *and no other zeroes elsewhere.*

Proof. We denote simply $u^{(1)} = u$.
Step 1. Let P_1 and $Q \in C$ are points on C corresponding to $u = u^{(1)}$ and v modulo Λ. The Brill-Noether matrix $B(D)$ associated to the divisor $D = P_1 + Q + 7\infty$ is given by

$$(12.2)\quad \begin{bmatrix} \Gamma^1\omega_1(u) & \Gamma^1\omega_2(u) & \Gamma^1\omega_3(u) & \Gamma^1\omega_4(u) & \Gamma^1\omega_5(u) & \Gamma^1\omega_6(u) & \Gamma^1\omega_7(u) & \Gamma^1\omega_8(u) & \Gamma^1\omega_9(u) & \Gamma^1\omega_{10}(u) \\ \Gamma^1\omega_1(v) & \Gamma^1\omega_2(v) & \Gamma^1\omega_3(v) & \Gamma^1\omega_4(v) & \Gamma^1\omega_5(v) & \Gamma^1\omega_6(v) & \Gamma^1\omega_7(v) & \Gamma^1\omega_8(v) & \Gamma^1\omega_9(v) & \Gamma^1\omega_{10}(v) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\operatorname{rank} B(D) = 7$. By 7.5 and 7.1 shows that $\sigma_{\langle j_1 j_2\rangle}(u+v) = 0$ for any j_1 and j_2. This, however, is proved by the peeling method. Therefore the translational relation holds for $\sigma_{\langle 973\rangle}(u+v)$. The principle of arguments for the logarithm of the translational relation shows that the zeroes of the function $v \mapsto \sigma_{\langle 973\rangle}(u+v)$ on $\kappa^{-1}\iota(C)$ has exactly $g = 10$ zeroes modulo Λ with their multiplicity.
Step 2. We consider the function

$$(12.7)\quad \frac{\sigma_\flat(u+v)\sigma_\flat(u+[\zeta]v)\sigma_\flat(u+[\zeta^2]v)\sigma_\flat(u+[\zeta^3]v)\sigma_\flat(u+[\zeta^4]v)}{\sigma_\sharp(u)^5\sigma_\sharp(v)^5}$$

of $u \in \kappa^{-1}\iota(C)$ and $v \in \kappa^{-1}\iota(C)$. By 11.1, the function (12.7), as a function of v (resp. of u), has only pole at $u \in \Lambda$ (resp. $v \in \Lambda$), and its order is $10\times 5 - 6\times 5 = 20$. Since every factors of (12.7) satisfy the translational relation, the function (12.7) of u and v is periodic with respect to Λ. If a factor, for example, $v \mapsto \sigma_\flat(u+[\zeta]v)$ vanishes at v_0, then $[\zeta^k]v_0$ is a zero of $v \mapsto \sigma_\flat(u+[\zeta^{1-k}]v)$. Hence (12.7), as a function of u (resp. v) on C, has divisor that is linearly equivalent to 0. Here we have used the relation $1+\zeta+\zeta^2+\zeta^3+\zeta^4 = 0$. Therefore, Abel-Jacobi theorem shows that (12.7) is a rational function of $x(u)$, $y(u)$, $x(v)$, and $y(v)$. If we consider their poles, we see that (12.7) is a polynomial of $x(u)$, $y(u)$, $x(v)$, and $y(v)$. Moreover (12.7) is invariant under $v \mapsto [\zeta^\nu]v$ (resp. $u \mapsto [\zeta^\nu]u$), and $u \leftrightarrow v$. It is homogeneous with respect to Sato weight. Thus, (12.7) is a symmetric polynomial of $x(v)$ and $x(u)$ of degree 4.
Step 3. By (2.6), we see

$$(12.8)\quad S_{\langle 973\rangle}(u+v) = {u_{\langle 1\rangle}}^{10}{v_{\langle 1\rangle}}^{6} + {u_{\langle 1\rangle}}^{9}{v_{\langle 1\rangle}}^{7} + {u_{\langle 1\rangle}}^{8}{v_{\langle 1\rangle}}^{8} + {u_{\langle 1\rangle}}^{7}{v_{\langle 1\rangle}}^{9} + {u_{\langle 1\rangle}}^{6}{v_{\langle 1\rangle}}^{10}.$$

Therefore, $\sigma_\flat(u+v)=\sigma_{\langle 973\rangle}(u+v)$ is not a constant function, and has only pole at the origins of order 6.

<u>*Step* 4</u>. If we consider (12.7) when $u=v$, namely,

$$u\mapsto\frac{\sigma_\flat(2u)\sigma_\flat(u+[\zeta]u)\sigma_\flat(u+[\zeta^2]u)\sigma_\flat(u+[\zeta^3]u)\sigma_\flat(u+[\zeta^4]u)}{\sigma_\sharp(u)^{10}}.\tag{12.9}$$

The 1st factor of the numerator of this has of degree 16 by (12.8), and the other factors in the numerator has of degree at least 26. Hence, (12.9) is a power series of $u_{\langle 1\rangle}$ of degree at least $(16+26\times 4)-10\times 10=120-100=20$. Since this is a polynomial of $x(u)$, it must be identically 0. This argument shows that the function

$$v\mapsto\frac{\sigma_\flat(u+v)\sigma_\flat(u+[\zeta]v)\sigma_\flat(u+[\zeta^2]v)\sigma_\flat(u+[\zeta^3]v)\sigma_\flat(u+[\zeta^4]v)}{\sigma_\sharp(u)^5\sigma_\sharp(v)^5}\tag{12.10}$$

vanishes at $v=u$.

<u>*Step* 5</u>. By our argument above, at least one factor of the numerator of (12.9) except the 1st factor vanishes when $v=u$. Suppose that

$$v\mapsto\sigma_\flat(u+[\zeta]v)\tag{12.11}$$

is such a factor. This means that for any value of $u_{\langle 1\rangle}$, we have

$$\begin{aligned}0&=\sigma_\flat(u+[\zeta]v)\big|_{v=u}\\&=\big(\sigma_\flat(u+[\zeta]v)-S_\flat(u+[\zeta]v)\big)\big|_{v=u}\\&=\big(c_{11,15}\lambda_1^2{u_{\langle 1\rangle}}^{15}(\zeta v_{\langle 1\rangle})^{11}+\cdots\big)\big|_{v=u}\quad(c_{k,\ell}\in\mathbf{Q}).\end{aligned}\tag{12.12}$$

Because this is an equation of indeterminates $\lambda_1,\ \cdots,\ \lambda_6,\ u_{\langle 1\rangle}$, it holds also when we change ζ by ζ^h $(h=2,\ 3,\ 4)$. Please do not confuse that this does not imply $c_{k,\ell}=0$ in (12.12). For example, we shall note that

$${u_{\langle 1\rangle}}^{15}{v_{\langle 1\rangle}}^{11}={u_{\langle 1\rangle}}^{14}{v_{\langle 1\rangle}}^{12}={u_{\langle 1\rangle}}^{13}{v_{\langle 1\rangle}}^{13}=\cdots\tag{12.13}$$

when $v=u$. Anyway, we have proved that (12.11) vanishes at 4 points $v=u$, $[\zeta]u$, $[\zeta^2]u$, $[\zeta^3]u$ modulo Λ. This fact shows that $v\mapsto\sigma_\flat(u+[\zeta^j]v)$ also vanishes at 4 points $v=[\zeta^i]u$ $(i+j\not\equiv 0 \bmod 5)$ modulo Λ. Summing up our arguments, we see that (12.11) has pole only at $v=(0,\cdots,0)$ modulo Λ, whose order is 6, and 4 zeroes at $v=[\zeta]u$, $[\zeta^2]u$, $[\zeta^3]u$, $[\zeta^4]u$ modulo Λ of order 1. The other factors of numerator in (12.7) has the same property. $\square$

By the proof above we see the assertion 7.7 holds for $n=2$. Now we know that (12.7) above is no other than $\big(x(u)-x(v)\big)^4$, so that we have proved the following.

Corollary 12.14. *Suppose u and v are two variables on $\kappa^{-1}\iota(C)$. Then we have*

$$\frac{\sigma_\flat(u+v)\sigma_\flat(u+[\zeta]v)\sigma_\flat(u+[\zeta^2]v)\sigma_\flat(u+[\zeta^3]v)\sigma_\flat(u+[\zeta^4]v)}{\sigma_\sharp(u)^5\sigma_\sharp(v)^5}=\begin{vmatrix}1 & x(u)\\ 1 & x(v)\end{vmatrix}^4.$$

13. The formula for $n=10$

Proposition 13.1 *We have the following equation*:

$$\frac{\sigma(u^{(1)}+u^{(2)}+\cdots+u^{(9)}+u^{(10)})\prod_{1\leqq i<j\leqq 10}\prod_{\nu=1}^{4}\sigma_\flat(u^{(i)}+[\zeta^\nu]u^{(j)})}{\prod_{j=1}^{10}\sigma_\sharp(u^{(j)})^{37}}$$

$$=\begin{vmatrix}
1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & y^2(u^{(1)}) & x^3(u^{(1)}) & x^2y(u^{(1)}) & xy^2(u^{(1)}) & y^3(u^{(1)})\\
1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & y^2(u^{(2)}) & x^3(u^{(2)}) & x^2y(u^{(2)}) & xy^2(u^{(2)}) & y^3(u^{(2)})\\
1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & xy(u^{(3)}) & y^2(u^{(3)}) & x^3(u^{(3)}) & x^2y(u^{(3)}) & xy^2(u^{(3)}) & y^3(u^{(3)})\\
1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & xy(u^{(4)}) & y^2(u^{(4)}) & x^3(u^{(4)}) & x^2y(u^{(4)}) & xy^2(u^{(4)}) & y^3(u^{(4)})\\
1 & x(u^{(5)}) & y(u^{(5)}) & x^2(u^{(5)}) & xy(u^{(5)}) & y^2(u^{(5)}) & x^3(u^{(5)}) & x^2y(u^{(5)}) & xy^2(u^{(5)}) & y^3(u^{(5)})\\
1 & x(u^{(6)}) & y(u^{(6)}) & x^2(u^{(6)}) & xy(u^{(6)}) & y^2(u^{(6)}) & x^3(u^{(6)}) & x^2y(u^{(6)}) & xy^2(u^{(6)}) & y^3(u^{(6)})\\
1 & x(u^{(7)}) & y(u^{(7)}) & x^2(u^{(7)}) & xy(u^{(7)}) & y^2(u^{(7)}) & x^3(u^{(7)}) & x^2y(u^{(7)}) & xy^2(u^{(7)}) & y^3(u^{(7)})\\
1 & x(u^{(8)}) & y(u^{(8)}) & x^2(u^{(8)}) & xy(u^{(8)}) & y^2(u^{(8)}) & x^3(u^{(8)}) & x^2y(u^{(8)}) & xy^2(u^{(8)}) & y^3(u^{(8)})\\
1 & x(u^{(9)}) & y(u^{(9)}) & x^2(u^{(9)}) & xy(u^{(9)}) & y^2(u^{(9)}) & x^3(u^{(9)}) & x^2y(u^{(9)}) & xy^2(u^{(9)}) & y^3(u^{(9)})\\
1 & x(u^{(10)}) & y(u^{(10)}) & x^2(u^{(10)}) & xy(u^{(10)}) & y^2(u^{(10)}) & x^3(u^{(10)}) & x^2y(u^{(10)}) & xy^2(u^{(10)}) & y^3(u^{(10)})
\end{vmatrix}$$

$$\cdot\begin{vmatrix}
1 & x(u^{(1)}) & x^2(u^{(1)}) & x^3(u^{(1)}) & x^4(u^{(1)}) & x^5(u^{(1)}) & x^6(u^{(1)}) & x^7(u^{(1)}) & x^8(u^{(1)}) & x^9(u^{(1)})\\
1 & x(u^{(2)}) & x^2(u^{(2)}) & x^3(u^{(2)}) & x^4(u^{(2)}) & x^5(u^{(2)}) & x^6(u^{(2)}) & x^7(u^{(2)}) & x^8(u^{(2)}) & x^9(u^{(2)})\\
1 & x(u^{(3)}) & x^2(u^{(3)}) & x^3(u^{(3)}) & x^4(u^{(3)}) & x^5(u^{(3)}) & x^6(u^{(3)}) & x^7(u^{(3)}) & x^8(u^{(3)}) & x^9(u^{(3)})\\
1 & x(u^{(4)}) & x^2(u^{(4)}) & x^3(u^{(4)}) & x^4(u^{(4)}) & x^5(u^{(4)}) & x^6(u^{(4)}) & x^7(u^{(4)}) & x^8(u^{(4)}) & x^9(u^{(4)})\\
1 & x(u^{(5)}) & x^2(u^{(5)}) & x^3(u^{(5)}) & x^4(u^{(5)}) & x^5(u^{(5)}) & x^6(u^{(5)}) & x^7(u^{(5)}) & x^8(u^{(5)}) & x^9(u^{(5)})\\
1 & x(u^{(6)}) & x^2(u^{(6)}) & x^3(u^{(6)}) & x^4(u^{(6)}) & x^5(u^{(6)}) & x^6(u^{(6)}) & x^7(u^{(6)}) & x^8(u^{(6)}) & x^9(u^{(6)})\\
1 & x(u^{(7)}) & x^2(u^{(7)}) & x^3(u^{(7)}) & x^4(u^{(7)}) & x^5(u^{(7)}) & x^6(u^{(7)}) & x^7(u^{(7)}) & x^8(u^{(7)}) & x^9(u^{(7)})\\
1 & x(u^{(8)}) & x^2(u^{(8)}) & x^3(u^{(8)}) & x^4(u^{(8)}) & x^5(u^{(8)}) & x^6(u^{(8)}) & x^7(u^{(8)}) & x^8(u^{(8)}) & x^9(u^{(8)})\\
1 & x(u^{(9)}) & x^2(u^{(9)}) & x^3(u^{(9)}) & x^4(u^{(9)}) & x^5(u^{(9)}) & x^6(u^{(9)}) & x^7(u^{(9)}) & x^8(u^{(9)}) & x^9(u^{(9)})\\
1 & x(u^{(10)}) & x^2(u^{(10)}) & x^3(u^{(10)}) & x^4(u^{(10)}) & x^5(u^{(10)}) & x^6(u^{(10)}) & x^7(u^{(10)}) & x^8(u^{(10)}) & x^9(u^{(10)})
\end{vmatrix}^3.$$

Proof. We denote simply $u^{(10)}=v$, and consider the two sides by regarding as functions of v mainly. A basic property of the function $\mathbf{C}^g \ni u \mapsto \sigma(u)$ ([**FK1**], p.290, Theorem and p.291, Theorem; or [**FK2**], p.308, Theorem and p.310, Theorem) shows that the divisor of the zeroes of the left hand side is linear equivalent to 0 (in $\mathrm{Pic}^0(C)$). The relation 5.1 of the Schur-Weierstrass polynomial $S(u)$ and $\sigma(u)$, this function $\sigma(u)$ is not a zero function 0. Since the left hand side is a periodic function of $v\in\kappa^{-1}\iota(C)$ (resp. $v\in\kappa^{-1}\iota(C)$ for $j=1,\cdots,9$) because of the translational relation, Abel's theorem ([**FK2**], p.93, III.6.3, Theorem) states that the left hand side is a rational function of $x(u^{(j)})$, $y(u^{(j)})$ $(j=1,\cdots,9)$, and $x(v)$, $y(v)$. Proposition 11.1 states that the left hand side has only pole at the points in Λ, it must be a polynomial of $x(u^{(j)})$, $y(u^{(j)})$ $(j=1,\cdots,9)$ and $x(v)$, $y(v)$. The order of the pole at $v=(0,0,\cdots,0)$ is

$$-(1+6\times 9\times 4-10\times 37)=-153 \tag{13.1}$$

by 11.1 and 11.2, The left hand side as a function of v is coincide with the right hand side up to non-zero multiplicative constant. Indeed, each pole of the left hand side as a function of v is of order $-5\times 9\times 3-6\times 3=-153$. This coincide with that of the right hand side. The right hand side as a function of v has zeroes at each

$v = u^{(i)}$ of order 4, and at each $v = [\zeta^j]u^{(i)}$ $(j = 1, \cdots, 4)$ of order 3. There are $4 \times 9 + 3 \times 9 \times 4 = 144$ zeroes totally. This zeroes coincide with those of the factors in the products of $\sigma_\flat$ in the numerator. Let $\alpha^{(k)}$ $(k = 1, \cdots, 9)$ be the remaining $153 - 144 = 9$ zeroes of the right hand side. We can prove that the left hand side also has zeroes at every $v = \beta^{(k)}$ of order 1 as follows. Abel's theorem shows that

$$
\begin{aligned}
&\alpha^{(1)} + \alpha^{(2)} + \cdots + \alpha^{(9)} \\
&\quad + 4u^{(1)} + 3([\zeta]u^{(1)} + [\zeta^2]u^{(1)} + [\zeta^3]u^{(1)} + [\zeta^4]u^{(1)}) \\
&\qquad + 4u^{(2)} + 3([\zeta]u^{(2)} + [\zeta^2]u^{(2)} + [\zeta^3]u^{(2)} + [\zeta^4]u^{(2)}) \\
&\qquad\quad + \quad \cdots\cdots\cdots \\
&\qquad\qquad + 4u^{(9)} + 3([\zeta]u^{(9)} + [\zeta^2]u^{(9)} + [\zeta^3]u^{(9)} + [\zeta^4]u^{(9)}) \in \Lambda.
\end{aligned} \tag{13.2}
$$

By using the relation $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$, we have

$$\alpha^{(1)} + \alpha^{(2)} + \cdots + \alpha^{(9)} + u^{(1)} + u^{(2)} + \cdots + u^{(9)} \in \Lambda. \tag{13.3}$$

Therefore, we see that

$$\sigma(u^{(1)} + u^{(2)} + \cdots + u^{(9)} + v) \tag{13.4}$$

has the same zeroes with

$$\sigma(v - \alpha^{(1)} - \alpha^{(2)} - \cdots - \alpha^{(9)}). \tag{13.5}$$

This function has 10 zeroes at $v = (0, 0, \cdots, 0)$, $\alpha^{(1)}$, $\cdots$, $\alpha^{(9)}$ by 4.12(3). The principle of arguments implies as in the proof of 12.1 that all these 10 zeroes are of order 1. Hence, the left hand side is a product of the right hand side and a function, say $a(u^{(1)}, u^{(2)}, \cdots, u^{(9)})$, which is independent of v. If we regard the left hand side as a function of each of $u^{(j)}$, we see that it has pole only at $u = (0, 0, \cdots, 0)$ modulo Λ. We know that it already has of order 153 which is the order of the pole of the right hand side. Thus the function $a(u^{(1)}, u^{(2)}, \cdots, u^{(9)})$ must be independent of $u^{(j)}$s, and be a constant, say a. Since the relation that we have obtained must be hold for any λ_j, $a(u^{(1)}, u^{(2)}, \cdots, u^{(9)})$ is independent of λ_j. Here, we note that the two sides has the same Sato weight. While we can show that $a = 1$ by calculating the expansions of the two sides, it is tedious for us. We postpone knowing the exact value of a and proceed without determining its value till the finish of the proof of 20.1. Then we might understand that $a = 1$. $\square$

Corollary 13.9 *The function* $v \mapsto \sigma(u^{(1)} + u^{(2)} + u^{(3)} + u^{(4)} + u^{(5)} + u^{(6)} + u^{(7)} + u^{(8)} + u^{(9)} + v)$ *has zeroes at* $(0, 0, \cdots, 0)$ *modulo* Λ*, and at the non-trivial zeroes of the function*

$$v \longmapsto \left|\begin{array}{cccccccccc} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & y^2(u^{(1)}) & x^3(u^{(1)}) & x^2y(u^{(1)}) & xy^2(u^{(1)}) & y^3(u^{(1)}) \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & y^2(u^{(2)}) & x^3(u^{(2)}) & x^2y(u^{(2)}) & xy^2(u^{(2)}) & y^3(u^{(2)}) \\ 1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & xy(u^{(3)}) & y^2(u^{(3)}) & x^3(u^{(3)}) & x^2y(u^{(3)}) & xy^2(u^{(3)}) & y^3(u^{(3)}) \\ 1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & xy(u^{(4)}) & y^2(u^{(4)}) & x^3(u^{(4)}) & x^2y(u^{(4)}) & xy^2(u^{(4)}) & y^3(u^{(4)}) \\ 1 & x(u^{(5)}) & y(u^{(5)}) & x^2(u^{(5)}) & xy(u^{(5)}) & y^2(u^{(5)}) & x^3(u^{(5)}) & x^2y(u^{(5)}) & xy^2(u^{(5)}) & y^3(u^{(5)}) \\ 1 & x(u^{(6)}) & y(u^{(6)}) & x^2(u^{(6)}) & xy(u^{(6)}) & y^2(u^{(6)}) & x^3(u^{(6)}) & x^2y(u^{(6)}) & xy^2(u^{(6)}) & y^3(u^{(6)}) \\ 1 & x(u^{(7)}) & y(u^{(7)}) & x^2(u^{(7)}) & xy(u^{(7)}) & y^2(u^{(7)}) & x^3(u^{(7)}) & x^2y(u^{(7)}) & xy^2(u^{(7)}) & y^3(u^{(7)}) \\ 1 & x(u^{(8)}) & y(u^{(8)}) & x^2(u^{(8)}) & xy(u^{(8)}) & y^2(u^{(8)}) & x^3(u^{(8)}) & x^2y(u^{(8)}) & xy^2(u^{(8)}) & y^3(u^{(8)}) \\ 1 & x(u^{(9)}) & y(u^{(9)}) & x^2(u^{(9)}) & xy(u^{(9)}) & y^2(u^{(9)}) & x^3(u^{(9)}) & x^2y(u^{(9)}) & xy^2(u^{(9)}) & y^3(u^{(9)}) \\ 1 & x(v) & y(v) & x^2(v) & xy(v) & y^2(v) & x^3(v) & x^2y(v) & xy^2(v) & y^3(v) \end{array}\right|,$$

namely, at the 9 *zeroes modulo* Λ *of this determinant except points* $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, $u^{(4)}$, $u^{(5)}$, $u^{(6)}$, $u^{(7)}$, $u^{(8)}$, $u^{(9)}$ *modulo* Λ*, with of order* 1 *for all zeroes. The function claimed above has no other zeroes. Especially, the assertion* 7.7 *holds for* $n = 10$.

14. The formula for $n=9$

Proposition 14.1 *We have the following equation:*

$$\frac{\sigma_{\natural^9}(u^{(1)}+u^{(2)}+\cdots+u^{(9)})\prod_{1\leqq i<j\leqq 9}\prod_{\nu=1}^{4}\sigma_\flat(u^{(i)}+[\zeta^\nu]u^{(j)})}{\prod_{j=1}^{9}\sigma_\sharp(u^{(j)})^{33}}$$

$$=\begin{vmatrix}
1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & y^2(u^{(1)}) & x^3(u^{(1)}) & x^2y(u^{(1)}) & xy^2(u^{(1)})\\
1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & y^2(u^{(2)}) & x^3(u^{(2)}) & x^2y(u^{(2)}) & xy^2(u^{(2)})\\
1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & xy(u^{(3)}) & y^2(u^{(3)}) & x^3(u^{(3)}) & x^2y(u^{(3)}) & xy^2(u^{(3)})\\
1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & xy(u^{(4)}) & y^2(u^{(4)}) & x^3(u^{(4)}) & x^2y(u^{(4)}) & xy^2(u^{(4)})\\
1 & x(u^{(5)}) & y(u^{(5)}) & x^2(u^{(5)}) & xy(u^{(5)}) & y^2(u^{(5)}) & x^3(u^{(5)}) & x^2y(u^{(5)}) & xy^2(u^{(5)})\\
1 & x(u^{(6)}) & y(u^{(6)}) & x^2(u^{(6)}) & xy(u^{(6)}) & y^2(u^{(6)}) & x^3(u^{(6)}) & x^2y(u^{(6)}) & xy^2(u^{(6)})\\
1 & x(u^{(7)}) & y(u^{(7)}) & x^2(u^{(7)}) & xy(u^{(7)}) & y^2(u^{(7)}) & x^3(u^{(7)}) & x^2y(u^{(7)}) & xy^2(u^{(7)})\\
1 & x(u^{(8)}) & y(u^{(8)}) & x^2(u^{(8)}) & xy(u^{(8)}) & y^2(u^{(8)}) & x^3(u^{(8)}) & x^2y(u^{(8)}) & xy^2(u^{(8)})\\
1 & x(u^{(9)}) & y(u^{(9)}) & x^2(u^{(9)}) & xy(u^{(9)}) & y^2(u^{(9)}) & x^3(u^{(9)}) & x^2y(u^{(9)}) & xy^2(u^{(9)})
\end{vmatrix}$$

$$\cdot\begin{vmatrix}
1 & x(u^{(1)}) & x^2(u^{(1)}) & x^3(u^{(1)}) & x^4(u^{(1)}) & x^5(u^{(1)}) & x^6(u^{(1)}) & x^7(u^{(1)}) & x^8(u^{(1)})\\
1 & x(u^{(2)}) & x^2(u^{(2)}) & x^3(u^{(2)}) & x^4(u^{(2)}) & x^5(u^{(2)}) & x^6(u^{(2)}) & x^7(u^{(2)}) & x^8(u^{(2)})\\
1 & x(u^{(3)}) & x^2(u^{(3)}) & x^3(u^{(3)}) & x^4(u^{(3)}) & x^5(u^{(3)}) & x^6(u^{(3)}) & x^7(u^{(3)}) & x^8(u^{(3)})\\
1 & x(u^{(4)}) & x^2(u^{(4)}) & x^3(u^{(4)}) & x^4(u^{(4)}) & x^5(u^{(4)}) & x^6(u^{(4)}) & x^7(u^{(4)}) & x^8(u^{(4)})\\
1 & x(u^{(5)}) & x^2(u^{(5)}) & x^3(u^{(5)}) & x^4(u^{(5)}) & x^5(u^{(5)}) & x^6(u^{(5)}) & x^7(u^{(5)}) & x^8(u^{(5)})\\
1 & x(u^{(6)}) & x^2(u^{(6)}) & x^3(u^{(6)}) & x^4(u^{(6)}) & x^5(u^{(6)}) & x^6(u^{(6)}) & x^7(u^{(6)}) & x^8(u^{(6)})\\
1 & x(u^{(7)}) & x^2(u^{(7)}) & x^3(u^{(7)}) & x^4(u^{(7)}) & x^5(u^{(7)}) & x^6(u^{(7)}) & x^7(u^{(7)}) & x^8(u^{(7)})\\
1 & x(u^{(8)}) & x^2(u^{(8)}) & x^3(u^{(8)}) & x^4(u^{(8)}) & x^5(u^{(8)}) & x^6(u^{(8)}) & x^7(u^{(8)}) & x^8(u^{(8)})\\
1 & x(u^{(9)}) & x^2(u^{(9)}) & x^3(u^{(9)}) & x^4(u^{(9)}) & x^5(u^{(9)}) & x^6(u^{(9)}) & x^7(u^{(9)}) & x^8(u^{(9)})
\end{vmatrix}^3.$$

Proof. In the sequel, we write simply

$$u^{[j]}=u^{(1)}+\cdots+u^{(j)} \tag{14.2}$$

for $u^{(1)},\ \cdots,\ u^{(j)}\in\kappa^{-1}(\Theta^{[1]})$. We expand the formula is 13.9 by $v_{\langle 1\rangle}$. Lemma 4.12 shows that $\sigma(u)$ vanishes on $\Theta^{[9]}$, we have

$$\sigma(u^{[9]}+v)=\sigma_{\langle 1\rangle}(u^{[9]})v_{\langle 1\rangle}+\cdots. \tag{14.3}$$

The expansion of the right hand side of 13.9 by $v_{\langle 1\rangle}$ is coincide with the right hand side of the claimed formula this Proposition because of 3.15. Summing up this with 11.1 and 12.1 gives the desired formula. $\square$

Corollary 14.4 *The function* $v \mapsto \sigma_{\natural^9}(u^{(1)} + u^{(2)} + u^{(3)} + u^{(4)} + u^{(5)} + u^{(6)} + u^{(7)} + u^{(8)} + v)$ *has zeroes only at* $(0, 0, \cdots, 0)$ *modulo* Λ, *and at the non-trivial zeroes of*

$$v \longmapsto \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & y^2(u^{(1)}) & x^3(u^{(1)}) & x^2y(u^{(1)}) & xy^2(u^{(1)}) \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & y^2(u^{(2)}) & x^3(u^{(2)}) & x^2y(u^{(2)}) & xy^2(u^{(2)}) \\ 1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & xy(u^{(3)}) & y^2(u^{(3)}) & x^3(u^{(3)}) & x^2y(u^{(3)}) & xy^2(u^{(3)}) \\ 1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & xy(u^{(4)}) & y^2(u^{(4)}) & x^3(u^{(4)}) & x^2y(u^{(4)}) & xy^2(u^{(4)}) \\ 1 & x(u^{(5)}) & y(u^{(5)}) & x^2(u^{(5)}) & xy(u^{(5)}) & y^2(u^{(5)}) & x^3(u^{(5)}) & x^2y(u^{(5)}) & xy^2(u^{(5)}) \\ 1 & x(u^{(6)}) & y(u^{(6)}) & x^2(u^{(6)}) & xy(u^{(6)}) & y^2(u^{(6)}) & x^3(u^{(6)}) & x^2y(u^{(6)}) & xy^2(u^{(6)}) \\ 1 & x(u^{(7)}) & y(u^{(7)}) & x^2(u^{(7)}) & xy(u^{(7)}) & y^2(u^{(7)}) & x^3(u^{(7)}) & x^2y(u^{(7)}) & xy^2(u^{(7)}) \\ 1 & x(u^{(8)}) & y(u^{(8)}) & x^2(u^{(8)}) & xy(u^{(8)}) & y^2(u^{(8)}) & x^3(u^{(8)}) & x^2y(u^{(8)}) & xy^2(u^{(8)}) \\ 1 & x(v) & y(v) & x^2(v) & xy(v) & y^2(v) & x^3(v) & x^2y(v) & xy^2(v) \end{vmatrix},$$

namely, 9 *zeroes modulo* Λ *of this determinant except for the points* $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, $u^{(4)}$, $u^{(5)}$, $u^{(6)}$, $u^{(7)}$, $u^{(8)}$ *modulo* Λ, *with of order* 1 *for all zeroes. The function claimed above has no other zeroes. Especially, the assertion* 7.7 *holds for* $n = 9$.

15. The formula for $n=8$

Proposition 15.1 *We have the following equation:*

$$\frac{\sigma_{\natural^8}(u^{(1)}+u^{(2)}+\cdots+u^{(8)})\prod_{1\leqq i<j\leqq 8}\prod_{\nu=1}^{4}\sigma_\flat(u^{(i)}+[\zeta^\nu]u^{(j)})}{\prod_{j=1}^{8}\sigma_\sharp(u^{(j)})^{29}}$$
$$= -\begin{vmatrix}
1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & y^2(u^{(1)}) & x^3(u^{(1)}) & x^2y(u^{(1)}) \\
1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & y^2(u^{(2)}) & x^3(u^{(2)}) & x^2y(u^{(2)}) \\
1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & xy(u^{(3)}) & y^2(u^{(3)}) & x^3(u^{(3)}) & x^2y(u^{(3)}) \\
1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & xy(u^{(4)}) & y^2(u^{(4)}) & x^3(u^{(4)}) & x^2y(u^{(4)}) \\
1 & x(u^{(5)}) & y(u^{(5)}) & x^2(u^{(5)}) & xy(u^{(5)}) & y^2(u^{(5)}) & x^3(u^{(5)}) & x^2y(u^{(5)}) \\
1 & x(u^{(6)}) & y(u^{(6)}) & x^2(u^{(6)}) & xy(u^{(6)}) & y^2(u^{(6)}) & x^3(u^{(6)}) & x^2y(u^{(6)}) \\
1 & x(u^{(7)}) & y(u^{(7)}) & x^2(u^{(7)}) & xy(u^{(7)}) & y^2(u^{(7)}) & x^3(u^{(7)}) & x^2y(u^{(7)}) \\
1 & x(u^{(8)}) & y(u^{(8)}) & x^2(u^{(8)}) & xy(u^{(8)}) & y^2(u^{(8)}) & x^3(u^{(8)}) & x^2y(u^{(8)})
\end{vmatrix}$$
$$\cdot\begin{vmatrix}
1 & x(u^{(1)}) & x^2(u^{(1)}) & x^3(u^{(1)}) & x^4(u^{(1)}) & x^5(u^{(1)}) & x^6(u^{(1)}) & x^7(u^{(1)}) \\
1 & x(u^{(2)}) & x^2(u^{(2)}) & x^3(u^{(2)}) & x^4(u^{(2)}) & x^5(u^{(2)}) & x^6(u^{(2)}) & x^7(u^{(2)}) \\
1 & x(u^{(3)}) & x^2(u^{(3)}) & x^3(u^{(3)}) & x^4(u^{(3)}) & x^5(u^{(3)}) & x^6(u^{(3)}) & x^7(u^{(3)}) \\
1 & x(u^{(4)}) & x^2(u^{(4)}) & x^3(u^{(4)}) & x^4(u^{(4)}) & x^5(u^{(4)}) & x^6(u^{(4)}) & x^7(u^{(4)}) \\
1 & x(u^{(5)}) & x^2(u^{(5)}) & x^3(u^{(5)}) & x^4(u^{(5)}) & x^5(u^{(5)}) & x^6(u^{(5)}) & x^7(u^{(5)}) \\
1 & x(u^{(6)}) & x^2(u^{(6)}) & x^3(u^{(6)}) & x^4(u^{(6)}) & x^5(u^{(6)}) & x^6(u^{(6)}) & x^7(u^{(6)}) \\
1 & x(u^{(7)}) & x^2(u^{(7)}) & x^3(u^{(7)}) & x^4(u^{(7)}) & x^5(u^{(7)}) & x^6(u^{(7)}) & x^7(u^{(7)}) \\
1 & x(u^{(8)}) & x^2(u^{(8)}) & x^3(u^{(8)}) & x^4(u^{(8)}) & x^5(u^{(8)}) & x^6(u^{(8)}) & x^7(u^{(8)})
\end{vmatrix}^3$$

Proof. We use the notation (14.2) here. We denote the variable $u^{(9)}$ in 15.1 simply $u^{(9)}=v=(v_{\langle 19\rangle},v_{\langle 14\rangle},\cdots,v_{\langle 1\rangle})$. Paying attention to the fact $\sigma_{\langle 1\rangle}(u^{[8]})=0$ of 14.1, we expand both sides of the equation in 14.1 by $v_{\langle 1\rangle}$. Then we have

$$\sigma_{\langle 1\rangle}(u^{[8]}+v)=\sigma_{\langle 11\rangle}(u^{[8]})v_{\langle 1\rangle}+\cdots. \tag{15.2}$$

Since $v_{\langle 2\rangle}=\frac{1}{2}{v_{\langle 1\rangle}}^2+\cdots$, the expansion of $\sigma(u^{[8]}+v)$ with respect to $v_{\langle 1\rangle}$ is of the form

$$\begin{aligned}\sigma(u^{[8]}+v)=\sigma_{\langle 1\rangle}(u^{[8]})&+\tfrac{1}{2!}\,\sigma_{\langle 11\rangle}(u^{[8]}){v_{\langle 1\rangle}}^2\\&+\tfrac{1}{1!}\,\sigma_{\langle 2\rangle}(u^{[8]})\big(\tfrac{1}{2}{v_{\langle 1\rangle}}^2+\cdots\big)+\cdots.\end{aligned} \tag{15.3}$$

Because $\sigma(u^{[8]}+v)$ is identically 0, the coefficients of the 2nd order terms relate as

$$\tfrac{1}{2!}\,\sigma_{\langle 11\rangle}(u^{[8]})=-\tfrac{1}{2}\,\sigma_{\langle 2\rangle}(u^{[8]}). \tag{15.4}$$

By (15.2) and (15.4), we see

$$\sigma_{\langle 1\rangle}(u^{[8]}+v)=-\sigma_{\langle 2\rangle}(u^{[8]})v_{\langle 1\rangle}+\big(d^\circ(v_{\langle 1\rangle})\geqq 2\big). \tag{15.5}$$

and the desired formula has been obtained. $\square$

Corollary 15.6 *The function* $v \mapsto \sigma_{\natural^8}(u^{(1)} + u^{(2)} + u^{(3)} + u^{(4)} + u^{(5)} + u^{(6)} + u^{(7)} + v)$ *has zeroes only at* $(0, 0, \cdots, 0)$ *modulo* Λ*, and non-trivial zeroes of*

$$v \quad \longmapsto \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & y^2(u^{(1)}) & x^3(u^{(1)}) & x^2y(u^{(1)}) \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & y^2(u^{(2)}) & x^3(u^{(2)}) & x^2y(u^{(2)}) \\ 1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & xy(u^{(3)}) & y^2(u^{(3)}) & x^3(u^{(3)}) & x^2y(u^{(3)}) \\ 1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & xy(u^{(4)}) & y^2(u^{(4)}) & x^3(u^{(4)}) & x^2y(u^{(4)}) \\ 1 & x(u^{(5)}) & y(u^{(5)}) & x^2(u^{(5)}) & xy(u^{(5)}) & y^2(u^{(5)}) & x^3(u^{(5)}) & x^2y(u^{(5)}) \\ 1 & x(u^{(6)}) & y(u^{(6)}) & x^2(u^{(6)}) & xy(u^{(6)}) & y^2(u^{(6)}) & x^3(u^{(6)}) & x^2y(u^{(6)}) \\ 1 & x(u^{(7)}) & y(u^{(7)}) & x^2(u^{(7)}) & xy(u^{(7)}) & y^2(u^{(7)}) & x^3(u^{(7)}) & x^2y(u^{(7)}) \\ 1 & x(v) & y(v) & x^2(v) & xy(v) & y^2(v) & x^3(v) & x^2y(v) \end{vmatrix},$$

namely, the 9 *zeroes modulo* Λ *except the points* $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, $u^{(4)}$, $u^{(5)}$, $u^{(6)}$, $u^{(7)}$ *modulo* Λ*. The function claimed above is no other zeroes elsewhere, and its all zeroes are of order* 1*. Especially, the assertion* 7.7 *holds for* $n = 8$*.*

16. The formula for $n=7$

Proposition 16.1 *We have the following equation:*

$$\frac{\sigma_{\natural^7}(u^{(1)}+u^{(2)}+\cdots+u^{(7)})\prod_{1\leqq i<j\leqq 7}\prod_{\nu=1}^{4}\sigma_\flat(u^{(i)}+[\zeta^\nu]u^{(j)})}{\prod_{j=1}^{7}\sigma_\sharp(u^{(j)})^{25}}$$
$$= \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & y^2(u^{(1)}) & x^3(u^{(1)}) \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & y^2(u^{(2)}) & x^3(u^{(2)}) \\ 1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & xy(u^{(3)}) & y^2(u^{(3)}) & x^3(u^{(3)}) \\ 1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & xy(u^{(4)}) & y^2(u^{(4)}) & x^3(u^{(4)}) \\ 1 & x(u^{(5)}) & y(u^{(5)}) & x^2(u^{(5)}) & xy(u^{(5)}) & y^2(u^{(5)}) & x^3(u^{(5)}) \\ 1 & x(u^{(6)}) & y(u^{(6)}) & x^2(u^{(6)}) & xy(u^{(6)}) & y^2(u^{(6)}) & x^3(u^{(6)}) \\ 1 & x(u^{(7)}) & y(u^{(7)}) & x^2(u^{(7)}) & xy(u^{(7)}) & y^2(u^{(7)}) & x^3(u^{(7)}) \end{vmatrix}$$
$$\times \begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & x^3(u^{(1)}) & x^4(u^{(1)}) & x^5(u^{(1)}) & x^6(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & x^3(u^{(2)}) & x^4(u^{(2)}) & x^5(u^{(2)}) & x^6(u^{(2)}) \\ 1 & x(u^{(3)}) & x^2(u^{(3)}) & x^3(u^{(3)}) & x^4(u^{(3)}) & x^5(u^{(3)}) & x^6(u^{(3)}) \\ 1 & x(u^{(4)}) & x^2(u^{(4)}) & x^3(u^{(4)}) & x^4(u^{(4)}) & x^5(u^{(4)}) & x^6(u^{(4)}) \\ 1 & x(u^{(5)}) & x^2(u^{(5)}) & x^3(u^{(5)}) & x^4(u^{(5)}) & x^5(u^{(5)}) & x^6(u^{(5)}) \\ 1 & x(u^{(6)}) & x^2(u^{(6)}) & x^3(u^{(6)}) & x^4(u^{(6)}) & x^5(u^{(6)}) & x^6(u^{(6)}) \\ 1 & x(u^{(7)}) & x^2(u^{(7)}) & x^3(u^{(7)}) & x^4(u^{(7)}) & x^5(u^{(7)}) & x^6(u^{(7)}) \end{vmatrix}^3 .$$

Proof. We use again the notation (14.2). The variable $u^{(9)}$ is simply denoted by $v=(v_{\langle 19\rangle}, v_{\langle 14\rangle}, \cdots, v_{\langle 1\rangle})$. By 15.1, we see that $\sigma_{\langle 1\rangle}(u^{[8]})=0$. Using this fact and Table 22.2 with the expansion

$$\begin{gathered}\big(\sigma_{\langle 11\rangle}(u^{[7]}+v)=\sigma_{\langle 111\rangle}(u^{[7]})v_{\langle 1\rangle}+\cdots,\big)\\ \sigma_{\langle 2\rangle}(u^{[7]}+v)=\sigma_{\langle 21\rangle}(u^{[7]})v_{\langle 1\rangle}+\cdots,\end{gathered} \tag{16.2}$$

we are going to expand both sides of the equation in 16.1 by $v_{\langle 1\rangle}$. We also use the expansion with respect to $v_{\langle 1\rangle}$ of $\sigma(u^{[7]}+v)$ by using $u_{\langle 2\rangle}=\frac{1}{2}{v_{\langle 1\rangle}}^2+\cdots$, namely,

$$\begin{aligned}\sigma(u^{[7]}+v)=\sigma_{\langle 1\rangle}(u^{[7]})v_{\langle 1\rangle}&+\tfrac{1}{2!}\,\sigma_{\langle 11\rangle}(u^{[7]}){v_{\langle 1\rangle}}^2\\ &+\tfrac{1}{1!}\,\sigma_{\langle 2\rangle}(u^{[7]}+v)\big(\tfrac{1}{2}{v_{\langle 1\rangle}}^2+\cdots\big)+\cdots.\end{aligned} \tag{16.3}$$

Since $\sigma(u^{[7]}+u)=0$ identically, the terms of degree 3 have the relation

$$\tfrac{1}{3!}\,\sigma_{\langle 111\rangle}(u^{[7]})+\tfrac{1}{2}\,\sigma_{\langle 21\rangle}(u^{[7]})+\tfrac{1}{3}\,\sigma_{\langle 3\rangle}(u^{[7]})=0. \tag{16.4}$$

Moreover, if we expand $\sigma_{\langle 1\rangle}(u^{[7]}+v)$ with respect to $v_{\langle 1\rangle}$ by using $v_{\langle 2\rangle}=\frac{1}{2}{v_{\langle 1\rangle}}^2+\cdots$, then we have

$$\begin{aligned}\sigma_{\langle 1\rangle}(u^{[7]}+u)=\sigma_{\langle 11\rangle}(u^{[7]})v_{\langle 1\rangle}&+\tfrac{1}{2!}\,\sigma_{\langle 111\rangle}(u^{[7]}){v_{\langle 1\rangle}}^2\\ &+\tfrac{1}{2!}\,\sigma_{\langle 21\rangle}(u^{[7]})\big(\tfrac{1}{2}{v_{\langle 1\rangle}}^2+\cdots\big)+\cdots.\end{aligned} \tag{16.5}$$

Because $\sigma(u^{[7]}+u)=0$ identically, the terms of degree 2 have the relation

$$\tfrac{1}{2!}\,\sigma_{\langle 111\rangle}(u^{[7]})+\tfrac{1}{2!}\,\sigma_{\langle 21\rangle}(u^{[7]})=0. \tag{16.6}$$

The equalities (16.4) and (16.6) give

$$\sigma_{\langle 111\rangle}(u^{[7]}) = \sigma_{\langle 21\rangle}(u^{[7]}) = \sigma_{\langle 3\rangle}(u^{[7]}). \tag{16.7}$$

Summing up the results above, we have the Table 22.3. Adding 9.1 and 10.1 with this, we have the desired statement. $\square$

Remark 16.8 The relation (16.7) is consistent with the uniqueness of functions on the 8th stratum up to multiplicative constants which have the translational relation, and with 6.4.

Corollary 16.9 *The function* $v \mapsto \sigma_{\natural^7}(u^{(1)} + u^{(2)} + u^{(3)} + u^{(4)} + u^{(5)} + u^{(6)} + v)$ *has zeroes only at points* $(0, 0, \cdots, 0)$ *modulo* Λ*, and the* 9 *zeroes modulo* Λ *of the determinant*

$$v \quad \longmapsto \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & y^2(u^{(1)}) & x^3(u^{(1)}) \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & y^2(u^{(2)}) & x^3(u^{(2)}) \\ 1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & xy(u^{(3)}) & y^2(u^{(3)}) & x^3(u^{(3)}) \\ 1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & xy(u^{(4)}) & y^2(u^{(4)}) & x^3(u^{(4)}) \\ 1 & x(u^{(5)}) & y(u^{(5)}) & x^2(u^{(5)}) & xy(u^{(5)}) & y^2(u^{(5)}) & x^3(u^{(5)}) \\ 1 & x(u^{(6)}) & y(u^{(6)}) & x^2(u^{(6)}) & xy(u^{(6)}) & y^2(u^{(6)}) & x^3(u^{(6)}) \\ 1 & x(v) & y(v) & x^2(v) & xy(v) & y^2(v) & x^3(v) \end{vmatrix}$$

except the trivial zeroes $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, $u^{(4)}$, $u^{(5)}$, $u^{(6)}$ *modulo* Λ *of this. The function claimed above has no other zeroes, and its all zeroes are of order* 1*. Especially, the assertion* 7.7 *holds for* $n = 7$*.*

17. The formula for $n=6$

Proposition 17.1 *We have the following equation*:

$$\frac{\sigma_{\natural^6}(u^{(1)}+u^{(2)}+\cdots+u^{(6)})\prod_{1\leqq i<j\leqq 6}\prod_{\nu=1}^{4}\sigma_\flat(u^{(i)}+[\zeta^\nu]u^{(j)})}{\prod_{j=1}^{6}\sigma_\sharp(u^{(j)})^{21}}$$

$$= -\begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & y^2(u^{(1)}) \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & y^2(u^{(2)}) \\ 1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & xy(u^{(3)}) & y^2(u^{(3)}) \\ 1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & xy(u^{(4)}) & y^2(u^{(4)}) \\ 1 & x(u^{(5)}) & y(u^{(5)}) & x^2(u^{(5)}) & xy(u^{(5)}) & y^2(u^{(5)}) \\ 1 & x(u^{(6)}) & y(u^{(6)}) & x^2(u^{(6)}) & xy(u^{(6)}) & y^2(u^{(6)}) \end{vmatrix}$$

$$\times \begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & x^3(u^{(1)}) & x^4(u^{(1)}) & x^5(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & x^3(u^{(2)}) & x^4(u^{(2)}) & x^5(u^{(2)}) \\ 1 & x(u^{(3)}) & x^2(u^{(3)}) & x^3(u^{(3)}) & x^4(u^{(3)}) & x^5(u^{(3)}) \\ 1 & x(u^{(4)}) & x^2(u^{(4)}) & x^3(u^{(4)}) & x^4(u^{(4)}) & x^5(u^{(4)}) \\ 1 & x(u^{(5)}) & x^2(u^{(5)}) & x^3(u^{(5)}) & x^4(u^{(5)}) & x^5(u^{(5)}) \\ 1 & x(u^{(6)}) & x^2(u^{(6)}) & x^3(u^{(6)}) & x^4(u^{(6)}) & x^5(u^{(6)}) \end{vmatrix}^3.$$

Proof. We remain using the notation (14.2). We again write simply $v=u^{(7)}$, and We are going to expand both side of the equation in 16.1 by $v_{\langle 1\rangle}=u^{(7)}_{\langle 1\rangle}$. Since the Table 22.3 indicates all the derivative by the coordinates of total Sato weight at most 2 of $\sigma(u)$ on $\Theta^{[6]}$ are identically 0, we see that

$$\sigma_{\langle 3\rangle}(u^{[6]}+v)=\sigma_{\langle 31\rangle}(u^{[6]})\tfrac{1}{3!}v_{\langle 1\rangle}+\cdots. \tag{17.2}$$

This and the Table 22.4 shows that

$$-\sigma_{\langle 1111\rangle}(u^{[6]})=\sigma_{\langle 211\rangle}(u^{[6]})=-\sigma_{\langle 22\rangle}(u^{[6]})=-\sigma_{\langle 31\rangle}(u^{[6]})=\sigma_{\langle 4\rangle}(u^{[6]}). \tag{17.3}$$

Now we have the desired equation. □

Corollary 17.4 *The function $v\mapsto\sigma_{\natural^6}(u^{(1)}+u^{(2)}+u^{(3)}+u^{(4)}+u^{(5)}+v)$ has the zeroes only at the points $(0,0,\cdots,0)$ modulo Λ of order 3 and at the 7 zeroes of order 1 modulo Λ of the determinant*

$$v \longmapsto \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & y^2(u^{(1)}) \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & y^2(u^{(2)}) \\ 1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & xy(u^{(3)}) & y^2(u^{(3)}) \\ 1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & xy(u^{(4)}) & y^2(u^{(4)}) \\ 1 & x(u^{(5)}) & y(u^{(5)}) & x^2(u^{(5)}) & xy(u^{(5)}) & y^2(u^{(5)}) \\ 1 & x(v) & y(v) & x^2(v) & xy(v) & y^2(v) \end{vmatrix},$$

except the trivial zeroes $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, $u^{(4)}$, $u^{(5)}$ modulo Λ of this determinant. The function claimed above has no other zeroes. Especially, the assertion 7.7 *holds for $n=6$.*

18. The formula for $n = 5$

Proposition 18.1 *We have the following equation*:

$$\frac{\sigma_{\natural^5}(u^{(1)}+u^{(2)}+\cdots+u^{(5)})\prod_{1\leqq i<j\leqq 5}\prod_{\nu=1}^{4}\sigma_\flat(u^{(i)}+[\zeta^\nu]u^{(j)})}{\prod_{j=1}^{5}\sigma_\sharp(u^{(j)})^{17}}$$
$$=\begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) \\ 1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & xy(u^{(3)}) \\ 1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & xy(u^{(4)}) \\ 1 & x(u^{(5)}) & y(u^{(5)}) & x^2(u^{(5)}) & xy(u^{(5)}) \end{vmatrix}\begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & x^3(u^{(1)}) & x^4(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & x^3(u^{(2)}) & x^4(u^{(2)}) \\ 1 & x(u^{(3)}) & x^2(u^{(3)}) & x^3(u^{(3)}) & x^4(u^{(3)}) \\ 1 & x(u^{(4)}) & x^2(u^{(4)}) & x^3(u^{(4)}) & x^4(u^{(4)}) \\ 1 & x(u^{(5)}) & x^2(u^{(5)}) & x^3(u^{(5)}) & x^4(u^{(5)}) \end{vmatrix}^3.$$

Proof. We still use the notation (14.2). We again write simply $v = u^{(6)}$, and We are going to expand both side of the equation in 17.1 by $v_{\langle 1\rangle} = u^{(6)}_{\langle 1\rangle}$. Since from the Table 22.1 to Table 22.6 indicates all the derivative by the coordinates of total Sato weight at most 5 of $\sigma(u)$ on $\Theta^{[5]}$ are identically 0, we see that

$$\begin{aligned} &\sigma_{\langle 4\rangle}(u^{[5]}+v) \\ (18.2)\qquad &= \tfrac{1}{3!}\sigma_{\langle 4111\rangle}(u^{[5]}){v_{\langle 1\rangle}}^3 + 2\tfrac{1}{2!}\sigma_{\langle 421\rangle}(u^{[5]})v_{\langle 1\rangle}v^{(2)} + \sigma_{\langle 43\rangle}(u^{(5)})v_{\langle 3\rangle} + \cdots \\ &= \big(\tfrac{1}{6}\sigma_{\langle 4111\rangle}(u^{[5]}) + 2\tfrac{1}{2}\sigma_{\langle 421\rangle}(u^{[5]}) + \tfrac{1}{3}\sigma_{\langle 43\rangle}(u^{(5)}){v_{\langle 1\rangle}}^3\big) + \cdots. \end{aligned}$$

This and the Table 22.7 shows that

$$(18.3)\qquad \sigma_{\langle 4111\rangle}(u^{[5]}) = \sigma_{\langle 421\rangle}(u^{[5]}) = \sigma_{\langle 43\rangle}(u^{[5]}) = -\sigma_{\langle 7\rangle}(u^{[5]}).$$

Adding 9.1 and 10.1 with this, we have

$$(18.4)\qquad \sigma_{\langle 4\rangle}(u^{[5]}+v) = -\sigma_{\langle 7\rangle}(u^{[5]}){v_{\langle 1\rangle}}^3 + \big(d^\circ(v_{\langle 1\rangle}) \geqq 4\big).$$

Hence, we have proved the desired formula. $\square$

Corollary 18.5 *The function* $v \mapsto \sigma_{\natural^5}(u^{(1)}+u^{(2)}+u^{(3)}+u^{(4)}+v)$ *has the zeroes only at* $(0, 0, \cdots, 0)$ *modulo* Λ *of order* 3, *and at the zeroes of non-trivial zeroes of the determinant*

$$v \quad\longmapsto\quad \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) \\ 1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) & xy(u^{(3)}) \\ 1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) & xy(u^{(4)}) \\ 1 & x(v) & y(v) & x^2(v) & xy(v) \end{vmatrix},$$

namely, at the 7 *zeroes modulo* Λ *except the trivial zeroes* $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, $u^{(4)}$ *modulo* Λ *of this determinant. The function claimed above has no other zeroes. Especially, the assertion of* 7.7 *holds for* $n = 5$.

19. The formula for $n = 4$

Proposition 19.1 *We have the following equation:*

$$\frac{\sigma_{\natural^4}(u^{(1)}+u^{(2)}+u^{(3)}+u^{(4)})\prod_{1\leqq i<j\leqq 4}\prod_{\nu=1}^{4}\sigma_{\flat}(u^{(i)}+[\zeta^{\nu}]u^{(j)})}{\prod_{j=1}^{4}\sigma_{\sharp}(u^{(j)})^{13}}$$
$$= \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) \\ 1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) \\ 1 & x(u^{(4)}) & y(u^{(4)}) & x^2(u^{(4)}) \end{vmatrix} \begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & x^3(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & x^3(u^{(2)}) \\ 1 & x(u^{(3)}) & x^2(u^{(3)}) & x^3(u^{(3)}) \\ 1 & x(u^{(4)}) & x^2(u^{(4)}) & x^3(u^{(4)}) \end{vmatrix}^3.$$

Proof. We still use the notation (14.2). We again write simply $v = u^{(5)}$, and We are going to expand both side of the equation in 18.1 by $v_{\langle 1\rangle} = u^{(5)}_{\langle 1\rangle}$. Since from the Table 22.1 to Table 22.9 indicates all the derivative by the coordinates of total Sato weight at most 9 of $\sigma(u)$ on $\Theta^{[4]}$ are identically 0, we see that

$$\begin{aligned}(19.2)\qquad &\sigma_{\langle 7\rangle}(u^{[4]}+v) \\ &= \tfrac{1}{3!}\sigma_{\langle 7111\rangle}(u^{[4]}){v_{\langle 1\rangle}}^3 + 2\tfrac{1}{2!}\sigma_{\langle 721\rangle}(u^{[4]})v_{\langle 1\rangle}v_{\langle 2\rangle} + \sigma_{\langle 73\rangle}(u^{[4]})v_{\langle 3\rangle} + \cdots \\ &= \big(\tfrac{1}{6}\sigma_{\langle 7111\rangle}(u^{[4]}) + \tfrac{1}{2}\sigma_{\langle 721\rangle}(u^{[4]}) + \tfrac{1}{3}\sigma_{\langle 73\rangle}(u^{[4]})\big){v_{\langle 1\rangle}}^3 + \cdots.\end{aligned}$$

By the Table 22.10, we see that

$$(19.3)\qquad \sigma_{\langle 7111\rangle}(u^{[4]}) = \sigma_{\langle 721\rangle}(u^{[4]}) = \sigma_{\langle 73\rangle}(u^{[4]}) = -\sigma_{\langle 82\rangle}(u^{[4]}).$$

Therefore, we have that

$$(19.4)\qquad \sigma_{\langle 7\rangle}(u^{[4]}+v) = -\sigma_{\langle 82\rangle}(u^{[4]}){v_{\langle 1\rangle}}^3 + \big(d^{\circ}(v_{\langle 1\rangle}) \geqq 4\big),$$

and showed the desired formula. □

Corollary 19.5 *The function $v \mapsto \sigma_{\natural^4}(u^{(1)}+u^{(2)}+u^{(3)}+v)$ has the zeroes only at $(0,0,\cdots,0)$ modulo Λ of order 3, and at the non-trivial zeroes of the determinant*

$$v \longmapsto \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) \\ 1 & x(u^{(3)}) & y(u^{(3)}) & x^2(u^{(3)}) \\ 1 & x(v) & y(v) & x^2(v) \end{vmatrix},$$

namely, at the 7 zeroes of order 1 modulo Λ except the zeroes $u^{(1)}$, $u^{(2)}$, $u^{(3)}$ modulo Λ of this determinant. The function claimed above has no other zeroes. Especially, the assertion in 7.7 holds for $n = 4$.

20. The formula for $n=3$

Proposition 20.1 *We have the following equation:*

$$\frac{\sigma_{\natural^3}(u^{(1)}+u^{(2)}+u^{(3)})\prod_{1\leqq i<j\leqq 3}\prod_{\nu=1}^{4}\sigma_\flat(u^{(i)}+[\zeta^\nu]u^{(j)})}{\prod_{j=1}^{3}\sigma_\sharp(u^{(j)})^9}$$
$$=\begin{vmatrix}1 & x(u^{(1)}) & y(u^{(1)})\\ 1 & x(u^{(2)}) & y(u^{(2)})\\ 1 & x(u^{(3)}) & y(u^{(3)})\end{vmatrix}\cdot\begin{vmatrix}1 & x(u^{(1)}) & x^2(u^{(1)})\\ 1 & x(u^{(2)}) & x^2(u^{(2)})\\ 1 & x(u^{(3)}) & x^2(u^{(3)})\end{vmatrix}^3.$$

Proof. We still use the notation (14.2). We again write simply $v=u^{(4)}$, and We are going to expand both side of the equation in 18.1 by $v_{\langle 1\rangle}=u^{(4)}_{\langle 1\rangle}$. Since from the Table 22.1 to Table 22.12 indicates all the derivative by the coordinates of total Sato weight at most 12 of $\sigma(u)$ on $\Theta^{[3]}$ are identically 0, we see that

$$\begin{aligned}&\sigma_{\langle 82\rangle}(u^{[3]}+v)\\ (20.2)\qquad &=\tfrac{1}{3!}\sigma_{\langle 82111\rangle}(u^{[3]}){v_{\langle 1\rangle}}^3+2\tfrac{1}{2!}\sigma_{\langle 8321\rangle}(u^{[3]})v_{\langle 1\rangle}v_{\langle 2\rangle}+\sigma_{\langle 833\rangle}(u^{[3]})v_{\langle 3\rangle}+\cdots\\ &=\big(\tfrac{1}{6}\sigma_{\langle 82111\rangle}(u^{[3]})+\tfrac{1}{2}\sigma_{\langle 8221\rangle}(u^{[3]})+\tfrac{1}{3}\sigma_{\langle 832\rangle}(u^{[3]})\big){v_{\langle 1\rangle}}^3+\big(d^\circ(v_{\langle 1\rangle})\geqq 4\big).\end{aligned}$$

This and the Table 22.13 imply that

$$(20.3)\qquad \sigma_{\langle 832\rangle}(u^{[3]})=\sigma_{\langle 8221\rangle}(u^{[3]})=\sigma_{\langle 82111\rangle}(u^{[3]})=\sigma_{\langle 931\rangle}(u^{[3]}).$$

Hence, we see that

$$(20.4)\qquad \sigma_{\langle 82\rangle}(u^{[3]}+v)=\sigma_{\langle 931\rangle}(u^{[3]}){v_{\langle 1\rangle}}^3+\big(d^\circ(v_{\langle 1\rangle})\geqq 4\big).$$

This shows the desired formula. $\square$

Corollary 20.5 *The function $v\mapsto\sigma_{\natural^3}(u^{(1)}+u^{(2)}+v)$ has the zeroes only at the points $(0,0,\cdots,0)$ modulo Λ of order 6, and at the non trivial zeroes of the determinant*

$$v\quad\longmapsto\quad\begin{vmatrix}1 & x(u^{(1)}) & y(u^{(1)})\\ 1 & x(u^{(2)}) & y(u^{(2)})\\ 1 & x(v) & y(v)\end{vmatrix},$$

namely, the 4 zeroes modulo Λ except the zeroes $u^{(1)}$ and $u^{(2)}$ modulo Λ of order 1. The function claimed above has no other zeroes. Especially, the assertion 7.7 holds for $n=3$.

21. Kiepert-type formulae

Proposition 21.1 *We have the following formula:*

$$\frac{\sigma_\flat(2u)}{\sigma_\sharp(u)^4} = 5y(u)^4.$$

Proof. From 11.1 (4) and 10.8, we have

$$\frac{\sigma_\flat(2u)}{\sigma_\sharp(u)^4} = \frac{5{u_{\langle 1\rangle}}^{16} + \cdots}{(u_{\langle 1\rangle}^{10} + \cdots)^4} = \frac{5}{{u_{\langle 1\rangle}}^{24}} + \cdots . \tag{21.2}$$

From 11.1 (3) and 12.1 (3), we see that (21.2) is a periodic function with respect to Λ, and that its poles are only at the points in Λ. By 4.15, we know that

$$\sigma_\flat([\zeta]u) = \zeta^4\sigma_\flat(u), \quad \sigma_\sharp([\zeta]u) = \sigma_\sharp(u).$$

Therefore, we see that

$$\frac{\sigma_\flat(2[\zeta]u)}{\sigma_\sharp([\zeta]u)^4} = \zeta^4\frac{\sigma_\flat(2u)}{\sigma_\sharp(u)^4}.$$

Thus, the assertion has been proved. □

Lemma 21.2 *We have the following:*

$$\lim_{v\to u}\frac{\sigma_\flat(u+[\zeta]v)\,\sigma_\flat(u+[\zeta^2]v)\,\sigma_\flat(u+[\zeta^3]v)\,\sigma_\flat(u+[\zeta^4]v)}{\sigma_\sharp(u)^3\sigma_\sharp(v)^3(u_{\langle 1\rangle}-v_{\langle 1\rangle})^4} = 5^3.$$

Proof. By 12.14 and 21.1, we have

$$\begin{aligned}
&5y(u)^4\left(\lim_{v\to u}\frac{\sigma_\flat(u+[\zeta]v)\,\sigma_\flat(u+[\zeta^2]v)\,\sigma_\flat(u+[\zeta^3]v)\,\sigma_\flat(u+[\zeta^4]v)}{\sigma_\sharp(u)^3\sigma_\sharp(v)^3(u_{\langle 1\rangle}-v_{\langle 1\rangle})^4}\right)\\
&\quad= \lim_{v\to u}\frac{\sigma_\flat(u+v)\,\sigma_\flat(u+[\zeta]v)\,\sigma_\flat(u+[\zeta]^2]v)\,\sigma_\flat(u+[\zeta]^3]v)\,\sigma_\flat(u+[\zeta]^4]v)}{\sigma_\sharp(u)^5\sigma_\sharp(v)^5(u_{\langle 1\rangle}-v_{\langle 1\rangle})^4}\\
&\quad= \lim_{v\to u}\left(\frac{x(u)-x(v)}{u_{\langle 1\rangle}-v_{\langle 1\rangle}}\right)^4\\
&\quad= \lim_{v\to u}\left(\frac{dx}{du_{\langle 1\rangle}}(u)\right)^4\\
&\quad= \left(\frac{5y(u)^4}{y(u)^3}\right)^4\\
&\quad= 5^4y(u)^4.
\end{aligned}$$

Hence the statement. □

Corollary 21.3 (Kiepert-type formula) *If $u \in \kappa^{-1}\iota(C)$, then we have the following*:

$$\psi_n(u) := \frac{\sigma_{\natural^n}(nu)}{\sigma_{\sharp}(u)^{n(4n-3)}} = \pm\, y^{n(n-1)/2}(u) \cdot \left|(x^{a_j}y^{b_j})^{(i-1)}\right|_{2\leqq i,j\leqq n}(u),$$

where $^{(i)}$ *means* $(\mathrm{d}/\mathrm{d}u_{\langle 1\rangle})^i$,

$$\text{“}\pm\text{”} = \begin{cases} \text{“}+\text{”} & (\textit{if } n = 6,\ 8,\ \geqq 10), \\ \text{“}-\text{”} & (\textit{otherwise}). \end{cases}$$

The determinant is of size $(n-1) \times (n-1)$.

Proof. For each factor $x(u^{(i)}) - x(u^{(j)})$ of the Vandermonde determinant in 9.7, we have

$$\lim_{v\to u} \frac{x(u) - x(v)}{u_{\langle 1\rangle} - v_{\langle 1\rangle}} = \frac{\mathrm{d}x}{\mathrm{d}u_{\langle 1\rangle}}(u) = 5y(u).$$

This and 21.2 implies the statement. $\square$

22. Tables of peelings

Table 22-1 (Omitted.)

Table 22-2 (Omitted.)

Table 22-3

Weight 3			1	2	3
Weight	Function	Stratum	$\langle 111\rangle$	$\langle 21\rangle$	$\langle 3\rangle$
0	$\langle\,\rangle$	9	1/3!	2/2!/2	1/1!/3
1	$\langle 1\rangle$	8	1/2!	1/1!/2	0
Result			1	-1	1

Table 22-4

Weight 4			1	2	3	4	5
Weight	Function	Stratum	$\langle 1111\rangle$	$\langle 211\rangle$	$\langle 22\rangle$	$\langle 31\rangle$	$\langle 4\rangle$
0	$\langle\,\rangle$	9	1/4!	3/3!/2	1/2!/4	2/2!/3	1/1!/4
1	$\langle 1\rangle$	8	1/3!	2/2!/2	0	1/1!/3	0
2	$\langle 11\rangle$	7	1/2!	1/1!/2	0	0	0
2	$\langle 2\rangle$	7	0	1/2!	1/1!/2	0	0
3	$\langle 111\rangle$	$7, \langle 3\rangle, \ \ 1$	1/1!	0	0	0	0
3	$\langle 21\rangle$	$7, \langle 3\rangle, -1$	0	1/1!	0	0	0
3	$\langle 3\rangle$	$7, \langle 3\rangle, \ \ 1$	0	0	0	1/1!	0
Result			-1	1	-1	-1	1

Table 22-5

Weight 5			1	2	3	4	5	6
Weight	Function	Stratum	$\langle 11111\rangle$	$\langle 2111\rangle$	$\langle 221\rangle$	$\langle 311\rangle$	$\langle 32\rangle$	$\langle 41\rangle$
0	$\langle\,\rangle$	9	1/5!	4/4!/2	3/3!/4	3/3!/3	2/2!/6	2/2!/4
1	$\langle 1\rangle$	8	1/4!	3/3!/2	1/2!/4	2/2!/3	0	1/1!/4
2	$\langle 11\rangle$	7	1/3!	2/2!/2	0	1/1!/3	0	0
2	$\langle 2\rangle$	7	0	1/3!	2/2!/2	0	1/1!/3	0
3	$\langle 111\rangle$	$6, \langle 3\rangle, \ \ 1$	1/2!	1/1!/2	0	0	0	0
3	$\langle 21\rangle$	$6, \langle 3\rangle, -1$	0	1/2!	1/1!/2	0	0	0
3	$\langle 3\rangle$	$6, \langle 3\rangle, \ \ 1$	0	0	0	1/2!	1/1!/2	0

Table 22-6

Weight 6			1	2	3	4	5	6	7	8	9
Weight	Function	Stratum	$\langle 111111\rangle$	$\langle 21111\rangle$	$\langle 2211\rangle$	$\langle 222\rangle$	$\langle 3111\rangle$	$\langle 321\rangle$	$\langle 33\rangle$	$\langle 411\rangle$	$\langle 42\rangle$
0	$\langle\,\rangle$	9	1/6!	5/5!/2	6/4!/4	1/3!/8	4/4!/3	6/3!/6	1/2!/9	3/3!/4	2/2!/8
1	$\langle 1\rangle$	8	1/5!	4/4!/2	3/3!/4	0	3/3!/3	2/2!/6	0	2/2!/2	0
2	$\langle 11\rangle$	7	1/4!	3/3!/2	1/2!/4	0	2/2!/3	0	0	1/1!	0
	$\langle 2\rangle$	7	0	1/4!	3/3!/2	1/2!/4	0	2/2!/3	0	0	1/1!/4
3	$\langle 111\rangle$	6	1/3!	2/2!/2	0	0	1/1!/3	0	0	0	0
	$\langle 21\rangle$	6	0	1/3!	2/2!/2	0	0	1/1!/3	0	0	0
	$\langle 3\rangle$	6	0	0	0	0	1/3!	2/2!/2	1/1!/3	0	0
Result			10	−4	2	0	1	−1	1	0	0
			0	−6	4	0	9	−3	0	−2	1

Table 22-7

Weight 7			1	2	3	4	5
Weight	Function	Stratum	$\langle 1^7 \rangle$	$\langle 211111 \rangle$	$\langle 22111 \rangle$	$\langle 2221 \rangle$	$\langle 31111 \rangle$
0	$\langle\,\rangle$	9	1/7!	6/6!/2	10/5!/4	4/4!/8	5/5!/3
1	$\langle 1 \rangle$	8	1/6!	5/5!/2	6/4!/4	1/3!/8	4/4!/3
2	$\langle 11 \rangle$	7	1/5!	4/4!/2	3/3!/4	0	3/3!/3
	$\langle 2 \rangle$	7	0	1/5!	4/4!/2	3/3!/4	0
3	$\langle 111 \rangle$	6	1/4!	3/3!/2	1/2!/4	0	2/2!/3
	$\langle 21 \rangle$	6	0	1/4!	3/3!/2	1/2!/4	0
	$\langle 3 \rangle$	6	0	0	0	0	1/4!
4	$\langle 1111 \rangle$	$6, \langle 4 \rangle, -1$	1/3!	2/2!/2	0	0	1/1!/3
	$\langle 211 \rangle$	$6, \langle 4 \rangle, 1$	0	1/3!	2/2!/2	0	0
	$\langle 22 \rangle$	$6, \langle 4 \rangle, -1$	0	0	1/3!	2/2!/2	0
	$\langle 31 \rangle$	$6, \langle 4 \rangle, -1$	0	0	0	0	1/3!
	$\langle 4 \rangle$	$6, \langle 4 \rangle, 1$	0	0	0	0	0
5	$\langle 11111 \rangle$	$6, \langle 41 \rangle, -5$	1/2!	1/1!/2	0	0	0
	$\langle 2111 \rangle$	$6, \langle 41 \rangle, 3$	0	1/2!	1/1!/2	0	0
	$\langle 221 \rangle$	$6, \langle 41 \rangle, -1$	0	0	1/2!	1/1!/2	0
	$\langle 311 \rangle$	$6, \langle 41 \rangle, -2$	0	0	0	0	1/2!
	$\langle 32 \rangle$	$6, \langle 41 \rangle, 0$	0	0	0	0	0
	$\langle 41 \rangle$	$6, \langle 41 \rangle, 1$	0	0	0	0	0
Result			15	−5	−1	3	3

	6	7	8	9	10	11	12
Function	$\langle 3211 \rangle$	$\langle 322 \rangle$	$\langle 331 \rangle$	$\langle 4111 \rangle$	$\langle 421 \rangle$	$\langle 43 \rangle$	$\langle 7 \rangle$
$\langle\,\rangle$	12/4!/6	3/3!/12	3/3!/9	4/4!/4	6/3!/8	2/2!/12	1/1!/7
$\langle 1 \rangle$	6/3!/6	0	1/2!/9	3/3!/4	2/2!/8	0	0
$\langle 11 \rangle$	2/2!/6	0	0	2/2!/4	0	0	0
$\langle 2 \rangle$	3/3!/3	2/2!/6	0	0	2/2!/4	0	0
$\langle 111 \rangle$	0	0	0	1/1!/4	0	0	0
$\langle 21 \rangle$	2/2!/3	0	0	0	1/1!/4	0	0
$\langle 3 \rangle$	3/3!/2	1/2!/4	2/2!/3	0	0	1/1!/4	0
$\langle 1111 \rangle$	0	0	0	0	0	0	0
$\langle 211 \rangle$	1/1!/3	0	0	0	0	0	0
$\langle 22 \rangle$	0	1/1!/3	0	0	0	0	0
$\langle 31 \rangle$	2/2!/2	0	1/1!/3	0	0	0	0
$\langle 4 \rangle$	0	0	0	1/3!	2/2!/2	1/1!/3	0
$\langle 11111 \rangle$	0	0	0	0	0	0	0
$\langle 2111 \rangle$	0	0	0	0	0	0	0
$\langle 221 \rangle$	0	0	0	0	0	0	0
$\langle 311 \rangle$	1/1!/2	0	0	0	0	0	0
$\langle 32 \rangle$	1/2!	1/1!/2	0	0	0	0	0
$\langle 41 \rangle$	0	0	0	1/2!	1/1!/2	0	0
	1	−1	0	−1	−1	−1	1

Table 22-8

Weight 8			1	2	3	4	5	6	7
Weight	Function	Stratum	$\langle 1^8\rangle$	⟨2111111⟩	⟨221111⟩	⟨22211⟩	⟨2222⟩	⟨311111⟩	⟨32111⟩
0	⟨⟩	9	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16	6/6!/3	20/5!/6
1	⟨1⟩	8	1/7!	6/6!/2	10/5!/4	6/4!/8	0	5/5!/3	12/4!/6
2	⟨11⟩	7	1/6!	5/5!/2	6/4!/4	1/3!/8	0	4/4!/3	6/3!/6
	⟨2⟩	7	0	1/6!	5/5!/2	6/4!/4	1/3!/8	0	4/4!/3
3	⟨111⟩	6	1/5!	4/4!/2	3/3!/4	0	0	3/3!/3	2/2!/6
	⟨21⟩	6	0	1/5!	4/4!/2	3/3!/4	0	0	3/3!/3
	⟨3⟩	6	0	0	0	0	0	1/5!	4/4!/2
4	⟨1111⟩	5	1/4!	3/3!/2	1/2!/4	0	0	2/2!/3	0
	⟨211⟩	5	0	1/4!	3/3!/2	1/2!/4	0	0	2/2!/3
	⟨22⟩	5	0	0	1/4!	3/3!/2	1/2!/4	0	0
	⟨31⟩	5	0	0	0	0	0	1/4!	3/3!/2
	⟨4⟩	5	0	0	0	0	0	0	0
5	⟨11111⟩	5, ⟨41⟩, −5	1/3!	2/2!/2	0	0	0	1/1!/3	0
	⟨2111⟩	5, ⟨41⟩, 3	0	1/3!	2/2!/2	0	0	0	1/1!/3
	⟨221⟩	5, ⟨41⟩, −1	0	0	1/3!	2/2!/2	0	0	0
	⟨311⟩	5, ⟨41⟩, −2	0	0	0	0	0	1/3!	2/2!/2
	⟨32⟩	5	0	0	0	0	0	0	1/3!
	⟨41⟩	5, ⟨41⟩, 1	0	0	0	0	0	0	0
6	⟨222⟩	5	0	0	0	1/2!	1/1!/2	0	0
7	⟨331⟩	5	0	0	0	0	0	0	0
	⟨43⟩	5, ⟨7⟩, −1	0	0	0	0	0	0	0
	⟨7⟩	5, ⟨7⟩, 1	0	0	0	0	0	0	0
Result			−120	40	−8	0	0	−15	1

	8	9	10	11	12	13	14	15	16	17
Function	⟨3221⟩	⟨3311⟩	⟨332⟩	⟨41111⟩	⟨4211⟩	⟨422⟩	⟨431⟩	⟨44⟩	⟨71⟩	⟨8⟩
⟨⟩	12/4!/12	6/4!/9	3/3!/18	5/5!/4	12/4!/8	3/3!/16	6/3!/12	1/2!/16	2/2!/7	1/1!/8
⟨1⟩	3/3!/12	3/3!/9	0	4/4!/4	6/3!/8	0	2/2!/12	0	1/1!/7	0
⟨11⟩	0	1/2!/9	0	3/3!/4	2/2!/8	0	0	0	0	0
⟨2⟩	6/3!/6	0	1/2!/9	0	3/3!/4	2/2!/8	0	0	0	0
⟨111⟩	0	0	0	2/2!/4	0	0	0	0	0	0
⟨21⟩	2/2!/6	0	0	0	2/2!/4	0	0	0	0	0
⟨3⟩	3/3!/4	3/3!/3	2/2!/6	0	0	0	2/2!/4	0	0	0
⟨1111⟩	0	0	0	1/1!/4	0	0	0	0	0	0
⟨211⟩	0	0	0	0	1/1!/4	0	0	0	0	0
⟨22⟩	2/2!/3	0	0	0	0	1/1!/4	0	0	0	0
⟨31⟩	1/2!/4	2/2!/3	0	0	0	0	1/1!/4	0	0	0
⟨4⟩	0	0	0	1/4!	3/3!/2	1/2!/4	2/2!/3	1/1!/4	0	0
⟨11111⟩	0	0	0	0	0	0	0	0	0	0
⟨2111⟩	0	0	0	0	0	0	0	0	0	0
⟨221⟩	1/1!/3	0	0	0	0	0	0	0	0	0
⟨311⟩	0	1/1!/3	0	0	0	0	0	0	0	0
⟨32⟩	2/2!/2	0	1/1!/3	0	0	0	0	0	0	0
⟨41⟩	0	0	0	1/3!	2/2!/2	0	1/1!/3	0	0	0
⟨222⟩	0	0	0	0	0	0	0	0	0	0
⟨331⟩	0	1/1!	0	0	0	0	0	0	0	0
⟨43⟩	0	0	0	0	0	0	1/1!	0	0	0
⟨7⟩	0	0	0	0	0	0	0	0	1/1!	0
	1	0	−2	4	0	0	1	−2	−1	1

Table 22-9 (1/5)

Weight 9			1	2	3	4	5	6
Weight	Function	Stratum	$\langle 1^{10}\rangle$	$\langle 21^{8}\rangle$	$\langle 221^{6}\rangle$	$\langle 2221^{4}\rangle$	$\langle 222211\rangle$	$\langle 2^{5}\rangle$
0	$\langle\,\rangle$	9	1/10!	9/9!/2	28/8!/4	35/7!/8	15/6!/16	1/5!/32
1	$\langle 1\rangle$	8	1/9!	8/8!/2	21/7!/4	20/6!/8	5/5!/16	0
2	$\langle 11\rangle$	7	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16	0
	$\langle 2\rangle$	7	0	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16
3	$\langle 111\rangle$	6	1/7!	6/6!/2	10/5!/4	4/4!/8	0	0
	$\langle 21\rangle$	6	0	1/7!	6/6!/2	10/5!/4	4/4!/8	0
	$\langle 3\rangle$	6	0	0	0	0	0	0
4	$\langle 1111\rangle$	5	1/6!	5/5!/2	6/4!/4	1/3!/8	0	0
	$\langle 211\rangle$	5	0	1/6!	5/5!/2	6/4!/4	1/3!/8	0
	$\langle 22\rangle$	5	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8
	$\langle 31\rangle$	5	0	0	0	0	0	0
	$\langle 4\rangle$	5	0	0	0	0	0	0
5	$\langle 11111\rangle$	5	1/5!	4/4!/2	3/3!/4	0	0	0
	$\langle 2111\rangle$	5	0	1/5!	4/4!/2	3/3!/4	0	0
	$\langle 221\rangle$	5	0	0	1/5!	4/4!/2	3/3!/4	0
	$\langle 311\rangle$	5	0	0	0	0	0	0
	$\langle 32\rangle$	5	0	0	0	0	0	0
	$\langle 41\rangle$	5	0	0	0	0	0	0
6	$\langle 111111\rangle$	5	1/4!	3/3!/2	0	0	0	0
	$\langle 21111\rangle$	5	0	1/4!	3/3!/2	0	0	0
	$\langle 2211\rangle$	5	0	0	1/4!	3/3!/2	0	0
	$\langle 222\rangle$	5	0	0	0	1/4!	3/3!/2	0
	$\langle 3111\rangle$	5	0	0	0	0	0	0
	$\langle 321\rangle$	5	0	0	0	0	0	0
	$\langle 33\rangle$	5	0	0	0	0	0	0
	$\langle 411\rangle$	5	0	0	0	0	0	0
	$\langle 42\rangle$	5	0	0	0	0	0	0
7	$\langle 331\rangle$	5	0	0	0	0	0	0
	$\langle 43\rangle$	$5, \langle 7\rangle, -1$	0	0	0	0	0	0
	$\langle 7\rangle$	$5, \langle 7\rangle, \ \ 1$	0	0	0	0	0	0
Result			0	0	0	0	0	0

Table 22-9 (2/5)

	7	8	9	10	11	12
Function	$\langle 31^7\rangle$	$\langle 321^5\rangle$	$\langle 3221^3\rangle$	$\langle 32221\rangle$	$\langle 331^4\rangle$	$\langle 33211\rangle$
$\langle\,\rangle$	8/8!/3	42/7!/6	60/6!/12	20/5!/24	15/6!/9	30/5!/18
$\langle 1\rangle$	7/7!/3	30/6!/6	30/5!/12	0	10/5!/9	12/4!/18
$\langle 11\rangle$	6/6!/3	20/5!/6	12/4!/12	0	6/4!/9	3/3!/18
$\langle 2\rangle$	0	6/6!/3	20/5!/6	12/4!/12	0	6/4!/9
$\langle 111\rangle$	5/5!/3	12/4!/6	3/3!/12	0	3/3!/9	0
$\langle 21\rangle$	0	5/5!/3	12/4!/6	3/3!/12	0	3/3!/9
$\langle 3\rangle$	1/7!	6/6!/2	10/5!/4	4/4!/8	5/5!/3	12/4!/6
$\langle 1111\rangle$	4/4!/3	6/3!/6	0	0	1/2!/9	0
$\langle 211\rangle$	0	4/4!/3	6/3!/6	0	0	1/2!/9
$\langle 22\rangle$	0	0	4/4!/3	6/3!/6	0	0
$\langle 31\rangle$	1/6!	5/5!/2	6/4!/4	1/3!/8	4/4!/3	6/3!/6
$\langle 4\rangle$	0	0	0	0	0	0
$\langle 11111\rangle$	3/3!/3	2/2!/6	0	0	0	0
$\langle 2111\rangle$	0	3/3!/3	2/2!/6	0	0	0
$\langle 221\rangle$	0	0	3/3!/3	2/2!/6	0	0
$\langle 311\rangle$	1/5!	4/4!/2	3/3!/4	0	3/3!/3	2/2!/6
$\langle 32\rangle$	0	1/5!	4/4!/2	3/3!/4	0	3/3!/3
$\langle 41\rangle$	0	0	0	0	0	0
$\langle 111111\rangle$	2/2!/3	0	0	0	0	0
$\langle 21111\rangle$	0	2/2!/3	0	0	0	0
$\langle 2211\rangle$	0	0	2/2!/3	0	0	0
$\langle 222\rangle$	0	0	0	2/2!/3	0	0
$\langle 3111\rangle$	1/4!	3/3!/2	0	0	2/2!/3	0
$\langle 321\rangle$	0	1/4!	0	1/2!/4	0	2/2!/3
$\langle 33\rangle$	0	0	0	0	1/4!	3/3!/2
$\langle 411\rangle$	0	0	0	0	0	0
$\langle 42\rangle$	0	0	0	0	0	0
$\langle 331\rangle$	0	0	0	0	1/3!	2/2!/2
$\langle 43\rangle$	0	0	0	0	0	0
$\langle 7\rangle$	0	0	0	0	0	0
	0	0	0	0	0	0

Table 22-9 (3/5)

	13	14	15	16	17	18
Function	$\langle 3322\rangle$	$\langle 3331\rangle$	$\langle 41^6\rangle$	$\langle 421^4\rangle$	$\langle 42211\rangle$	$\langle 4222\rangle$
$\langle\,\rangle$	6/4!/36	4/4!/27	7/7!/4	30/6!/8	30/5!/16	4/4!/32
$\langle 1\rangle$	0	1/3!/27	6/6!/4	20/5!/8	12/4!/16	0
$\langle 11\rangle$	0	0	5/5!/4	12/4!/8	3/3!/16	0
$\langle 2\rangle$	3/3!/18	0	0	5/5!/4	12/4!/8	3/3!/16
$\langle 111\rangle$	0	0	4/4!/4	6/3!/8	0	0
$\langle 21\rangle$	0	0	0	4/4!/4	6/3!/8	0
$\langle 3\rangle$	3/3!/12	3/3!/9	0	0	0	0
$\langle 1111\rangle$	0	0	3/3!/4	2/2!/8	0	0
$\langle 211\rangle$	0	0	0	3/3!/4	2/2!/8	0
$\langle 22\rangle$	1/2!/9	0	0	0	3/3!/4	2/2!/8
$\langle 31\rangle$	0	1/2!/9	0	0	0	0
$\langle 4\rangle$	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8
$\langle 11111\rangle$	0	0	2/2!/4	0	0	0
$\langle 2111\rangle$	0	0	0	2/2!/4	0	0
$\langle 221\rangle$	0	0	0	0	2/2!/4	0
$\langle 311\rangle$	0	0	0	0	0	0
$\langle 32\rangle$	2/2!/6	0	0	0	0	0
$\langle 41\rangle$	0	0	1/5!	4/4!/2	3/3!/4	0
$\langle 111111\rangle$	0	0	0	0	0	0
$\langle 21111\rangle$	0	0	0	0	0	0
$\langle 2211\rangle$	0	0	0	0	0	0
$\langle 222\rangle$	0	0	0	0	0	0
$\langle 3111\rangle$	0	0	0	0	0	0
$\langle 321\rangle$	0	0	0	0	0	0
$\langle 33\rangle$	1/2!/4	2/2!/3	0	0	0	0
$\langle 411\rangle$	0	0	1/4!	3/3!/2	1/2!/4	0
$\langle 42\rangle$	0	0	0	1/4!	3/3!/2	1/2!/4
$\langle 331\rangle$	0	1/1!/3	0	0	0	0
$\langle 43\rangle$	0	0	0	0	0	0
$\langle 7\rangle$	0	0	0	0	0	0
	0	0	0	0	0	0

Table 22-9 (4/5)

	19	20	21	22	23	24
Function	$\langle 43111\rangle$	$\langle 4321\rangle$	$\langle 433\rangle$	$\langle 4411\rangle$	$\langle 442\rangle$	$a7111$
$\langle\,\rangle$	20/5!/12	24/4!/24	3/3!/36	6/4!/16	3/3!/32	4/4!/7
$\langle 1\rangle$	12/4!/12	6/3!/24	0	3/3!/16	0	3/3!/7
$\langle 11\rangle$	6/3!/12	0	0	1/2!/16	0	2/2!/7
$\langle 2\rangle$	0	6/3!/12	0	0	1/2!/16	0
$\langle 111\rangle$	2/2!/12	0	0	0	0	1/1!/7
$\langle 21\rangle$	0	2/2!/12	0	0	0	0
$\langle 3\rangle$	4/4!/4	6/3!/8	2/2!/12	0	0	0
$\langle 1111\rangle$	0	0	0	0	0	0
$\langle 211\rangle$	0	0	0	0	0	0
$\langle 22\rangle$	0	0	0	0	0	0
$\langle 31\rangle$	3/3!/4	2/2!/8	0	0	0	0
$\langle 4\rangle$	4/4!/3	6/3!/6	1/2!/9	3/3!/4	2/2!/8	0
$\langle 11111\rangle$	0	0	0	0	0	0
$\langle 2111\rangle$	0	0	0	0	0	0
$\langle 221\rangle$	0	0	0	0	0	0
$\langle 311\rangle$	2/2!/4	0	0	0	0	0
$\langle 32\rangle$	0	2/2!/4	0	0	0	0
$\langle 41\rangle$	3/3!/3	2/2!/6	0	2/2!/4	0	0
$\langle 111111\rangle$	0	0	0	0	0	0
$\langle 21111\rangle$	0	0	0	0	0	0
$\langle 2211\rangle$	0	0	0	0	0	0
$\langle 222\rangle$	0	0	0	0	0	0
$\langle 3111\rangle$	0	0	0	0	0	0
$\langle 321\rangle$	0	1/1!/4	0	0	0	0
$\langle 33\rangle$	0	0	1/1!/4	0	0	0
$\langle 411\rangle$	2/2!/3	0	0	1/1!/4	0	0
$\langle 42\rangle$	0	2/2!/3	0	0	1/1!/4	0
$\langle 331\rangle$	0	0	0	0	0	0
$\langle 43\rangle$	1/3!	2/2!/2	1/1!/3	0	0	0
$\langle 7\rangle$	0	0	0	0	0	0
	0	0	0	0	0	0

Table 22-9 (5/5)

	25	26	27	28	29
Function	⟨721⟩	⟨73⟩	⟨811⟩	⟨82⟩	⟨91⟩
⟨ ⟩	6/3!/14	2/2!/21	3/3!/8	2/2!/16	2/2!/9
⟨1⟩	2/2!/14	0	2/2!/8	0	1/1!/9
⟨11⟩	0	0	1/1!/8	0	0
⟨2⟩	2/2!/7	0	0	1/1!/8	0
⟨111⟩	0	0	0	0	0
⟨21⟩	1/1!/7	0	0	0	0
⟨3⟩	0	1/1!/7	0	0	0
⟨1111⟩	0	0	0	0	0
⟨211⟩	0	0	0	0	0
⟨22⟩	0	0	0	0	0
⟨31⟩	0	0	0	0	0
⟨4⟩	0	0	0	0	0
⟨11111⟩	0	0	0	0	0
⟨2111⟩	0	0	0	0	0
⟨221⟩	0	0	0	0	0
⟨311⟩	0	0	0	0	0
⟨32⟩	0	0	0	0	0
⟨41⟩	0	0	0	0	0
⟨111111⟩	0	0	0	0	0
⟨21111⟩	0	0	0	0	0
⟨2211⟩	0	0	0	0	0
⟨222⟩	0	0	0	0	0
⟨3111⟩	0	0	0	0	0
⟨321⟩	0	0	0	0	0
⟨33⟩	0	0	0	0	0
⟨411⟩	0	0	0	0	0
⟨42⟩	0	0	0	0	0
⟨331⟩	0	0	0	0	0
⟨43⟩	0	0	0	0	0
⟨7⟩	0	0	0	0	0
	0	0	0	0	0

Table 22-10 (1/5)

Weight 10			1	2	3	4	5	6
Weight	Function	Stratum	$\langle 1^{10}\rangle$	$\langle 21^{8}\rangle$	$\langle 221^{6}\rangle$	$\langle 2221^{4}\rangle$	$\langle 222211\rangle$	$\langle 22222\rangle$
0	$\langle\,\rangle$	9	1/10!	9/9!/2	28/8!/4	35/7!/8	15/6!/16	1/5!/32
1	$\langle 1\rangle$	8	1/9!	8/8!/2	21/7!/4	20/6!/8	5/5!/16	0
2	$\langle 11\rangle$	7	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16	0
	$\langle 2\rangle$	7	0	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16
3	$\langle 111\rangle$	7	1/7!	6/6!/2	10/5!/4	4/4!/8	0	0
	$\langle 21\rangle$	6	0	1/7!	6/6!/2	10/5!/4	4/4!/8	0
	$\langle 3\rangle$	6	0	0	0	0	0	0
4	$\langle 1111\rangle$	5	1/6!	5/5!/2	6/4!/4	1/3!/8	0	0
	$\langle 211\rangle$	5	0	1/6!	5/5!/2	6/4!/4	1/3!/8	0
	$\langle 22\rangle$	5	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8
	$\langle 31\rangle$	5	0	0	0	0	0	0
	$\langle 4\rangle$	5	0	0	0	0	0	0
5	$\langle 11111\rangle$	$6, \langle 41\rangle, -5$	1/5!	4/4!/2	3/3!/4	0	0	0
	$\langle 2111\rangle$	$6, \langle 41\rangle,\ \ 3$	0	1/5!	4/4!/2	3/3!/4	0	0
	$\langle 221\rangle$	$6, \langle 41\rangle - 1$	0	0	1/5!	4/4!/2	3/3!/4	0
	$\langle 311\rangle$	$6, \langle 41\rangle, -2$	0	0	0	0	0	0
	$\langle 32\rangle$	6	0	0	0	0	0	0
	$\langle 41\rangle$	$6, \langle 41\rangle,\ \ 1$	0	0	0	0	0	0
6	$\langle 111111\rangle$	6	1/4!	3/3!/2	1/2!/4	0	0	0
	$\langle 21111\rangle$	6	0	1/4!	3/3!/2	1/2!/4	0	0
	$\langle 2211\rangle$	6	0	0	1/4!	3/3!/2	1/2!/4	0
	$\langle 222\rangle$	6	0	0	0	1/4!	3/3!/2	1/2!/4
	$\langle 3111\rangle$	6	0	0	0	0	0	0
	$\langle 321\rangle$	6	0	0	0	0	0	0
	$\langle 33\rangle$	6	0	0	0	0	0	0
	$\langle 411\rangle$	6	0	0	0	0	0	0
	$\langle 42\rangle$	6	0	0	0	0	0	0
7	$\langle 1111111\rangle$	$5, \langle 7\rangle,\ 15$	1/3!	2/2!/2	0	0	0	0
	$\langle 211111\rangle$	$5, \langle 7\rangle, -5$	0	1/3!	2/2!/2	0	0	0
	$\langle 22111\rangle$	$5, \langle 7\rangle, -1$	0	0	1/3!	2/2!/2	0	0
	$\langle 2221\rangle$	$5, \langle 7\rangle,\ \ 3$	0	0	0	1/3!	2/2!/2	0
	$\langle 31111\rangle$	$5, \langle 7\rangle,\ \ 3$	0	0	0	0	0	0
	$\langle 3211\rangle$	$5, \langle 7\rangle,\ \ 1$	0	0	0	0	0	0
	$\langle 322\rangle$	$5, \langle 7\rangle, -1$	0	0	0	0	0	0
	$\langle 331\rangle$	$5, \langle 7\rangle,\ \ 0$	0	0	0	0	0	0
	$\langle 4111\rangle$	$5, \langle 7\rangle, -1$	0	0	0	0	0	0
	$\langle 421\rangle$	$5, \langle 7\rangle, -1$	0	0	0	0	0	0
	$\langle 43\rangle$	$5, \langle 7\rangle, -1$	0	0	0	0	0	0
	$\langle 7\rangle$	$5, \langle 7\rangle,\ \ 1$	0	0	0	0	0	0
Result			−225	55	−5	3	−9	15

Table 22-10 (2/5)

	7	8	9	10	12	13
Function	$\langle 31^7\rangle$	$\langle 3211111\rangle$	$\langle 322111\rangle$	$\langle 32221\rangle$	$\langle 331111\rangle$	$\langle 33211\rangle$
$\langle\,\rangle$	8/8!/3	42/7!/6	60/6!/12	20/5!/24	15/6!/9	30/5!/18
$\langle 1\rangle$	7/7!/3	30/6!/6	30/5!/12	4/4!/24	10/5!/9	12/4!/18
$\langle 11\rangle$	6/6!/3	20/5!/6	12/4!/12	0	6/4!/9	3/3!/18
$\langle 2\rangle$	0	6/6!/3	20/5!/6	12/4!/12	0	6/4!/9
$\langle 111\rangle$	5/5!/3	12/4!/6	3/3!/12	0	3/3!/9	0
$\langle 21\rangle$	0	5/5!/3	12/4!/6	3/3!/12	0	3/3!/9
$\langle 3\rangle$	1/7!	6/6!/2	10/5!/4	4/4!/8	5/5!/3	12/4!/6
$\langle 1111\rangle$	4/4!/3	6/3!/6	0	0	1/2!/9	0
$\langle 211\rangle$	0	4/4!/3	6/3!/6	0	0	1/2!/9
$\langle 22\rangle$	0	0	4/4!/3	6/3!/6	0	0
$\langle 31\rangle$	1/6!	5/5!/2	6/4!/4	1/3!/8	4/4!/3	6/3!/6
$\langle 4\rangle$	0	0	0	0	0	0
$\langle 11111\rangle$	3/3!/3	2/2!/6	0	0	0	0
$\langle 2111\rangle$	0	3/3!/3	2/2!/6	0	0	0
$\langle 221\rangle$	0	0	3/3!/3	2/2!/6	0	0
$\langle 311\rangle$	1/5!	4/4!/2	3/3!/4	0	3/3!/3	2/2!/6
$\langle 32\rangle$	0	1/5!	4/4!/2	3/3!/4	0	3/3!/3
$\langle 41\rangle$	0	0	0	0	0	0
$\langle 111111\rangle$	2/2!/3	0	0	0	0	0
$\langle 21111\rangle$	0	2/2!/3	0	0	0	0
$\langle 2211\rangle$	0	0	2/2!/3	0	0	0
$\langle 222\rangle$	0	0	0	2/2!/3	0	0
$\langle 3111\rangle$	1/4!	3/3!/2	1/2!/4	0	2/2!/3	0
$\langle 321\rangle$	0	1/4!	3/3!/2	1/2!/4	0	2/2!/3
$\langle 33\rangle$	0	0	0	0	1/4!	3/3!/2
$\langle 411\rangle$	0	0	0	0	0	0
$\langle 42\rangle$	0	0	0	0	0	0
$\langle 1111111\rangle$	1/1!/3	0	0	0	0	0
$\langle 211111\rangle$	0	1/1!/3	0	0	0	0
$\langle 22111\rangle$	0	0	1/1!/3	0	0	0
$\langle 2221\rangle$	0	0	0	1/1!/3	0	0
$\langle 31111\rangle$	1/3!	2/2!/2	0	0	1/1!/3	0
$\langle 3211\rangle$	0	1/3!	2/2!/2	0	0	1/1!/3
$\langle 322\rangle$	0	0	1/3!	2/2!/2	0	0
$\langle 331\rangle$	0	0	0	0	1/3!	2/2!/2
$\langle 4111\rangle$	0	0	0	0	0	0
$\langle 421\rangle$	0	0	0	0	0	0
$\langle 43\rangle$	0	0	0	0	0	0
$\langle 7\rangle$	0	0	0	0	0	0
Result	−15	−5	1	3	6	−2

Table 22-10 (3/5)

	14	15	16	17	18	19
Function	$\langle 3322\rangle$	$\langle 3331\rangle$	$\langle 41^6\rangle$	$\langle 421111\rangle$	$\langle 42211\rangle$	$\langle 4222\rangle$
$\langle\,\rangle$	6/4!/36	4/4!/27	7/7!/4	30/6!/8	30/5!/16	4/4!/32
$\langle 1\rangle$	0	1/3!/27	6/6!/4	20/5!/8	12/4!/16	0
$\langle 11\rangle$	0	0	5/5!/4	12/4!/8	3/3!/16	0
$\langle 2\rangle$	3/3!/18	0	0	5/5!/4	12/4!/8	3/3!/16
$\langle 111\rangle$	0	0	4/4!/4	6/3!/8	0	0
$\langle 21\rangle$	0	0	0	4/4!/4	6/3!/8	0
$\langle 3\rangle$	3/3!/12	3/3!/9	0	0	0	0
$\langle 1111\rangle$	0	0	3/3!/4	2/2!/8	0	0
$\langle 211\rangle$	0	0	0	3/3!/4	2/2!/8	0
$\langle 22\rangle$	1/2!/9	0	0	0	3/3!/4	2/2!/8
$\langle 31\rangle$	0	1/2!/9	0	0	0	0
$\langle 4\rangle$	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8
$\langle 11111\rangle$	0	0	2/2!/4	0	0	0
$\langle 2111\rangle$	0	0	0	2/2!/4	0	0
$\langle 221\rangle$	0	0	0	0	2/2!/4	0
$\langle 311\rangle$	0	0	0	0	0	0
$\langle 32\rangle$	2/2!/6	0	0	0	0	0
$\langle 41\rangle$	0	0	1/5!	4/4!/2	3/3!/4	0
$\langle 111111\rangle$	0	0	1/1!/4	0	0	0
$\langle 21111\rangle$	0	0	0	1/1!/4	0	0
$\langle 2211\rangle$	0	0	0	0	1/1!/4	0
$\langle 222\rangle$	0	0	0	0	0	1/1!/4
$\langle 3111\rangle$	0	0	0	0	0	0
$\langle 321\rangle$	0	0	0	0	0	0
$\langle 33\rangle$	1/2!/4	2/2!/3	0	0	0	0
$\langle 411\rangle$	0	0	1/4!	3/3!/2	1/2!/4	0
$\langle 42\rangle$	0	0	0	1/4!	3/3!/2	1/2!/4
$\langle 1111111\rangle$	0	0	0	0	0	0
$\langle 211111\rangle$	0	0	0	0	0	0
$\langle 22111\rangle$	0	0	0	0	0	0
$\langle 2221\rangle$	0	0	0	0	0	0
$\langle 31111\rangle$	0	0	0	0	0	0
$\langle 3211\rangle$	0	0	0	0	0	0
$\langle 322\rangle$	1/1!/3	0	0	0	0	0
$\langle 331\rangle$	0	1/1!/3	0	0	0	0
$\langle 4111\rangle$	0	0	1/3!	2/2!/2	0	0
$\langle 421\rangle$	0	0	0	1/3!	2/2!/2	0
$\langle 43\rangle$	0	0	0	0	0	0
$\langle 7\rangle$	0	0	0	0	0	0
Result	−2	0	5	1	1	−3

Table 22-10 (4/5)

	20	21	22	23	24	25
Function	$\langle 43111\rangle$	$\langle 4321\rangle$	$\langle 433\rangle$	$\langle 4411\rangle$	$\langle 442\rangle$	$\langle 7111\rangle$
$\langle\ \rangle$	20/5!/12	24/4!/24	3/3!/36	6/4!/16	3/3!/32	4/4!/7
$\langle 1\rangle$	12/4!/12	6/3!/24	0	3/3!/16	0	3/3!/7
$\langle 11\rangle$	6/3!/12	0	0	1/2!/16	0	2/2!/7
$\langle 2\rangle$	0	6/3!/12	0	0	1/2!/16	0
$\langle 111\rangle$	2/2!/12	0	0	0	0	1/1!/7
$\langle 21\rangle$	0	2/2!/12	0	0	0	0
$\langle 3\rangle$	4/4!/4	6/3!/8	2/2!/12	0	0	0
$\langle 1111\rangle$	0	0	0	0	0	0
$\langle 211\rangle$	0	0	0	0	0	0
$\langle 22\rangle$	0	0	0	0	0	0
$\langle 31\rangle$	3/3!/4	2/2!/8	0	0	0	0
$\langle 4\rangle$	4/4!/3	6/3!/6	1/2!/9	3/3!/4	2/2!/8	0
$\langle 11111\rangle$	0	0	0	0	0	0
$\langle 2111\rangle$	0	0	0	0	0	0
$\langle 221\rangle$	0	0	0	0	0	0
$\langle 311\rangle$	2/2!/4	0	0	0	0	0
$\langle 32\rangle$	0	2/2!/4	0	0	0	0
$\langle 41\rangle$	3/3!/3	2/2!/6	0	2/2!/4	0	0
$\langle 111111\rangle$	0	0	0	0	0	0
$\langle 21111\rangle$	0	0	0	0	0	0
$\langle 2211\rangle$	0	0	0	0	0	0
$\langle 222\rangle$	0	0	0	0	0	0
$\langle 3111\rangle$	1/1!/4	0	0	0	0	0
$\langle 321\rangle$	0	1/1!/4	0	0	0	0
$\langle 33\rangle$	0	0	1/1!/4	0	0	0
$\langle 411\rangle$	2/2!/3	0	0	1/1!/4	0	0
$\langle 42\rangle$	0	2/2!/3	0	0	1/1!/4	0
$\langle 1111111\rangle$	0	0	0	0	0	0
$\langle 211111\rangle$	0	0	0	0	0	0
$\langle 22111\rangle$	0	0	0	0	0	0
$\langle 2221\rangle$	0	0	0	0	0	0
$\langle 31111\rangle$	0	0	0	0	0	0
$\langle 3211\rangle$	0	0	0	0	0	0
$\langle 322\rangle$	0	0	0	0	0	0
$\langle 331\rangle$	0	0	0	0	0	0
$\langle 4111\rangle$	1/1!/3	0	0	0	0	0
$\langle 421\rangle$	0	1/1!/3	0	0	0	0
$\langle 43\rangle$	1/3!	2/2!/2	1/1!/3	0	0	0
$\langle 7\rangle$	0	0	0	0	0	1/3!
Result	−1	1	2	−1	−1	−1

Table 22-10 (5/5)

	26	27	28	29	30
Function	$\langle 721\rangle$	$\langle 73\rangle$	$\langle 811\rangle$	$\langle 82\rangle$	$\langle 91\rangle$
$\langle\,\rangle$	6/3!/14	2/2!/21	3/3!/8	2/2!/16	2/2!/9
$\langle 1\rangle$	2/2!/14	0	2/2!/8	0	1/1!/9
$\langle 11\rangle$	0	0	1/1!/8	0	0
$\langle 2\rangle$	2/2!/7	0	0	1/1!/8	0
$\langle 111\rangle$	0	0	0	0	0
$\langle 21\rangle$	1/1!/7	0	0	0	0
$\langle 3\rangle$	0	1/1!/7	0	0	0
$\langle 1111\rangle$	0	0	0	0	0
$\langle 211\rangle$	0	0	0	0	0
$\langle 22\rangle$	0	0	0	0	0
$\langle 31\rangle$	0	0	0	0	0
$\langle 4\rangle$	0	0	0	0	0
$\langle 11111\rangle$	0	0	0	0	0
$\langle 2111\rangle$	0	0	0	0	0
$\langle 221\rangle$	0	0	0	0	0
$\langle 311\rangle$	0	0	0	0	0
$\langle 32\rangle$	0	0	0	0	0
$\langle 41\rangle$	0	0	0	0	0
$\langle 111111\rangle$	0	0	0	0	0
$\langle 21111\rangle$	0	0	0	0	0
$\langle 2211\rangle$	0	0	0	0	0
$\langle 222\rangle$	0	0	0	0	0
$\langle 3111\rangle$	0	0	0	0	0
$\langle 321\rangle$	0	0	0	0	0
$\langle 33\rangle$	0	0	0	0	0
$\langle 411\rangle$	0	0	0	0	0
$\langle 42\rangle$	0	0	0	0	0
$\langle 1111111\rangle$	0	0	0	0	0
$\langle 211111\rangle$	0	0	0	0	0
$\langle 22111\rangle$	0	0	0	0	0
$\langle 2221\rangle$	0	0	0	0	0
$\langle 31111\rangle$	0	0	0	0	0
$\langle 3211\rangle$	0	0	0	0	0
$\langle 322\rangle$	0	0	0	0	0
$\langle 331\rangle$	0	0	0	0	0
$\langle 4111\rangle$	0	0	0	0	0
$\langle 421\rangle$	0	0	0	0	0
$\langle 43\rangle$	0	0	0	0	0
$\langle 7\rangle$	2/2!/2	1/1!/3	0	0	0
Result	-1	-1	1	1	0

Table 22-11 (1/6)

Weight 11			1	2	3	4	5	6
Weight	Function	Stratum	$\langle 1^{11}\rangle$	$\langle 21^{9}\rangle$	$\langle 221^{7}\rangle$	$\langle 2221^{5}\rangle$	$\langle 22221^{3}\rangle$	$\langle 2^{5}1\rangle$
0	$\langle\ \rangle$	9	1/11!	10/10!/2	36/9!/4	56/8!/8	35/7!/16	6/6!/32
1	$\langle 1\rangle$	8	1/10!	9/9!/2	28/8!/4	35/7!/8	15/6!/16	1/5!/32
2	$\langle 11\rangle$	7	1/9!	8/8!/2	21/7!/4	20/6!/8	5/5!/16	0
	$\langle 2\rangle$	7	0	1/9!	8/8!/2	21/7!/4	20/6!/8	5/5!/16
3	$\langle 111\rangle$	6	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16	0
	$\langle 21\rangle$	6	0	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16
	$\langle 3\rangle$	6	0	0	0	0	0	0
4	$\langle 1111\rangle$	5	1/7!	6/6!/2	10/5!/4	4/4!/8	0	0
	$\langle 211\rangle$	5	0	1/7!	6/6!/2	10/5!/4	4/4!/8	0
	$\langle 22\rangle$	5	0	0	1/7!	6/6!/2	10/5!/4	4/4!/8
	$\langle 31\rangle$	5	0	0	0	0	0	0
	$\langle 4\rangle$	5	0	0	0	0	0	0
5	$\langle 11111\rangle$	5	1/6!	5/5!/2	6/4!/4	1/3!/8	0	0
	$\langle 2111\rangle$	5	0	1/6!	5/5!/2	6/4!/4	1/3!/8	0
	$\langle 221\rangle$	5	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8
	$\langle 311\rangle$	5	0	0	0	0	0	0
	$\langle 32\rangle$	5	0	0	0	0	0	0
	$\langle 41\rangle$	5	0	0	0	0	0	0
6	$\langle 111111\rangle$	5	1/5!	4/4!/2	3/3!/4	0	0	0
	$\langle 21111\rangle$	5	0	1/5!	4/4!/2	3/3!/4	0	0
	$\langle 2211\rangle$	5	0	0	1/5!	4/4!/2	3/3!/4	0
	$\langle 222\rangle$	5	0	0	0	1/5!	4/4!/2	3/3!/4
	$\langle 3111\rangle$	5	0	0	0	0	0	0
	$\langle 321\rangle$	5	0	0	0	0	0	0
	$\langle 33\rangle$	5	0	0	0	0	0	0
	$\langle 411\rangle$	5	0	0	0	0	0	0
	$\langle 42\rangle$	5	0	0	0	0	0	0
7	$\langle 1111111\rangle$	5	1/4!	3/3!/2	0	0	0	0
	$\langle 211111\rangle$	5	0	1/4!	3/3!/2	0	0	0
	$\langle 22111\rangle$	5	0	0	1/4!	3/3!/2	0	0
	$\langle 2221\rangle$	5	0	0	0	1/4!	3/3!/2	0
	$\langle 31111\rangle$	5	0	0	0	0	0	0
	$\langle 3211\rangle$	5	0	0	0	0	0	0
	$\langle 322\rangle$	5	0	0	0	0	0	0
	$\langle 331\rangle$	5	0	0	0	0	0	0
	$\langle 4111\rangle$	5	0	0	0	0	0	0
	$\langle 421\rangle$	5	0	0	0	0	0	0
	$\langle 43\rangle$	5	0	0	0	0	0	0
	$\langle 7\rangle$	5	0	0	0	0	0	0
Result			0	0	0	0	0	0

Table 22-11 (2/6)

	7	8	9	10	11	12	13
Function	$\langle 31^8\rangle$	$\langle 321^6\rangle$	$\langle 3221^4\rangle$	$\langle 32221^2\rangle$	$\langle 32^4\rangle$	$\langle 331^5\rangle$	$\langle 3321^3\rangle$
$\langle\,\rangle$	9/9!/3	56/8!/6	105/7!/12	60/6!/24	5/5!/48	21/7!/9	60/6!/18
$\langle 1\rangle$	8/8!/3	42/7!/6	60/6!/12	20/5!/24	0	15/6!/9	30/5!/18
$\langle 11\rangle$	7/7!/3	30/6!/6	30/5!/12	0	0	10/5!/9	12/4!/18
$\langle 2\rangle$	0	7/7!/3	30/6!/6	30/5!/12	4/4!/24	0	10/5!/9
$\langle 111\rangle$	6/6!/3	20/5!/6	12/4!/12	0	0	6/4!/9	3/3!/18
$\langle 21\rangle$	0	6/6!/3	20/5!/6	12/4!/12	0	0	6/4!/9
$\langle 3\rangle$	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16	6/6!/3	20/5!/6
$\langle 1111\rangle$	5/5!/3	12/4!/6	3/3!/12	0	0	3/3!/9	0
$\langle 211\rangle$	0	5/5!/3	12/4!/6	3/3!/12	0	0	3/3!/9
$\langle 22\rangle$	0	0	5/5!/3	12/4!/6	3/3!/12	0	0
$\langle 31\rangle$	1/7!	6/6!/2	10/5!/4	4/4!/8	0	5/5!/3	12/4!/6
$\langle 4\rangle$	0	0	0	0	0	0	0
$\langle 11111\rangle$	4/4!/3	6/3!/6	0	0	0	1/2!/9	0
$\langle 2111\rangle$	0	4/4!/3	6/3!/6	0	0	0	1/2!/9
$\langle 221\rangle$	0	0	4/4!/3	6/3!/6	0	0	0
$\langle 311\rangle$	1/6!	5/5!/2	6/4!/4	1/3!/8	0	4/4!/3	6/3!/6
$\langle 32\rangle$	0	1/6!	5/5!/2	6/4!/4	1/3!/8	0	0
$\langle 41\rangle$	0	0	0	0	0	0	0
$\langle 111111\rangle$	3/3!/3	2/2!/6	0	0	0	0	0
$\langle 21111\rangle$	0	3/3!/3	2/2!/6	0	0	0	0
$\langle 2211\rangle$	0	0	3/3!/3	2/2!/6	0	0	0
$\langle 222\rangle$	0	0	0	3/3!/3	2/2!/6	0	0
$\langle 3111\rangle$	1/5!	4/4!/2	3/3!/4	0	0	3/3!/3	2/2!/6
$\langle 321\rangle$	0	1/5!	4/4!/2	3/3!/4	0	0	3/3!/3
$\langle 33\rangle$	0	0	0	0	0	1/5!	4/4!/2
$\langle 411\rangle$	0	0	0	0	0	0	0
$\langle 42\rangle$	0	0	0	0	0	0	0
$\langle 1111111\rangle$	2/2!/3	0	0	0	0	0	0
$\langle 211111\rangle$	0	2/2!/3	0	0	0	0	0
$\langle 22111\rangle$	0	0	2/2!/3	0	0	0	0
$\langle 2221\rangle$	0	0	0	2/2!/3	0	0	0
$\langle 31111\rangle$	1/4!	3/3!/2	0	0	0	2/2!/3	0
$\langle 3211\rangle$	0	1/4!	0	1/2!/4	0	0	2/2!/3
$\langle 322\rangle$	0	0	1/4!	3/3!/2	1/2!/4	0	0
$\langle 331\rangle$	0	0	0	0	0	1/4!	3/3!/2
$\langle 4111\rangle$	0	0	0	0	0	0	0
$\langle 421\rangle$	0	0	0	0	0	0	0
$\langle 43\rangle$	0	0	0	0	0	0	0
$\langle 7\rangle$	0	0	0	0	0	0	0
Result	0	0	0	0	0	0	0

Table 22-11 (3/6)

	14	15	16	17	18	19	20
Function	$\langle 33221\rangle$	$\langle 3331^2\rangle$	$\langle 3332\rangle$	$\langle 41^7\rangle$	$\langle 421^5\rangle$	$\langle 4221^3\rangle$	$\langle 42221\rangle$
$\langle\ \rangle$	30/5!/36	10/5!/27	4/4!/54	8/8!/4	42/7!/8	60/6!/16	20/5!/32
$\langle 1\rangle$	6/4!/36	4/4!/27	0	7/7!/4	30/6!/8	30/5!/16	4/4!/32
$\langle 11\rangle$	0	1/3!/27	0	6/6!/4	20/5!/8	12/4!/16	0
$\langle 2\rangle$	12/4!/18	0	1/3!/27	0	6/6!/4	20/5!/8	12/4!/16
$\langle 111\rangle$	0	0	0	5/5!/4	12/4!/8	3/3!/16	0
$\langle 21\rangle$	3/3!/18	0	0	0	5/5!/4	12/4!/8	3/3!/16
$\langle 3\rangle$	12/4!/12	6/4!/9	3/3!/18	0	0	0	0
$\langle 1111\rangle$	0	0	0	4/4!/4	6/3!/8	0	0
$\langle 211\rangle$	0	0	0	0	4/4!/4	6/3!/8	0
$\langle 22\rangle$	3/3!/9	0	0	0	0	4/4!/4	6/3!/8
$\langle 31\rangle$	3/3!/12	3/3!/9	0	0	0	0	0
$\langle 4\rangle$	0	0	0	1/7!	6/6!/2	10/5!/4	4/4!/8
$\langle 11111\rangle$	0	0	0	3/3!/4	2/2!/8	0	0
$\langle 2111\rangle$	0	0	0	0	3/3!/4	2/2!/8	0
$\langle 221\rangle$	1/2!/9	0	0	0	0	3/3!/4	2/2!/8
$\langle 311\rangle$	0	1/2!/9	0	0	0	0	0
$\langle 32\rangle$	4/4!/3	0	1/2!/9	0	0	0	0
$\langle 41\rangle$	0	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8
$\langle 111111\rangle$	0	0	0	2/2!/4	0	0	0
$\langle 21111\rangle$	0	0	0	0	2/2!/4	0	0
$\langle 2211\rangle$	0	0	0	0	0	2/2!/4	0
$\langle 222\rangle$	0	0	0	0	0	0	2/2!/4
$\langle 3111\rangle$	0	0	0	0	0	0	0
$\langle 321\rangle$	2/2!/6	0	0	0	0	0	0
$\langle 33\rangle$	3/3!/4	3/3!/3	2/2!/6	0	0	0	0
$\langle 411\rangle$	0	0	0	1/5!	4/4!/2	3/3!/4	0
$\langle 42\rangle$	0	0	0	0	1/5!	4/4!/2	3/3!/4
$\langle 1111111\rangle$	0	0	0	0	0	0	0
$\langle 211111\rangle$	0	0	0	0	0	0	0
$\langle 22111\rangle$	0	0	0	0	0	0	0
$\langle 2221\rangle$	0	0	0	0	0	0	0
$\langle 31111\rangle$	0	0	0	0	0	0	0
$\langle 3211\rangle$	0	0	0	0	0	0	0
$\langle 322\rangle$	1/2!/3	0	0	0	0	0	0
$\langle 331\rangle$	1/2!/4	2/2!/3	0	0	0	0	0
$\langle 4111\rangle$	0	0	0	1/4!	3/3!/2	1/2!/4	0
$\langle 421\rangle$	0	0	0	0	1/4!	3/3!/2	1/2!/4
$\langle 43\rangle$	0	0	0	0	0	0	0
$\langle 7\rangle$	0	0	0	0	0	0	0
Result	0	0	0	0	0	0	0

Table 22-11 (4/6)

	21	22	23	24	25	26
Function	$\langle 431^4\rangle$	$\langle 43211\rangle$	$\langle 4322\rangle$	$\langle 4331\rangle$	$\langle 441^3\rangle$	$\langle 4421\rangle$
$\langle\,\rangle$	30/6!/12	60/5!/24	12/4!/48	12/4!/36	10/5!/16	12/4!/32
$\langle 1\rangle$	20/5!/12	24/4!/24	0	3/3!/36	6/4!/16	3/3!/32
$\langle 11\rangle$	12/4!/12	6/3!/24	0	0	3/3!/16	0
$\langle 2\rangle$	0	12/4!/12	6/3!/24	0	0	3/3!/16
$\langle 111\rangle$	6/3!/12	0	0	0	1/2!/16	0
$\langle 21\rangle$	0	6/3!/12	0	0	0	1/2!/16
$\langle 3\rangle$	5/5!/4	12/4!/8	3/3!/16	6/3!/12	0	0
$\langle 1111\rangle$	2/2!/12	0	0	0	0	0
$\langle 211\rangle$	0	2/2!/12	0	0	0	0
$\langle 22\rangle$	0	0	2/2!/12	0	0	0
$\langle 31\rangle$	4/4!/4	6/3!/8	0	2/2!/12	0	0
$\langle 4\rangle$	5/5!/3	12/4!/6	3/3!/12	3/3!/9	0	4/4!/4
$\langle 11111\rangle$	0	0	0	0	0	0
$\langle 2111\rangle$	0	0	0	0	0	0
$\langle 221\rangle$	0	0	0	0	0	0
$\langle 311\rangle$	3/3!/4	2/2!/8	0	0	0	0
$\langle 32\rangle$	0	3/3!/4	2/2!/8	0	0	0
$\langle 41\rangle$	4/4!/3	6/3!/6	0	1/2!/9	3/3!/4	2/2!/8
$\langle 111111\rangle$	0	0	0	0	0	0
$\langle 21111\rangle$	0	0	0	0	0	0
$\langle 2211\rangle$	0	0	0	0	0	0
$\langle 222\rangle$	0	0	0	0	0	0
$\langle 3111\rangle$	2/2!/4	0	0	0	0	0
$\langle 321\rangle$	0	2/2!/4	0	0	0	0
$\langle 33\rangle$	0	0	0	2/2!/4	0	0
$\langle 411\rangle$	3/3!/3	2/2!/6	0	0	2/2!/4	0
$\langle 42\rangle$	0	3/3!/3	2/2!/6	0	0	2/2!/4
$\langle 1111111\rangle$	0	0	0	0	0	0
$\langle 211111\rangle$	0	0	0	0	0	0
$\langle 22111\rangle$	0	0	0	0	0	0
$\langle 2221\rangle$	0	0	0	0	0	0
$\langle 31111\rangle$	0	0	0	0	0	0
$\langle 3211\rangle$	0	1/1!/4	0	0	0	0
$\langle 322\rangle$	0	0	1/1!/4	0	0	0
$\langle 331\rangle$	0	0	0	1/1!/4	0	0
$\langle 4111\rangle$	2/2!/3	0	0	0	1/1!/4	0
$\langle 421\rangle$	0	2/2!/3	0	0	0	1/1!/4
$\langle 43\rangle$	1/4!	3/3!/2	1/2!/4	2/2!/3	0	0
$\langle 7\rangle$	0	0	0	0	0	0
Result	0	0	0	0	0	0

Table 22-11 (5/6)

	27	28	29	30	31	32	33
Function	$\langle 443\rangle$	$\langle 71^4\rangle$	$\langle 721^2\rangle$	$\langle 722\rangle$	$\langle 731\rangle$	$\langle 74\rangle$	$\langle 81^3\rangle$
$\langle\,\rangle$	3/3!/48	5/5!/7	12/4!/14	3/3!/28	6/3!/21	2/2!/28	4/4!/8
$\langle 1\rangle$	0	4/4!/7	6/3!/14	0	2/2!/21	0	3/3!/8
$\langle 11\rangle$	0	3/3!/7	2/2!/14	0	0	0	2/2!/8
$\langle 2\rangle$	0	0	3/3!/7	2/2!/14	0	0	0
$\langle 111\rangle$	0	2/2!/7	0	0	0	0	1/1!/8
$\langle 21\rangle$	0	0	2/2!/7	0	0	0	0
$\langle 3\rangle$	1/2!/16	0	0	0	2/2!/7	0	0
$\langle 1111\rangle$	0	1/1!/7	0	0	0	0	0
$\langle 211\rangle$	0	0	1/1!/7	0	0	0	0
$\langle 22\rangle$	0	0	0	1/1!/7	0	0	0
$\langle 31\rangle$	0	0	0	0	1/1!/7	0	0
$\langle 4\rangle$	0	0	0	0	0	1/1!/7	0
$\langle 11111\rangle$	0	0	0	0	0	0	0
$\langle 2111\rangle$	0	0	0	0	0	0	0
$\langle 221\rangle$	0	0	0	0	0	0	0
$\langle 311\rangle$	0	0	0	0	0	0	0
$\langle 32\rangle$	0	0	0	0	0	0	0
$\langle 41\rangle$	0	0	0	0	0	0	0
$\langle 111111\rangle$	0	0	0	0	0	0	0
$\langle 21111\rangle$	0	0	0	0	0	0	0
$\langle 2211\rangle$	0	0	0	0	0	0	0
$\langle 222\rangle$	0	0	0	0	0	0	0
$\langle 3111\rangle$	0	0	0	0	0	0	0
$\langle 321\rangle$	0	0	0	0	0	0	0
$\langle 33\rangle$	0	0	0	0	0	0	0
$\langle 411\rangle$	0	0	0	0	0	0	0
$\langle 42\rangle$	0	0	0	0	0	0	0
$\langle 1111111\rangle$	0	0	0	0	0	0	0
$\langle 211111\rangle$	0	0	0	0	0	0	0
$\langle 22111\rangle$	0	0	0	0	0	0	0
$\langle 2221\rangle$	0	0	0	0	0	0	0
$\langle 31111\rangle$	0	0	0	0	0	0	0
$\langle 3211\rangle$	0	0	0	0	0	0	0
$\langle 322\rangle$	0	0	0	0	0	0	0
$\langle 331\rangle$	0	0	0	0	0	0	0
$\langle 4111\rangle$	0	0	0	0	0	0	0
$\langle 421\rangle$	0	0	0	0	0	0	0
$\langle 43\rangle$	1/1!/4	0	0	0	0	0	0
$\langle 7\rangle$	0	1/4!	3/3!/2	1/2!/4	2/2!/3	1/1!/4	0
Result	0	0	0	0	0	0	0

Table 22-11 (6/6)

	34	35	36	37
Function	⟨821⟩	⟨83⟩	⟨911⟩	⟨92⟩
⟨ ⟩	6/3!/16	2/2!/24	3/3!/9	2/2!/18
⟨1⟩	2/2!/16	0	2/2!/9	0
⟨11⟩	0	0	1/1!/9	0
⟨2⟩	2/2!/8	0	0	1/1!/9
⟨111⟩	0	0	0	0
⟨21⟩	1/1!/8	0	0	0
⟨3⟩	0	1/1!/8	0	0
⟨1111⟩	0	0	0	0
⟨211⟩	0	0	0	0
⟨22⟩	0	0	0	0
⟨31⟩	0	0	0	0
⟨4⟩	0	0	0	0
⟨11111⟩	0	0	0	0
⟨2111⟩	0	0	0	0
⟨221⟩	0	0	0	0
⟨311⟩	0	0	0	0
⟨32⟩	0	0	0	0
⟨41⟩	0	0	0	0
⟨111111⟩	0	0	0	0
⟨21111⟩	0	0	0	0
⟨2211⟩	0	0	0	0
⟨222⟩	0	0	0	0
⟨3111⟩	0	0	0	0
⟨321⟩	0	0	0	0
⟨33⟩	0	0	0	0
⟨411⟩	0	0	0	0
⟨42⟩	0	0	0	0
⟨1111111⟩	0	0	0	0
⟨211111⟩	0	0	0	0
⟨22111⟩	0	0	0	0
⟨2221⟩	0	0	0	0
⟨31111⟩	0	0	0	0
⟨3211⟩	0	0	0	0
⟨322⟩	0	0	0	0
⟨331⟩	0	0	0	0
⟨4111⟩	0	0	0	0
⟨421⟩	0	0	0	0
⟨43⟩	0	0	0	0
⟨7⟩	0	0	0	0
Result	0	0	0	0

Table 22-12 (1/5)

Weight 12, 4th Stratum			1	2	3	4	5	6	7	8
Weight	Function	Stratum	$\langle 1^{12}\rangle$	$\langle 21^{10}\rangle$	$\langle 221^{8}\rangle$	$\langle 2221^{6}\rangle$	$\langle 22221^{4}\rangle$	$\langle 2^{5}11\rangle$	$\langle 2^{6}\rangle$	$\langle 31^{9}\rangle$
0	$\langle\,\rangle$	9	1/12!	11/11!/2	45/10!/4	84/9!/8	70/8!/16	21/7!/32	1/6!/64	10/10!/3
1	$\langle 1\rangle$	8	1/11!	10/10!/2	36/9!/4	56/8!/8	35/7!/16	6/6!/32	0	9/9!/3
2	$\langle 11\rangle$	7	1/10!	9/9!/2	28/8!/4	35/7!/8	15/6!/16	1/5!/32	0	8/8!/3
	$\langle 2\rangle$	7	0	1/10!	9/9!/2	28/8!/4	35/7!/8	15/6!/16	1/5!/32	0
3	$\langle 111\rangle$	6	1/9!	8/8!/2	21/7!/4	20/6!/8	5/5!/16	0	0	7/7!/3
	$\langle 21\rangle$	6	0	1/9!	8/8!/2	21/7!/4	20/6!/8	5/5!/16	0	0
	$\langle 3\rangle$	6	0	0	0	0	0	0	0	1/9!
4	$\langle 1111\rangle$	5	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16	0	0	6/6!/3
	$\langle 211\rangle$	5	0	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16	0	0
	$\langle 22\rangle$	5	0	0	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16	0
	$\langle 31\rangle$	5	0	0	0	0	0	0	0	1/8!
	$\langle 4\rangle$	5	0	0	0	0	0	0	0	0
5	$\langle 11111\rangle$	5	1/7!	6/6!/2	10/5!/4	4/4!/8	0	0	0	5/5!/3
	$\langle 2111\rangle$	5	0	1/7!	6/6!/2	10/5!/4	4/4!/8	0	0	0
	$\langle 221\rangle$	5	0	0	1/7!	6/6!/2	10/5!/4	4/4!/8	0	0
	$\langle 311\rangle$	5	0	0	0	0	0	0	0	1/7!
	$\langle 32\rangle$	5	0	0	0	0	0	0	0	0
	$\langle 41\rangle$	5	0	0	0	0	0	0	0	0
6	$\langle 111111\rangle$	5	1/6!	5/5!/2	6/4!/4	1/3!/8	0	0	0	4/4!/3
	$\langle 21111\rangle$	5	0	1/6!	5/5!/2	6/4!/4	1/3!/8	0	0	0
	$\langle 2211\rangle$	5	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8	0	0
	$\langle 222\rangle$	5	0	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8	0
	$\langle 3111\rangle$	5	0	0	0	0	0	0	0	1/6!
	$\langle 321\rangle$	5	0	0	0	0	0	0	0	0
	$\langle 33\rangle$	5	0	0	0	0	0	0	0	0
	$\langle 411\rangle$	5	0	0	0	0	0	0	0	0
	$\langle 42\rangle$	5	0	0	0	0	0	0	0	0
7	$\langle 1111111\rangle$	5	1/5!	4/4!/2	3/3!/4	0	0	0	0	3/3!/3
	$\langle 211111\rangle$	5	0	1/5!	4/4!/2	3/3!/4	0	0	0	0
	$\langle 22111\rangle$	5	0	0	1/5!	4/4!/2	3/3!/4	0	0	0
	$\langle 2221\rangle$	5	0	0	0	1/5!	4/4!/2	3/3!/4	0	0
	$\langle 31111\rangle$	5	0	0	0	0	0	0	0	1/5!
	$\langle 3211\rangle$	5	0	0	0	0	0	0	0	0
	$\langle 322\rangle$	5	0	0	0	0	0	0	0	0
	$\langle 331\rangle$	5	0	0	0	0	0	0	0	0
	$\langle 4111\rangle$	5	0	0	0	0	0	0	0	0
	$\langle 421\rangle$	5	0	0	0	0	0	0	0	0
	$\langle 43\rangle$	5	0	0	0	0	0	0	0	0
	$\langle 7\rangle$	5	0	0	0	0	0	0	0	0
8	$\langle 11111111\rangle$	3	1/4!	3/3!/2	0	0	0	0	0	2/2!/3
	$\langle 2111111\rangle$	3	0	1/4!	3/3!/2	0	0	0	0	0
	$\langle 221111\rangle$	3	0	0	1/4!	3/3!/2	0	0	0	0
	$\langle 22211\rangle$	3	0	0	0	1/4!	3/3!/2	0	0	0
	$\langle 2222\rangle$	3	0	0	0	0	1/4!	3/3!/2	2/2!/4	0
	$\langle 311111\rangle$	3	0	0	0	0	0	0	0	1/4!
	$\langle 32111\rangle$	3	0	0	0	0	0	0	0	0
	$\langle 3221\rangle$	3	0	0	0	0	0	0	0	0
	$\langle 3311\rangle$	3	0	0	0	0	0	0	0	0
	$\langle 332\rangle$	3	0	0	0	0	0	0	0	0
	$\langle 41111\rangle$	3	0	0	0	0	0	0	0	0
	$\langle 4211\rangle$	3	0	0	0	0	0	0	0	0
	$\langle 422\rangle$	3	0	0	0	0	0	0	0	0
	$\langle 431\rangle$	3	0	0	0	0	0	0	0	0
	$\langle 44\rangle$	3	0	0	0	0	0	0	0	0
	$\langle 71\rangle$	3	0	0	0	0	0	0	0	0
	$\langle 8\rangle$	3	0	0	0	0	0	0	0	0
Result			0	0	0	0	0	0	0	0

Table 22-12 (2/5)

	9	10	11	12	13	14	15	16	17
Function	$\langle 321^7\rangle$	$\langle 3221^5\rangle$	$\langle 32221^3\rangle$	$\langle 32^41\rangle$	$\langle 331^6\rangle$	$\langle 3321^4\rangle$	$\langle 332211\rangle$	$\langle 33222\rangle$	$\langle 3331^3\rangle$
$\langle\,\rangle$	72/9!/6	168/8!/12	140/7!/24	30/6!/48	28/8!/9	105/7!/18	90/6!/36	10/5!/72	20/6!/27
$\langle 1\rangle$	56/8!/6	105/7!/12	60/6!/24	5/5!/48	21/7!/9	60/6!/18	30/5!/36	0	10/5!/27
$\langle 11\rangle$	42/7!/6	60/6!/12	20/5!/24	0	15/6!/9	30/5!/18	6/4!/36	0	4/4!/27
$\langle 2\rangle$	8/8!/3	42/7!/6	60/6!/12	20/5!/24	0	15/6!/9	30/5!/18	6/4!/36	0
$\langle 111\rangle$	30/6!/6	30/5!/12	0	0	10/5!/9	12/4!/18	0	0	1/3!/27
$\langle 21\rangle$	7/7!/3	30/6!/6	30/5!/12	4/4!/24	0	10/5!/9	12/4!/18	0	0
$\langle 3\rangle$	8/8!/2	21/7!/4	20/6!/8	5/5!/16	7/7!/3	30/6!/6	30/5!/12	4/4!/24	10/5!/9
$\langle 1111\rangle$	20/5!/6	12/4!/12	0	0	6/4!/9	3/3!/18	0	0	0
$\langle 211\rangle$	6/6!/3	20/5!/6	12/4!/12	0	0	6/4!/9	3/3!/18	0	0
$\langle 22\rangle$	0	6/6!/3	20/5!/6	12/4!/12	0	0	6/4!/9	3/3!/18	0
$\langle 31\rangle$	7/7!/2	15/6!/4	10/5!/8	1/4!/16	6/6!/3	20/5!/6	12/4!/12	0	6/4!/9
$\langle 4\rangle$	0	0	0	0	0	0	0	0	0
$\langle 11111\rangle$	12/4!/6	3/3!/12	0	0	3/3!/9	0	0	0	0
$\langle 2111\rangle$	5/5!/3	12/4!/6	3/3!/12	0	0	3/3!/9	0	0	0
$\langle 221\rangle$	0	5/5!/3	12/4!/6	3/3!/12	0	0	3/3!/9	0	0
$\langle 311\rangle$	6/6!/2	10/5!/4	4/4!/8	0	5/5!/3	12/4!/6	3/3!/12	0	3/3!/9
$\langle 32\rangle$	1/7!	6/6!/2	10/5!/4	4/4!/8	0	5/5!/3	12/4!/6	3/3!/12	0
$\langle 41\rangle$	0	0	0	0	0	0	0	0	0
$\langle 111111\rangle$	6/3!/6	0	0	0	1/2!/9	0	0	0	0
$\langle 21111\rangle$	4/4!/3	6/3!/6	0	0	0	1/2!/9	0	0	0
$\langle 2211\rangle$	0	4/4!/3	6/3!/6	0	0	0	1/2!/9	0	0
$\langle 222\rangle$	0	0	4/4!/3	6/3!/6	0	0	0	1/2!/9	0
$\langle 3111\rangle$	5/5!/2	6/4!/4	1/3!/8	0	4/4!/3	6/3!/6	0	0	1/2!/9
$\langle 321\rangle$	1/6!	5/5!/2	6/4!/4	1/3!/8	0	0	4/4!/3	6/3!/6	0
$\langle 33\rangle$	0	0	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8	4/4!/3
$\langle 411\rangle$	0	0	0	0	0	0	0	0	0
$\langle 42\rangle$	0	0	0	0	0	0	0	0	0
$\langle 1111111\rangle$	2/2!/6	0	0	0	0	0	0	0	0
$\langle 211111\rangle$	3/3!/3	2/2!/6	0	0	0	0	0	0	0
$\langle 22111\rangle$	0	3/3!/3	2/2!/6	0	0	0	0	0	0
$\langle 2221\rangle$	0	0	3/3!/3	2/2!/6	0	0	0	0	0
$\langle 31111\rangle$	4/4!/2	3/3!/4	0	0	3/3!/3	2/2!/6	0	0	0
$\langle 3211\rangle$	1/5!	4/4!/2	3/3!/4	0	0	3/3!/3	2/2!/6	0	0
$\langle 322\rangle$	0	1/5!	4/4!/2	3/3!/4	0	0	3/3!/3	2/2!/6	0
$\langle 331\rangle$	0	0	0	0	1/5!	4/4!/2	3/3!/4	0	3/3!/3
$\langle 4111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 421\rangle$	0	0	0	0	0	0	0	0	0
$\langle 43\rangle$	0	0	0	0	0	0	0	0	0
$\langle 7\rangle$	0	0	0	0	0	0	0	0	0
$\langle 11111111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 2111111\rangle$	2/2!/3	0	0	0	0	0	0	0	0
$\langle 221111\rangle$	0	2/2!/3	0	0	0	0	0	0	0
$\langle 22211\rangle$	0	0	2/2!/3	0	0	0	0	0	0
$\langle 2222\rangle$	0	0	0	2/2!/3	0	0	0	0	0
$\langle 311111\rangle$	3/3!/2	0	0	0	2/2!/3	0	0	0	0
$\langle 32111\rangle$	1/4!	0	1/2!/4	0	0	2/2!/3	0	0	0
$\langle 3221\rangle$	0	1/4!	3/3!/2	1/2!/4	0	0	1/2!/3	0	0
$\langle 3311\rangle$	0	0	0	0	1/4!	3/3!/2	1/2!/4	0	2/2!/3
$\langle 332\rangle$	0	0	0	0	0	1/4!	3/3!/2	1/2!/4	0
$\langle 41111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 4211\rangle$	0	0	0	0	0	0	0	0	0
$\langle 422\rangle$	0	0	0	0	0	0	0	0	0
$\langle 431\rangle$	0	0	0	0	0	0	0	0	0
$\langle 44\rangle$	0	0	0	0	0	0	0	0	0
$\langle 71\rangle$	0	0	0	0	0	0	0	0	0
$\langle 8\rangle$	0	0	0	0	0	0	0	0	0
Result	0	0	0	0	0	0	0	0	0

Table 22-12 (3/5)

	18	19	20	21	22	23	24	25	26	27
Function	$\langle 33321\rangle$	$\langle 3333\rangle$	$\langle 41^8\rangle$	$\langle 421^6\rangle$	$\langle 4221^4\rangle$	$\langle 422211\rangle$	$\langle 42222\rangle$	$\langle 431^5\rangle$	$\langle 432111\rangle$	$\langle 43221\rangle$
$\langle\,\rangle$	20/5!/54	1/4!/81	9/9!/4	56/8!/8	168/8!/16	60/6!/32	5/5!/64	42/7!/12	120/6!/24	60/5!/48
$\langle 1\rangle$	4/4!/54	0	8/8!/4	42/7!/8	60/6!/16	20/5!/32	0	30/6!/12	60/5!/24	12/4!/48
$\langle 11\rangle$	0	0	7/7!/4	30/6!/8	30/5!/16	4/4!/32	0	20/5!/12	24/4!/24	0
$\langle 2\rangle$	4/4!/27	0	0	7/7!/4	30/6!/8	30/5!/16	4/4!/32	0	20/5!/12	24/4!/24
$\langle 111\rangle$	0	0	6/6!/4	20/5!/8	12/4!/16	0	0	12/4!/12	6/3!/24	0
$\langle 21\rangle$	1/3!/27	0	0	6/6!/4	20/5!/8	12/4!/16	0	0	12/4!/12	6/3!/24
$\langle 3\rangle$	12/4!/18	1/3!/27	0	0	0	0	0	6/6!/4	20/5!/8	12/4!/16
$\langle 1111\rangle$	0	0	5/5!/4	12/4!/8	3/3!/16	0	0	6/3!/12	0	0
$\langle 211\rangle$	0	0	0	5/5!/4	12/4!/8	3/3!/16	0	0	6/3!/12	0
$\langle 22\rangle$	0	0	0	0	5/5!/4	12/4!/8	3/3!/16	0	0	6/3!/12
$\langle 31\rangle$	3/3!/18	0	0	0	0	0	0	5/5!/4	12/4!/8	3/3!/16
$\langle 4\rangle$	0	0	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16	6/6!/3	20/5!/6	12/4!/12
$\langle 11111\rangle$	0	0	4/4!/4	6/3!/8	0	0	0	2/2!/12	0	0
$\langle 2111\rangle$	0	0	0	4/4!/4	6/3!/8	0	0	0	2/2!/12	0
$\langle 221\rangle$	0	0	0	0	4/4!/4	6/3!/8	0	0	0	2/2!/12
$\langle 311\rangle$	0	0	0	0	0	0	0	4/4!/4	6/3!/8	0
$\langle 32\rangle$	3/3!/9	0	0	0	0	0	0	0	4/4!/4	6/3!/8
$\langle 41\rangle$	0	0	1/7!	6/6!/2	10/5!/4	4/4!/8	0	5/5!/3	12/4!/6	3/3!/12
$\langle 111111\rangle$	0	0	3/3!/4	2/2!/8	0	0	0	0	0	0
$\langle 21111\rangle$	0	0	0	3/3!/4	2/2!/8	0	0	0	0	0
$\langle 2211\rangle$	0	0	0	0	3/3!/4	2/2!/8	0	0	0	0
$\langle 222\rangle$	0	0	0	0	0	3/3!/4	2/2!/8	0	0	0
$\langle 3111\rangle$	0	0	0	0	0	0	0	3/3!/4	2/2!/8	0
$\langle 321\rangle$	1/2!/9	0	0	0	0	0	0	0	3/3!/4	2/2!/8
$\langle 33\rangle$	6/3!/6	1/2!/9	0	0	0	0	0	0	0	0
$\langle 411\rangle$	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8	0	4/4!/3	6/3!/6	0
$\langle 42\rangle$	0	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8	0	4/4!/3	6/3!/6
$\langle 1111111\rangle$	0	0	2/2!/4	0	0	0	0	0	0	0
$\langle 211111\rangle$	0	0	0	2/2!/4	0	0	0	0	0	0
$\langle 22111\rangle$	0	0	0	0	2/2!/4	0	0	0	0	0
$\langle 2221\rangle$	0	0	0	0	0	2/2!/4	0	0	0	0
$\langle 31111\rangle$	0	0	0	0	0	0	0	2/2!/4	0	0
$\langle 3211\rangle$	0	0	0	0	0	0	0	0	2/2!/4	0
$\langle 322\rangle$	0	0	0	0	0	0	0	0	0	2/2!/4
$\langle 331\rangle$	2/2!/6	0	0	0	0	0	0	0	0	0
$\langle 4111\rangle$	0	0	1/5!	4/4!/2	3/3!/4	0	0	3/3!/3	2/2!/6	0
$\langle 421\rangle$	0	0	0	1/5!	4/4!/2	3/3!/4	0	0	3/3!/3	2/2!/6
$\langle 43\rangle$	0	0	0	0	0	0	0	1/5!	4/4!/2	3/3!/4
$\langle 7\rangle$	0	0	0	0	0	0	0	0	0	0
$\langle 11111111\rangle$	0	0	0	0	0	0	0	0	0	0
$\langle 2111111\rangle$	0	0	0	0	0	0	0	0	0	0
$\langle 221111\rangle$	0	0	0	0	0	0	0	0	0	0
$\langle 22211\rangle$	0	0	0	0	0	0	0	0	0	0
$\langle 2222\rangle$	0	0	0	0	0	0	1/1!/4	0	0	0
$\langle 311111\rangle$	0	0	0	0	0	0	0	0	0	0
$\langle 32111\rangle$	0	0	0	0	0	0	0	0	1/1!/4	0
$\langle 3221\rangle$	0	0	0	0	0	0	0	0	0	1/1!/4
$\langle 3311\rangle$	0	0	0	0	0	0	0	0	0	0
$\langle 332\rangle$	2/2!/3	0	0	0	0	0	0	0	0	0
$\langle 41111\rangle$	0	0	1/4!	3/3!/2	1/2!/4	0	0	2/2!/3	0	0
$\langle 4211\rangle$	0	0	0	1/4!	3/3!/2	1/2!/4	0	0	2/2!/3	0
$\langle 422\rangle$	0	0	0	0	1/4!	3/3!/2	1/2!/4	0	0	2/2!/3
$\langle 431\rangle$	0	0	0	0	0	0	0	1/4!	3/3!/2	1/2!/4
$\langle 44\rangle$	0	0	0	0	0	0	0	0	0	0
$\langle 71\rangle$	0	0	0	0	0	0	0	0	0	0
$\langle 8\rangle$	0	0	0	0	0	0	0	0	0	0
Result	0	0	0	0	0	0	0	0	0	0

Table 22-12 (4/5)

	28	29	30	31	32	33	34	35	36	37
Function	⟨43311⟩	⟨4332⟩	$\langle 441^4\rangle$	⟨44211⟩	⟨4422⟩	⟨4431⟩	⟨444⟩	$\langle 71^5\rangle$	$\langle 721^3\rangle$	⟨7221⟩
⟨ ⟩	30/5!/36	12/4!/72	15/6!/16	30/5!/32	6/4!/64	12/4!/48	1/3!/64	6/6!/7	20/5!/14	2/4!/28
⟨1⟩	12/4!/36	0	10/5!/16	12/4!/32	0	3/3!/48	0	5/5!/7	12/4!/14	3/3!/28
⟨11⟩	3/3!/36	0	6/4!/16	3/3!/32	0	0	0	4/4!/7	6/3!/14	0
⟨2⟩	0	3/3!/36	0	6/4!/16	3/3!/32	0	0	0	4/4!/7	6/3!/14
⟨111⟩	0	0	3/3!/16	0	0	0	0	3/3!/7	2/2!/14	0
⟨21⟩	0	0	0	3/3!/16	0	0	0	0	3/3!/7	2/2!/14
⟨3⟩	12/4!/12	6/3!/24	0	0	0	3/3!/16	0	0	0	0
⟨1111⟩	0	0	1/2!/16	0	0	0	0	2/2!/7	0	0
⟨211⟩	0	0	0	1/2!/16	0	0	0	0	2/2!/7	0
⟨22⟩	0	0	0	0	1/2!/16	0	0	0	0	2/2!/7
⟨31⟩	6/3!/12	0	0	0	0	1/2!/16	0	0	0	0
⟨4⟩	6/4!/9	3/3!/18	0	5/5!/4	12/4!/8	3/3!/16	6/3!/12	0	0	0
⟨11111⟩	0	0	0	0	0	0	0	1/1!/7	0	0
⟨2111⟩	0	0	0	0	0	0	0	0	1/1!/7	0
⟨221⟩	0	0	0	0	0	0	0	0	0	1/1!/7
⟨311⟩	2/2!/12	0	0	0	0	0	0	0	0	0
⟨32⟩	0	2/2!/12	0	0	0	0	0	0	0	0
⟨41⟩	3/3!/9	0	0	4/4!/4	6/3!/8	0	2/2!/12	0	0	0
⟨111111⟩	0	0	0	0	0	0	0	0	0	0
⟨21111⟩	0	0	0	0	0	0	0	0	0	0
⟨2211⟩	0	0	0	0	0	0	0	0	0	0
⟨222⟩	0	0	0	0	0	0	0	0	0	0
⟨3111⟩	0	0	0	0	0	0	0	0	0	0
⟨321⟩	0	0	0	0	0	0	0	0	0	0
⟨33⟩	3/3!/4	2/2!/8	0	0	0	0	0	0	0	0
⟨411⟩	1/2!/9	0	3/3!/4	2/2!/8	0	0	0	0	0	0
⟨42⟩	0	1/2!/9	0	3/3!/4	2/2!/8	0	0	0	0	0
⟨1111111⟩	0	0	0	0	0	0	0	0	0	0
⟨211111⟩	0	0	0	0	0	0	0	0	0	0
⟨22111⟩	0	0	0	0	0	0	0	0	0	0
⟨2221⟩	0	0	0	0	0	0	0	0	0	0
⟨31111⟩	0	0	0	0	0	0	0	0	0	0
⟨3211⟩	0	0	0	0	0	0	0	0	0	0
⟨322⟩	0	0	0	0	0	0	0	0	0	0
⟨331⟩	2/2!/4	0	0	0	0	0	0	0	0	0
⟨4111⟩	0	0	2/2!/4	0	0	0	0	0	0	0
⟨421⟩	0	0	0	2/2!/4	0	0	0	0	0	0
⟨43⟩	3/3!/3	2/2!/6	0	0	0	2/2!/4	0	0	0	0
⟨7⟩	0	0	0	0	0	0	0	1/5!	4/4!/2	3/3!/4
⟨11111111⟩	0	0	0	0	0	0	0	0	0	0
⟨2111111⟩	0	0	0	0	0	0	0	0	0	0
⟨221111⟩	0	0	0	0	0	0	0	0	0	0
⟨22211⟩	0	0	0	0	0	0	0	0	0	0
⟨2222⟩	0	0	0	0	0	0	0	0	0	0
⟨311111⟩	0	0	0	0	0	0	0	0	0	0
⟨32111⟩	0	0	0	0	0	0	0	0	0	0
⟨3221⟩	0	0	0	0	0	0	0	0	0	0
⟨3311⟩	1/1!/4	0	0	0	0	0	0	0	0	0
⟨332⟩	0	1/1!/4	0	0	0	0	0	0	0	0
⟨41111⟩	0	0	1/1!/4	0	0	0	0	0	0	0
⟨4211⟩	0	0	0	1/1!/4	0	0	0	0	0	0
⟨422⟩	0	0	0	0	1/1!/4	0	0	0	0	0
⟨431⟩	2/2!/3	0	0	0	0	1/1!/4	0	0	0	0
⟨44⟩	0	0	1/4!	3/3!/2	2/2!/4	2/2!/3	1/1!/4	0	0	0
⟨71⟩	0	0	0	0	0	0	0	1/4!	3/3!/2	1/2!/4
⟨8⟩	0	0	0	0	0	0	0	0	0	0
Result	0	0	0	0	0	0	0	0	0	0

Table 22-12 (5/5)

	38	39	40	41	42	43	44	45	46	47	48
Function	$\langle 7311\rangle$	$\langle 732\rangle$	$\langle 741\rangle$	$\langle 81^4\rangle$	$\langle 8211\rangle$	$\langle 822\rangle$	$\langle 831\rangle$	$\langle 84\rangle$	$\langle 9111\rangle$	$\langle 921\rangle$	$\langle 93\rangle$
$\langle\,\rangle$	12/4!/21	6/3!/42	6/3!/28	5/5!/8	12/4!/16	12/4!/32	6/3!/24	2/2!/32	4/4!/9	6/3!/18	2/2!/27
$\langle 1\rangle$	6/3!/21	0	2/2!/28	4/4!/8	6/3!/16	3/3!/32	2/2!/24	0	3/3!/9	2/2!/18	0
$\langle 11\rangle$	2/2!/21	0	0	3/3!/8	2/2!/16	0	0	0	2/2!/9	0	0
$\langle 2\rangle$	0	2/2!/21	0	0	3/3!/8	2/2!/16	0	0	0	2/2!/9	0
$\langle 111\rangle$	0	0	0	2/2!/8	0	0	0	0	1/1!/9	0	0
$\langle 21\rangle$	0	0	0	0	2/2!/8	0	0	0	0	1/1!/9	0
$\langle 3\rangle$	3/3!/7	2/2!/14	0	0	0	0	2/2!/8	0	0	0	1/1!/9
$\langle 1111\rangle$	0	0	0	1/1!/8	0	0	0	0	0	0	0
$\langle 211\rangle$	0	0	0	0	1/1!/8	0	0	0	0	0	0
$\langle 22\rangle$	0	0	0	0	0	1/1!/8	0	0	0	0	0
$\langle 31\rangle$	2/2!/7	0	0	0	0	0	1/1!/8	0	0	0	0
$\langle 4\rangle$	0	0	2/2!/7	0	0	0	0	1/1!/8	0	0	0
$\langle 11111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 2111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 221\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 311\rangle$	1/1!/7	0	0	0	0	0	0	0	0	0	0
$\langle 32\rangle$	0	1/1!/7	0	0	0	0	0	0	0	0	0
$\langle 41\rangle$	0	0	1/1!/7	0	0	0	0	0	0	0	0
$\langle 111111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 21111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 2211\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 222\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 3111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 321\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 33\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 411\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 42\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 1111111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 211111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 22111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 2221\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 31111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 3211\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 322\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 331\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 4111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 421\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 43\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 7\rangle$	3/3!/3	2/2!/6	2/2!/4	0	0	0	0	0	0	0	0
$\langle 11111111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 2111111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 221111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 22211\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 2222\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 311111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 32111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 3221\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 3311\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 332\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 41111\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 4211\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 422\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 431\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 44\rangle$	0	0	0	0	0	0	0	0	0	0	0
$\langle 71\rangle$	2/2!/3	0	1/1!/4	0	0	0	0	0	0	0	0
$\langle 8\rangle$	0	0	0	1/4!	3/3!/2	1/2!/4	2/2!/3	1/1!/4	0	0	0
Result	0	0	0	0	0	0	0	0	0	0	0

Table 22-13 (1/5)

Weight 13, 3rd Stratum			1	2	3	4	5	6	7	8	9	10
Weight	Function	Stratum	$\langle 1^{13}\rangle$	$\langle 21^{11}\rangle$	$\langle 221^{9}\rangle$	$\langle 2221^{7}\rangle$	$\langle 22221^{5}\rangle$	$\langle 2^{5}111\rangle$	$\langle 2^{6}1\rangle$	$\langle 31^{10}\rangle$	$\langle 321^{8}\rangle$	$\langle 3221^{6}\rangle$
0	$\langle\rangle$	9	1/13!	12/12!/2	55/11!/4	120/10!/8	126/9!/16	56/8!/32	7/7!/64	11/11!/3	90/10!/6	252/9!/12
1	$\langle 1\rangle$	8	1/12!	11/11!/2	45/10!/4	84/9!/8	70/8!/16	21/7!/32	1/6!/64	10/10!/3	72/9!/6	168/8!/12
2	$\langle 11\rangle$	7	1/11!	10/10!/2	36/9!/4	56/8!/8	35/7!/16	6/6!/32	0	9/9!/3	56/8!/6	105/7!/12
2	$\langle 2\rangle$	7	0	1/11!	10/10!/2	36/9!/4	56/8!/8	35/7!/16	6/6!/32	0	9/9!/3	56/8!/6
3	$\langle 111\rangle$	6	1/10!	9/9!/2	28/8!/4	35/7!/8	15/6!/16	1/5!/32	0	8/8!/3	42/7!/6	60/6!/12
3	$\langle 21\rangle$	6	0	1/10!	9/9!/2	28/8!/4	35/7!/8	15/6!/16	1/5!/32	0	8/8!/3	42/7!/6
3	$\langle 3\rangle$	6	0	0	0	0	0	0	0	1/10!	9/9!/2	28/8!/4
4	$\langle 1111\rangle$	5	1/9!	8/8!/2	21/7!/4	20/6!/8	5/5!/16	0	0	7/7!/3	30/6!/6	30/5!/12
4	$\langle 211\rangle$	5	0	1/9!	8/8!/2	21/7!/4	20/6!/8	5/5!/16	0	0	7/7!/3	30/6!/6
4	$\langle 22\rangle$	5	0	0	1/9!	8/8!/2	21/7!/4	20/6!/8	5/5!/16	0	0	7/7!/3
4	$\langle 31\rangle$	5	0	0	0	0	0	0	0	1/9!	8/8!/2	21/7!/4
4	$\langle 4\rangle$	5	0	0	0	0	0	0	0	0	0	0
5	$\langle 11111\rangle$	5	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16	0	0	6/6!/3	20/5!/6	12/4!/12
5	$\langle 2111\rangle$	5	0	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16	0	0	6/6!/3	20/5!/6
5	$\langle 221\rangle$	5	0	0	1/8!	7/7!/2	15/6!/4	10/5!/8	1/4!/16	0	0	6/6!/3
5	$\langle 311\rangle$	5	0	0	0	0	0	0	0	1/8!	7/7!/2	15/6!/4
5	$\langle 32\rangle$	5	0	0	0	0	0	0	0	0	1/8!	7/7!/2
5	$\langle 41\rangle$	5	0	0	0	0	0	0	0	0	0	0
6	$\langle 111111\rangle$	5	1/7!	6/6!/2	10/5!/4	4/4!/8	0	0	0	5/5!/3	12/4!/6	3/3!/12
6	$\langle 21111\rangle$	5	0	1/7!	6/6!/2	10/5!/4	4/4!/8	0	0	0	5/5!/3	12/4!/6
6	$\langle 2211\rangle$	5	0	0	1/7!	6/6!/2	10/5!/4	4/4!/8	0	0	0	5/5!/3
6	$\langle 222\rangle$	5	0	0	0	1/7!	6/6!/2	10/5!/4	4/4!/8	0	0	0
6	$\langle 3111\rangle$	5	0	0	0	0	0	0	0	1/7!	6/6!/2	10/5!/4
6	$\langle 321\rangle$	5	0	0	0	0	0	0	0	0	1/7!	6/6!/2
6	$\langle 33\rangle$	5	0	0	0	0	0	0	0	0	0	0
6	$\langle 411\rangle$	5	0	0	0	0	0	0	0	0	0	0
6	$\langle 42\rangle$	5	0	0	0	0	0	0	0	0	0	0
7	$\langle 1111111\rangle$	5	1/6!	5/5!/2	6/4!/4	1/3!/8	0	0	0	4/4!/3	6/3!/6	0
7	$\langle 211111\rangle$	5	0	1/6!	5/5!/2	6/4!/4	1/3!/8	0	0	0	4/4!/3	6/3!/6
7	$\langle 22111\rangle$	5	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8	0	0	0	4/4!/3
7	$\langle 2221\rangle$	5	0	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8	0	0	0
7	$\langle 31111\rangle$	5	0	0	0	0	0	0	0	1/6!	5/5!/2	6/4!/4
7	$\langle 3211\rangle$	5	0	0	0	0	0	0	0	0	1/6!	5/5!/2
7	$\langle 322\rangle$	5	0	0	0	0	0	0	0	0	0	1/6!
7	$\langle 331\rangle$	5	0	0	0	0	0	0	0	0	0	0
7	$\langle 4111\rangle$	5	0	0	0	0	0	0	0	0	0	0
7	$\langle 421\rangle$	5	0	0	0	0	0	0	0	0	0	0
7	$\langle 43\rangle$	5	0	0	0	0	0	0	0	0	0	0
7	$\langle 7\rangle$	5	0	0	0	0	0	0	0	0	0	0
8	$\langle 11111111\rangle$	$4, \langle 8\rangle, -120$	1/5!	4/4!/2	3/3!/4	0	0	0	0	3/3!/3	2/2!/6	0
8	$\langle 2111111\rangle$	$4, \langle 8\rangle, 40$	0	1/5!	4/4!/2	3/3!/4	0	0	0	0	3/3!/3	2/2!/6
8	$\langle 221111\rangle$	$4, \langle 8\rangle, -8$	0	0	1/5!	4/4!/2	3/3!/4	0	0	0	0	3/3!/3
8	$\langle 22211\rangle$	4	0	0	0	1/5!	4/4!/2	3/3!/4	0	0	0	0
8	$\langle 2222\rangle$	4	0	0	0	0	1/5!	4/4!/2	3/3!/4	0	0	0
8	$\langle 311111\rangle$	$4, \langle 8\rangle, -15$	0	0	0	0	0	0	0	1/5!	4/4!/2	3/3!/4
8	$\langle 32111\rangle$	$4, \langle 8\rangle, 1$	0	0	0	0	0	0	0	0	1/5!	4/4!/2
8	$\langle 3221\rangle$	$4, \langle 8\rangle, 1$	0	0	0	0	0	0	0	0	0	1/5!
8	$\langle 3311\rangle$	$4, \langle 8\rangle, 0$	0	0	0	0	0	0	0	0	0	0
8	$\langle 332\rangle$	$4, \langle 8\rangle, -2$	0	0	0	0	0	0	0	0	0	0
8	$\langle 41111\rangle$	$4, \langle 8\rangle, 4$	0	0	0	0	0	0	0	0	0	0
8	$\langle 4211\rangle$	$4, \langle 8\rangle, 0$	0	0	0	0	0	0	0	0	0	0
8	$\langle 422\rangle$	$4, \langle 8\rangle, 0$	0	0	0	0	0	0	0	0	0	0
8	$\langle 431\rangle$	$4, \langle 8\rangle, 1$	0	0	0	0	0	0	0	0	0	0
8	$\langle 44\rangle$	$4, \langle 8\rangle, -2$	0	0	0	0	0	0	0	0	0	0
8	$\langle 71\rangle$	$4, \langle 8\rangle, -1$	0	0	0	0	0	0	0	0	0	0
8	$\langle 8\rangle$	$4, \langle 8\rangle, 1$	0	0	0	0	0	0	0	0	0	0
10	$\langle 1^{10}\rangle$	$4, \langle 91\rangle, -225$	1/3!	2/2!/2	0	0	0	0	0	1/1!/3	0	0
10	$\langle 21^{8}\rangle$	$4, \langle 91\rangle, 55$	0	1/3!	2/2!/2	0	0	0	0	0	1/1!/3	0
10	$\langle 221^{6}\rangle$	$4, \langle 91\rangle, -5$	0	0	1/3!	2/2!/2	0	0	0	0	0	1/1!/3
10	$\langle 2221^{4}\rangle$	$4, \langle 91\rangle, 3$	0	0	0	1/3!	2/2!/2	0	0	0	0	0
10	$\langle 222211\rangle$	$4, \langle 91\rangle, -9$	0	0	0	0	1/3!	2/2!/2	0	0	0	0
10	$\langle 22222\rangle$	$4, \langle 91\rangle, 15$	0	0	0	0	0	1/3!	2/2!/2	0	0	0
10	$\langle 31^{7}\rangle$	$4, \langle 91\rangle, -15$	0	0	0	0	0	0	0	1/3!	2/2!/2	0
10	$\langle 3211111\rangle$	$4, \langle 91\rangle, -5$	0	0	0	0	0	0	0	0	1/3!	2/2!/2
10	$\langle 322111\rangle$	$4, \langle 91\rangle, 1$	0	0	0	0	0	0	0	0	0	1/3!
10	$\langle 32221\rangle$	$4, \langle 91\rangle, 3$	0	0	0	0	0	0	0	0	0	0
10	$\langle 331111\rangle$	$4, \langle 91\rangle, 6$	0	0	0	0	0	0	0	0	0	0
10	$\langle 33211\rangle$	$4, \langle 91\rangle, -2$	0	0	0	0	0	0	0	0	0	0
10	$\langle 3322\rangle$	$4, \langle 91\rangle, -2$	0	0	0	0	0	0	0	0	0	0
10	$\langle 3331\rangle$	$4, \langle 91\rangle, 0$	0	0	0	0	0	0	0	0	0	0
10	$\langle 41^{6}\rangle$	$4, \langle 91\rangle, 5$	0	0	0	0	0	0	0	0	0	0
10	$\langle 421111\rangle$	$4, \langle 91\rangle, 1$	0	0	0	0	0	0	0	0	0	0
10	$\langle 42211\rangle$	$4, \langle 91\rangle, 1$	0	0	0	0	0	0	0	0	0	0
10	$\langle 4222\rangle$	$4, \langle 91\rangle, -3$	0	0	0	0	0	0	0	0	0	0
10	$\langle 43111\rangle$	$4, \langle 91\rangle, -1$	0	0	0	0	0	0	0	0	0	0
10	$\langle 4321\rangle$	$4, \langle 91\rangle, 1$	0	0	0	0	0	0	0	0	0	0
10	$\langle 433\rangle$	$4, \langle 91\rangle, 2$	0	0	0	0	0	0	0	0	0	0
10	$\langle 4411\rangle$	$4, \langle 91\rangle, -1$	0	0	0	0	0	0	0	0	0	0
10	$\langle 442\rangle$	$4, \langle 91\rangle, -1$	0	0	0	0	0	0	0	0	0	0
10	$\langle 7111\rangle$	$4, \langle 91\rangle, -1$	0	0	0	0	0	0	0	0	0	0
10	$\langle 721\rangle$	$4, \langle 91\rangle, -1$	0	0	0	0	0	0	0	0	0	0
10	$\langle 73\rangle$	$4, \langle 91\rangle, -1$	0	0	0	0	0	0	0	0	0	0
10	$\langle 811\rangle$	$4, \langle 91\rangle, 1$	0	0	0	0	0	0	0	0	0	0
10	$\langle 82\rangle$	$4, \langle 91\rangle, 1$	0	0	0	0	0	0	0	0	0	0
10	$\langle 91\rangle$	$4, \langle 91\rangle, 0$	0	0	0	0	0	0	0	0	0	0
Result			−3575	825	−155	45	−15	−15	45	−125	−15	−5

Table 22-13 (2/5)

	11	12	13	14	15	16	17	18	19	20	21	22	23	24
Function	$\langle 32221^4\rangle$	$\langle 32^411\rangle$	$\langle 32^5\rangle$	$\langle 331^7\rangle$	$\langle 3321^5\rangle$	⟨3322111⟩	⟨332221⟩	$\langle 3331^4\rangle$	⟨333211⟩	⟨33322⟩	⟨33331⟩	$\langle 41^9\rangle$	$\langle 421^7\rangle$	$\langle 4221^5\rangle$
⟨ ⟩	280/8!/24	105/7!/48	6/6!/96	36/9!/9	168/8!/18	210/7!/36	60/6!/72	35/7!/27	60/6!/54	10/5!/108	5/5!/81	10/10!/4	72/9!/8	168/8!/16
⟨1⟩	140/7!/24	30/6!/48	0	28/8!/9	105/7!/18	90/6!/36	10/5!/72	20/6!/27	20/5!/54	0	1/4!/81	9/9!/4	56/8!/8	105/7!/16
⟨11⟩	60/6!/24	5/5!/48	0	21/7!/9	60/6!/18	30/5!/36	0	10/5!/27	4/4!/54	0	0	8/8!/4	42/7!/8	60/6!/16
⟨2⟩	105/7!/12	60/6!/24	5/5!/48	0	21/7!/9	60/6!/18	30/5!/36	0	10/5!/27	4/4!/54	0	0	8/8!/4	42/7!/8
⟨111⟩	20/5!/24	0	0	15/6!/9	30/5!/18	6/4!/36	0	4/4!/27	0	0	0	7/7!/4	30/6!/8	30/5!/16
⟨21⟩	60/6!/12	20/5!/24	0	0	15/6!/9	30/5!/18	6/4!/36	0	4/4!/27	0	0	0	7/7!/4	30/6!/8
⟨3⟩	35/7!/8	15/6!/16	1/5!/32	8/8!/3	42/7!/6	60/6!/12	20/5!/24	15/6!/9	30/5!/18	6/4!/36	4/4!/27	0	0	0
⟨1111⟩	4/4!/24	0	0	10/5!/9	12/4!/18	0	0	1/3!/27	0	0	0	6/6!/4	20/5!/8	12/4!/16
⟨211⟩	30/5!/12	4/4!/24	0	0	10/5!/9	12/4!/18	0	0	1/3!/27	0	0	0	6/6!/4	20/5!/8
⟨22⟩	30/6!/6	30/5!/12	4/4!/24	0	0	10/5!/9	12/4!/18	0	0	1/3!/27	0	0	0	6/6!/4
⟨31⟩	20/6!/8	5/5!/16	0	7/7!/3	30/6!/6	30/5!/12	4/4!/24	10/5!/9	12/4!/18	0	1/3!/27	0	0	0
⟨4⟩	0	0	0	0	0	0	0	0	0	0	0	1/9!	8/8!/2	21/7!/4
⟨11111⟩	0	0	0	6/4!/9	3/3!/18	0	0	0	0	0	0	5/5!/4	12/4!/8	3/3!/16
⟨2111⟩	12/4!/12	0	0	0	6/4!/9	3/3!/18	0	0	0	0	0	0	5/5!/4	12/4!/8
⟨221⟩	20/5!/6	12/4!/12	0	0	0	6/4!/9	3/3!/18	0	0	0	0	0	0	5/5!/4
⟨311⟩	10/5!/8	1/4!/16	0	6/6!/3	20/5!/6	12/4!/12	0	6/4!/9	3/3!/18	0	0	0	0	0
⟨32⟩	15/6!/4	10/5!/8	1/4!/16	0	6/6!/3	20/5!/6	12/4!/12	0	6/4!/9	3/3!/18	0	0	0	0
⟨41⟩	0	0	0	0	0	0	0	0	0	0	0	1/8!	7/7!/2	15/6!/4
⟨111111⟩	0	0	0	3/3!/9	0	0	0	0	0	0	0	4/4!/4	6/3!/8	0
⟨21111⟩	3/3!/12	0	0	0	3/3!/9	0	0	0	0	0	0	0	4/4!/4	6/3!/8
⟨2211⟩	12/4!/6	3/3!/12	0	0	0	3/3!/9	0	0	0	0	0	0	0	4/4!/4
⟨222⟩	5/5!/3	12/4!/6	3/3!/12	0	0	0	3/3!/9	0	0	0	0	0	0	0
⟨3111⟩	4/4!/8	0	0	5/5!/3	12/4!/6	3/3!/12	0	3/3!/9	0	0	0	0	0	0
⟨321⟩	10/5!/4	4/4!/8	0	0	5/5!/3	12/4!/6	3/3!/12	0	3/3!/9	0	0	0	0	0
⟨33⟩	0	0	0	1/7!	6/6!/2	10/5!/4	4/4!/8	5/5!/3	12/4!/6	3/3!/12	3/3!/9	0	0	0
⟨411⟩	0	0	0	0	0	0	0	0	0	0	0	1/7!	6/6!/2	10/5!/4
⟨42⟩	0	0	0	0	0	0	0	0	0	0	0	0	1/7!	6/6!/2
⟨1111111⟩	0	0	0	1/2!/9	0	0	0	0	0	0	0	3/3!/4	2/2!/8	0
⟨211111⟩	0	0	0	0	1/2!/9	0	0	0	0	0	0	0	3/3!/4	2/2!/8
⟨22111⟩	6/3!/6	0	0	0	0	1/2!/9	0	0	0	0	0	0	0	3/3!/4
⟨2221⟩	4/4!/3	6/3!/6	0	0	0	0	1/2!/9	0	0	0	0	0	0	0
⟨31111⟩	1/3!/8	0	0	4/4!/3	6/3!/6	0	0	1/2!/9	0	0	0	0	0	0
⟨3211⟩	6/4!/4	1/3!/8	0	0	4/4!/3	6/3!/6	0	0	1/2!/9	0	0	0	0	0
⟨322⟩	5/5!/2	6/4!/4	1/3!/8	0	0	4/4!/3	6/3!/6	0	0	1/2!/9	0	0	0	0
⟨331⟩	0	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8	4/4!/3	6/3!/6	0	1/2!/9	0	0	0
⟨4111⟩	0	0	0	0	0	0	0	0	0	0	0	1/6!	5/5!/2	6/4!/4
⟨421⟩	0	0	0	0	0	0	0	0	0	0	0	0	1/6!	5/5!/2
⟨43⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨7⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨11111111⟩	0	0	0	0	0	0	0	0	0	0	0	2/2!/4	0	0
⟨2111111⟩	0	0	0	0	0	0	0	0	0	0	0	0	2/2!/4	0
⟨221111⟩	2/2!/6	0	0	0	0	0	0	0	0	0	0	0	0	2/2!/4
⟨22211⟩	3/3!/3	2/2!/6	0	0	0	0	0	0	0	0	0	0	0	0
⟨2222⟩	0	3/3!/3	2/2!/6	0	0	0	0	0	0	0	0	0	0	0
⟨311111⟩	0	0	0	3/3!/3	2/2!/6	0	0	0	0	0	0	0	0	0
⟨32111⟩	3/3!/4	0	0	0	3/3!/3	2/2!/6	0	0	0	0	0	0	0	0
⟨3221⟩	4/4!/2	3/3!/4	0	0	0	3/3!/3	2/2!/6	0	0	0	0	0	0	0
⟨3311⟩	0	0	0	1/5!	4/4!/2	3/3!/4	0	3/3!/3	2/2!/6	0	0	0	0	0
⟨332⟩	0	0	0	0	1/5!	4/4!/2	3/3!/4	0	3/3!/3	2/2!/6	0	0	0	0
⟨41111⟩	0	0	0	0	0	0	0	0	0	0	0	1/5!	4/4!/2	3/3!/4
⟨4211⟩	0	0	0	0	0	0	0	0	0	0	0	0	1/5!	4/4!/2
⟨422⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	1/5!
⟨431⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨44⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨71⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨8⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\langle 1^{10}\rangle$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\langle 21^8\rangle$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\langle 221^6\rangle$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\langle 2221^4\rangle$	1/1!/3	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨222211⟩	0	1/1!/3	0	0	0	0	0	0	0	0	0	0	0	0
⟨22222⟩	0	0	1/1!/3	0	0	0	0	0	0	0	0	0	0	0
$\langle 31^7\rangle$	0	0	0	1/1!/3	0	0	0	0	0	0	0	0	0	0
⟨3211111⟩	0	0	0	0	1/1!/3	0	0	0	0	0	0	0	0	0
⟨322111⟩	2/2!/2	0	0	0	0	1/1!/3	0	0	0	0	0	0	0	0
⟨32221⟩	1/3!	2/2!/2	0	0	0	0	1/1!/3	0	0	0	0	0	0	0
⟨331111⟩	0	0	0	1/3!	2/2!/2	0	0	1/1!/3	0	0	0	0	0	0
⟨33211⟩	0	0	0	0	1/3!	2/2!/2	0	0	1/1!/3	0	0	0	0	0
⟨3322⟩	0	0	0	0	0	1/3!	2/2!/2	0	0	1/1!/3	0	0	0	0
⟨3331⟩	0	0	0	0	0	0	0	1/3!	2/2!/2	0	1/1!/3	0	0	0
$\langle 41^6\rangle$	0	0	0	0	0	0	0	0	0	0	0	1/3!	2/2!/2	0
⟨421111⟩	0	0	0	0	0	0	0	0	0	0	0	0	1/3!	2/2!/2
⟨42211⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	1/3!
⟨4222⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨43111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨4321⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨433⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨4411⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨442⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨7111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨721⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨73⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨811⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨82⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨91⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Result	9	3	−15	40	0	−8	0	−2	6	−2	−8	15	15	−5

Table 22-13 (3/5)

	25	26	27	28	29	30	31	32	33	34	35	36	37
Function	⟨4222111⟩	⟨422221⟩	⟨431^6⟩	⟨4321111⟩	⟨432211⟩	⟨43222⟩	⟨433111⟩	⟨43321⟩	⟨4333⟩	⟨441^5⟩	⟨442111⟩	⟨44221⟩	⟨44311⟩
⟨ ⟩	140/7!/32	30/6!/64	56/8!/12	210/7!/24	180/6!/48	20/5!/96	60/6!/36	60/5!/72	4/4!/108	21/7!/16	60/6!/32	30/5!/64	30/5!/48
⟨1⟩	60/6!/32	5/5!/64	42/7!/12	120/6!/24	60/5!/48	0	30/5!/36	12/4!/72	0	15/6!/16	30/5!/32	6/4!/64	12/4!/48
⟨11⟩	20/5!/32	0	30/6!/12	60/5!/24	12/4!/48	0	12/4!/36	0	0	10/5!/16	12/4!/32	0	3/3!/48
⟨2⟩	60/6!/16	20/5!/32	0	30/6!/12	60/5!/24	12/4!/48	0	12/4!/36	0	0	10/5!/16	12/4!/32	0
⟨111⟩	4/4!/32	0	20/5!/12	24/4!/24	0	0	3/3!/36	0	0	6/4!/16	3/3!/32	0	0
⟨21⟩	30/5!/16	4/4!/32	0	20/5!/12	24/4!/24	0	0	3/3!/36	0	0	6/4!/16	3/3!/32	0
⟨3⟩	0	0	7/7!/4	30/6!/8	30/5!/16	4/4!/32	20/5!/12	24/4!/24	3/3!/36	0	0	0	6/4!/16
⟨1111⟩	0	0	12/4!/12	6/3!/24	0	0	0	0	0	3/3!/16	0	0	0
⟨211⟩	12/4!/16	0	0	12/4!/12	6/3!/24	0	0	0	0	0	3/3!/16	0	0
⟨22⟩	20/5!/8	12/4!/16	0	0	12/4!/12	6/3!/24	0	0	0	0	0	3/3!/16	0
⟨31⟩	0	0	6/6!/4	20/5!/8	12/4!/16	0	12/4!/12	6/3!/24	0	0	0	0	3/3!/16
⟨4⟩	20/6!/8	5/5!/16	7/7!/3	30/6!/6	30/5!/12	4/4!/24	10/5!/9	12/4!/18	1/3!/27	6/6!/4	20/5!/8	12/4!/16	12/4!/12
⟨11111⟩	0	0	6/3!/12	0	0	0	0	0	0	1/2!/16	0	0	0
⟨2111⟩	3/3!/16	0	0	6/3!/12	0	0	0	0	0	0	1/2!/16	0	0
⟨221⟩	12/4!/8	3/3!/16	0	0	6/3!/12	0	0	0	0	0	0	1/2!/16	0
⟨311⟩	0	0	5/5!/4	12/4!/8	3/3!/16	0	6/3!/12	0	0	0	0	0	1/2!/16
⟨32⟩	0	0	0	5/5!/4	12/4!/8	3/3!/16	0	6/3!/12	0	0	0	0	0
⟨41⟩	10/5!/8	1/4!/16	6/6!/3	20/5!/6	12/4!/12	0	6/4!/9	3/3!/18	0	5/5!/4	12/4!/8	3/3!/16	6/3!/12
⟨111111⟩	0	0	2/2!/12	0	0	0	0	0	0	0	0	0	0
⟨21111⟩	0	0	0	2/2!/12	0	0	0	0	0	0	0	0	0
⟨2211⟩	6/3!/8	0	0	0	2/2!/12	0	0	0	0	0	0	0	0
⟨222⟩	4/4!/4	6/3!/8	0	0	0	2/2!/12	0	0	0	0	0	0	0
⟨3111⟩	0	0	4/4!/4	6/3!/8	0	0	2/2!/12	0	0	0	0	0	0
⟨321⟩	0	0	0	4/4!/4	6/3!/8	0	0	2/2!/12	0	0	0	0	0
⟨33⟩	0	0	0	0	0	0	4/4!/4	6/3!/8	2/2!/12	0	0	0	0
⟨411⟩	4/4!/8	0	5/5!/3	12/4!/6	3/3!/12	0	3/3!/9	0	0	4/4!/4	6/3!/8	0	2/2!/12
⟨42⟩	10/5!/4	4/4!/8	0	5/5!/3	12/4!/6	3/3!/12	0	3/3!/9	0	0	4/4!/4	6/3!/8	0
⟨1111111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨211111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨22111⟩	2/2!/8	0	0	0	0	0	0	0	0	0	0	0	0
⟨2221⟩	3/3!/4	2/2!/8	0	0	0	0	0	0	0	0	0	0	0
⟨31111⟩	0	0	3/3!/4	2/2!/8	0	0	0	0	0	0	0	0	0
⟨3211⟩	0	0	0	3/3!/4	2/2!/8	0	0	0	0	0	0	0	0
⟨322⟩	0	0	0	0	3/3!/4	2/2!/8	0	0	0	0	0	0	0
⟨331⟩	0	0	0	0	0	0	3/3!/4	2/2!/8	0	0	0	0	0
⟨4111⟩	1/3!/8	0	4/4!/3	6/3!/6	0	0	1/2!/9	0	0	3/3!/4	2/2!/8	0	0
⟨421⟩	6/4!/4	1/3!/8	0	4/4!/3	6/3!/6	0	0	1/2!/9	0	0	3/3!/4	2/2!/8	0
⟨43⟩	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8	4/4!/3	6/3!/6	1/2!/9	0	0	0	3/3!/4
⟨7⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨11111111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨2111111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨221111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨22211⟩	2/2!/4	0	0	0	0	0	0	0	0	0	0	0	0
⟨2222⟩	0	2/2!/4	0	0	0	0	0	0	0	0	0	0	0
⟨311111⟩	0	0	2/2!/4	0	0	0	0	0	0	0	0	0	0
⟨32111⟩	0	0	0	2/2!/4	0	0	0	0	0	0	0	0	0
⟨3221⟩	0	0	0	0	2/2!/4	0	0	0	0	0	0	0	0
⟨3311⟩	0	0	0	0	0	0	2/2!/4	0	0	0	0	0	0
⟨332⟩	0	0	0	0	0	0	0	2/2!/4	0	0	0	0	0
⟨41111⟩	0	0	3/3!/3	2/2!/6	0	0	0	0	0	2/2!/4	0	0	0
⟨4211⟩	3/3!/4	0	0	3/3!/3	2/2!/6	0	0	0	0	0	2/2!/4	0	0
⟨422⟩	4/4!/2	3/3!/4	0	0	3/3!/3	2/2!/6	0	0	0	0	0	2/2!/4	0
⟨431⟩	0	0	1/5!	4/4!/2	3/3!/4	0	3/3!/3	2/2!/6	0	0	0	0	2/2!/4
⟨44⟩	0	0	0	0	0	0	0	0	0	1/5!	4/4!/2	3/3!/4	3/3!/3
⟨71⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨8⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨1^{10}⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨21^8⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨221^6⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨2221^4⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨222211⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨22222⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨31^7⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨3211111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨322111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨32221⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨331111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨33211⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨3322⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨3331⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨41^6⟩	0	0	1/1!/3	0	0	0	0	0	0	0	0	0	0
⟨421111⟩	0	0	0	1/1!/3	0	0	0	0	0	0	0	0	0
⟨42211⟩	2/2!/2	0	0	0	1/1!/3	0	0	0	0	0	0	0	0
⟨4222⟩	1/3!	2/2!/2	0	0	0	1/1!/3	0	0	0	0	0	0	0
⟨43111⟩	0	0	1/3!	2/2!/2	0	0	1/1!/3	0	0	0	0	0	0
⟨4321⟩	0	0	0	1/3!	2/2!/2	0	0	1/1!/3	0	0	0	0	0
⟨433⟩	0	0	0	0	0	0	1/3!	2/2!/2	1/1!/3	0	0	0	0
⟨4411⟩	0	0	0	0	0	0	0	0	0	1/3!	2/2!/2	0	1/1!/3
⟨442⟩	0	0	0	0	0	0	0	0	0	0	1/3!	2/2!/2	0
⟨7111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨721⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨73⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨811⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨82⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨91⟩	0	0	0	0	0	0	0	0	0	0	0	0	0
Result	3	−9	−15	3	1	3	0	0	6	5	−3	1	−1

Table 22-13 (4/5)

	38	39	40	41	42	43	44	45	46	47	48	49	50	51
Function	⟨4432⟩	⟨4441⟩	$\langle 71^{6}\rangle$	$\langle 721^{4}\rangle$	⟨72211⟩	⟨7222⟩	⟨73111⟩	⟨7321⟩	⟨733⟩	⟨7411⟩	⟨742⟩	$\langle 81^{5}\rangle$	⟨82111⟩	⟨8221⟩
⟨ ⟩	12/4!/96	4/4!/64	7/7!/7	30/6!/14	30/5!/28	4/4!/56	20/5!/21	24/4!/42	3/3!/63	12/4!/28	6/3!/56	6/6!/8	20/5!/16	12/4!/32
⟨1⟩	0	1/3!/64	6/6!/7	20/5!/14	12/4!/28	0	12/4!/21	6/3!/42	0	6/3!/28	0	5/5!/8	12/4!/16	3/3!/32
⟨11⟩	0	0	5/5!/7	12/4!/14	3/3!/28	0	6/3!/21	0	0	2/2!/28	0	4/4!/8	6/3!/16	0
⟨2⟩	3/3!/48	0	0	5/5!/7	12/4!/14	3/3!/28	0	6/3!/21	0	0	2/2!/28	0	4/4!/8	6/3!/16
⟨111⟩	0	0	4/4!/7	6/3!/14	0	0	2/2!/21	0	0	0	0	3/3!/8	2/2!/16	0
⟨21⟩	0	0	0	4/4!/7	6/3!/14	0	0	2/2!/21	0	0	0	0	3/3!/8	2/2!/16
⟨3⟩	3/3!/32	0	0	0	0	0	4/4!/7	6/3!/14	2/2!/21	0	0	0	0	0
⟨1111⟩	0	0	3/3!/7	2/2!/14	0	0	0	0	0	0	0	2/2!/8	0	0
⟨211⟩	0	0	0	3/3!/7	2/2!/14	0	0	0	0	0	0	0	2/2!/8	0
⟨22⟩	0	0	0	0	3/3!/7	2/2!/14	0	0	0	0	0	0	0	2/2!/8
⟨31⟩	0	0	0	0	0	0	3/3!/7	2/2!/14	0	0	0	0	0	0
⟨4⟩	6/3!/24	3/3!/16	0	0	0	0	0	0	0	3/3!/7	2/2!/14	0	0	0
⟨11111⟩	0	0	2/2!/7	0	0	0	0	0	0	0	0	1/1!/8	0	0
⟨2111⟩	0	0	0	2/2!/7	0	0	0	0	0	0	0	0	1/1!/8	0
⟨221⟩	0	0	0	0	2/2!/7	0	0	0	0	0	0	0	0	1/1!/8
⟨311⟩	0	0	0	0	0	0	2/2!/7	0	0	0	0	0	0	0
⟨32⟩	1/2!/16	0	0	0	0	0	0	2/2!/7	0	0	0	0	0	0
⟨41⟩	0	1/2!/16	0	0	0	0	0	0	0	2/2!/7	0	0	0	0
⟨111111⟩	0	0	1/1!/7	0	0	0	0	0	0	0	0	0	0	0
⟨21111⟩	0	0	0	1/1!/7	0	0	0	0	0	0	0	0	0	0
⟨2211⟩	0	0	0	0	1/1!/7	0	0	0	0	0	0	0	0	0
⟨222⟩	0	0	0	0	0	1/1!/7	0	0	0	0	0	0	0	0
⟨3111⟩	0	0	0	0	0	0	1/1!/7	0	0	0	0	0	0	0
⟨321⟩	0	0	0	0	0	0	0	1/1!/7	0	0	0	0	0	0
⟨33⟩	0	0	0	0	0	0	0	0	1/1!/7	0	0	0	0	0
⟨411⟩	0	0	0	0	0	0	0	0	0	1/1!/7	0	0	0	0
⟨42⟩	2/2!/12	0	0	0	0	0	0	0	0	0	1/1!/7	0	0	0
⟨1111111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨211111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨22111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨2221⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨31111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨3211⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨322⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨331⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨4111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨421⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨43⟩	2/2!/8	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨7⟩	0	0	1/6!	5/5!/2	6/4!/4	1/3!/8	4/4!/3	6/3!/6	1/2!/9	3/3!/4	2/2!/8	0	0	0
⟨11111111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨2111111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨221111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨22211⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨2222⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨311111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨32111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨3221⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨3311⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨332⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨41111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨4211⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨422⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨431⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨44⟩	2/2!/6	2/2!/4	0	0	0	0	0	0	0	0	0	0	0	0
⟨71⟩	0	0	1/5!	4/4!/2	3/3!/4	0	3/3!/3	2/2!/6	0	2/2!/4	0	0	0	0
⟨8⟩	0	0	0	0	0	0	0	0	0	0	0	1/5!	4/4!/2	3/3!/4
$\langle 1^{10}\rangle$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\langle 21^{8}\rangle$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\langle 221^{6}\rangle$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\langle 2221^{4}\rangle$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨222211⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨22222⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\langle 31^{7}\rangle$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨3211111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨322111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨32221⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨331111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨33211⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨3322⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨3331⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\langle 41^{6}\rangle$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨421111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨42211⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨4222⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨43111⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨4321⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨433⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨4411⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨442⟩	1/1!/3	0	0	0	0	0	0	0	0	0	0	0	0	0
⟨7111⟩	0	0	1/3!	2/2!/2	0	0	1/1!/3	0	0	0	0	0	0	0
⟨721⟩	0	0	0	1/3!	2/2!/2	0	0	1/1!/3	0	0	0	0	0	0
⟨73⟩	0	0	0	0	0	0	1/3!	2/2!/2	1/1!/3	0	0	0	0	0
⟨811⟩	0	0	0	0	0	0	0	0	0	0	0	1/3!	2/2!/2	0
⟨82⟩	0	0	0	0	0	0	0	0	0	0	0	0	1/3!	2/2!/2
⟨91⟩	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Result	−3	3	−5	−1	−1	3	1	−1	−2	1	1	5	1	1

Table 22-13 (5/5)

	52	53	54	55	56	57	58	59	60
Function	$\langle 8311\rangle$	$\langle 832\rangle$	$\langle 841\rangle$	$\langle 91111\rangle$	$\langle 9211\rangle$	$\langle 922\rangle$	$\langle 931\rangle$	$\langle 94\rangle$	$\langle 13\rangle$
$\langle\,\rangle$	12/4!/24	6/3!/48	6/3!/32	5/5!/9	12/4!/18	3/3!/36	6/3!/27	2/2!/36	1/1!/13
$\langle 1\rangle$	6/3!/24	0	2/2!/32	4/4!/9	6/3!/18	0	2/2!/27	0	0
$\langle 11\rangle$	2/2!/24	0	0	3/3!/9	2/2!/18	0	0	0	0
$\langle 2\rangle$	0	2/2!/24	0	0	3/3!/9	2/2!/18	0	0	0
$\langle 111\rangle$	0	0	0	2/2!/9	0	0	0	0	0
$\langle 21\rangle$	0	0	0	0	2/2!/9	0	0	0	0
$\langle 3\rangle$	3/3!/8	2/2!/16	0	0	0	0	2/2!/9	0	0
$\langle 1111\rangle$	0	0	0	1/1!/9	0	0	0	0	0
$\langle 211\rangle$	0	0	0	0	1/1!/9	0	0	0	0
$\langle 22\rangle$	0	0	0	0	0	1/1!/9	0	0	0
$\langle 31\rangle$	2/2!/8	0	0	0	0	0	1/1!/9	0	0
$\langle 4\rangle$	0	0	2/2!/8	0	0	0	0	1/1!/9	0
$\langle 11111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 2111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 221\rangle$	0	0	0	0	0	0	0	0	0
$\langle 311\rangle$	1/1!/8	0	0	0	0	0	0	0	0
$\langle 32\rangle$	0	1/1!/8	0	0	0	0	0	0	0
$\langle 41\rangle$	0	0	1/1!/8	0	0	0	0	0	0
$\langle 111111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 21111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 2211\rangle$	0	0	0	0	0	0	0	0	0
$\langle 222\rangle$	0	0	0	0	0	0	0	0	0
$\langle 3111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 321\rangle$	0	0	0	0	0	0	0	0	0
$\langle 33\rangle$	0	0	0	0	0	0	0	0	0
$\langle 411\rangle$	0	0	0	0	0	0	0	0	0
$\langle 42\rangle$	0	0	0	0	0	0	0	0	0
$\langle 1111111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 211111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 22111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 2221\rangle$	0	0	0	0	0	0	0	0	0
$\langle 31111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 3211\rangle$	0	0	0	0	0	0	0	0	0
$\langle 322\rangle$	0	0	0	0	0	0	0	0	0
$\langle 331\rangle$	0	0	0	0	0	0	0	0	0
$\langle 4111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 421\rangle$	0	0	0	0	0	0	0	0	0
$\langle 43\rangle$	0	0	0	0	0	0	0	0	0
$\langle 7\rangle$	0	0	0	0	0	0	0	0	0
$\langle 11111111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 2111111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 221111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 22211\rangle$	0	0	0	0	0	0	0	0	0
$\langle 2222\rangle$	0	0	0	0	0	0	0	0	0
$\langle 311111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 32111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 3221\rangle$	0	0	0	0	0	0	0	0	0
$\langle 3311\rangle$	0	0	0	0	0	0	0	0	0
$\langle 332\rangle$	0	0	0	0	0	0	0	0	0
$\langle 41111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 4211\rangle$	0	0	0	0	0	0	0	0	0
$\langle 422\rangle$	0	0	0	0	0	0	0	0	0
$\langle 431\rangle$	0	0	0	0	0	0	0	0	0
$\langle 44\rangle$	0	0	0	0	0	0	0	0	0
$\langle 71\rangle$	0	0	0	0	0	0	0	0	0
$\langle 8\rangle$	3/3!/3	2/2!/6	2/2!/4	0	0	0	0	0	0
$\langle 1^{10}\rangle$	0	0	0	0	0	0	0	0	0
$\langle 21^{8}\rangle$	0	0	0	0	0	0	0	0	0
$\langle 221^{6}\rangle$	0	0	0	0	0	0	0	0	0
$\langle 2221^{4}\rangle$	0	0	0	0	0	0	0	0	0
$\langle 222211\rangle$	0	0	0	0	0	0	0	0	0
$\langle 22222\rangle$	0	0	0	0	0	0	0	0	0
$\langle 31^{7}\rangle$	0	0	0	0	0	0	0	0	0
$\langle 3211111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 322111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 32221\rangle$	0	0	0	0	0	0	0	0	0
$\langle 331111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 33211\rangle$	0	0	0	0	0	0	0	0	0
$\langle 3322\rangle$	0	0	0	0	0	0	0	0	0
$\langle 3331\rangle$	0	0	0	0	0	0	0	0	0
$\langle 41^{6}\rangle$	0	0	0	0	0	0	0	0	0
$\langle 421111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 42211\rangle$	0	0	0	0	0	0	0	0	0
$\langle 4222\rangle$	0	0	0	0	0	0	0	0	0
$\langle 43111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 4321\rangle$	0	0	0	0	0	0	0	0	0
$\langle 433\rangle$	0	0	0	0	0	0	0	0	0
$\langle 4411\rangle$	0	0	0	0	0	0	0	0	0
$\langle 442\rangle$	0	0	0	0	0	0	0	0	0
$\langle 7111\rangle$	0	0	0	0	0	0	0	0	0
$\langle 721\rangle$	0	0	0	0	0	0	0	0	0
$\langle 73\rangle$	0	0	0	0	0	0	0	0	0
$\langle 811\rangle$	1/1!/3	0	0	0	0	0	0	0	0
$\langle 82\rangle$	0	1/1!/3	0	0	0	0	0	0	0
$\langle 91\rangle$	0	0	0	1/3!	2/2!/2	0	1/1!/3	0	0
Result	−1	1	−1	−2	0	−2	1	0	0

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