ON A GENERALIZATION OF JACOBI'S DERIVATIVE FORMULA TO HYPERELLIPTIC CURVES

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Abstract. In this paper we give a weak generalization of so-called Jacobi's derivative formula to hyperelliptic curves of any genus.

§0. Introduction

Let E be an elliptic curve over the complex number field $\mathbf{C}$. Let $(\omega',\, \omega'')$ be a basis of the lattice of periods of E such that the imaginary part of $Z := {\omega'}^{-1}\omega''$ is positive. Let $\sigma(u) = \sigma(u; E)$, $(u \in \mathbf{C})$, be Weierstrass' sigma function (whose explicit definition is given by (1.2) below). Then so-called Jacobi's derivative formula states

$$(1) \qquad \frac{d}{du}\sigma(u;E)\Big|_{u=0} = \frac{2\pi}{\omega'}\theta\begin{bmatrix}0\\0\end{bmatrix}(0,Z)\cdot\theta\begin{bmatrix}0\\1/2\end{bmatrix}(0,Z)\cdot\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(0,Z),$$

where $\theta\begin{bmatrix}a\\b\end{bmatrix}(z,Z)$ is the theta series with characteristic $\begin{bmatrix}a\\b\end{bmatrix}$. This formula (1) evaluates the coefficient of the lead term of the Taylor expansion of $\sigma(u)$ at the origin.

This function $\sigma(u)$ was generalized for hyperelliptic curves ([**B1**, **B2**] or (1.2) bellow). We call it the *hyperelliptic sigma function.* Let C be a hyperelliptic curve of genus $g \geq 1$. The hyperelliptic sigma function of C, denoted by $\sigma(u)$ or $\sigma(u_1, \cdots, u_g; C)$, is a function of g variables. In the Taylor expansion of $\sigma(u)$ at the origin, the form of the terms of lowest degree is independent to the curve C up to a multiplicative constant, say $\gamma(C)$ (defined explicitly in (2.2) bellow), and depends only on the genus of C (see (2.1) below). So we want to give a generalization of (1) which evaluates $\gamma(C)$. In the main result (3.3) of this paper, we evaluates not $\gamma(C)$ itself but certain power of it by the fourth power of a product of several "Thetanullwerten" of even characteristic (i.e. special values of even theta functions of C at the origin).

Grant's paper [**G1**] is a start of this paper. He treated only the case of genus two and gave a stronger formula than ours in the case. But we treat all the hyperelliptic curves over the complex number field.

We furthermore explain the hyperelliptic sigma function. The function is essentially a singled out theta series, but has a particular role in the theory of hyperelliptic abelian functions ([**B1**, **B2**, **B3**, **G1**, **G2**]). That is characterized by several second logarithmic derivatives $\dfrac{\partial^2}{\partial u_j \partial u_g} \log \sigma(u)$ $(j = 1, \cdots, g)$ being certain fundamental abelian functions on the Jacobian variety of C. We refer the reader to [**B1**] for detail.

Though our formula is a simple result from the formula of Thomae and Baker etc., it gives one of the steps to write down the coefficient $\gamma(C)$ by suitable values related to Thetanullwerten.

Another generalization of Jacobi's formula was given by Igusa ([**I1**, **I2**, **I3**]) which evaluates a determinant of the Jacobian matrix of sets of theta series on a higher dimensional Abelian variety at the origin.

§1. Preliminaries

Let C be a smooth projective model of the curve of genus $g(>0)$ defined by $y^2 = f(x)$ over the complex number field $\mathbf{C}$, where

$$f(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_{2g+1} x^{2g+1}.$$

We arbitrarily fix an ordering $\prec$ of the roots of $f(x) = 0$, say

$$c_1 \prec a_1 \prec \cdots \prec c_g \prec a_g \prec c_{g+1}. \tag{1.1}$$

Of course, we have

$$f(x) = \lambda_{2g+1}(x - a_1) \cdots (x - a_g)(x - c_1) \cdots (x - c_g)(x - c_{g+1}).$$

In this paper we denote the discriminant of quantities $X_1, X_2, \cdots, X_m$ by $\Delta(X_1, X_2, \cdots, X_m)$:

$$\Delta(X_1, X_2, \cdots, X_m) = \prod_{1 \le i < j \le m} (X_i - X_j)^2.$$

We define

$$\Delta(C) := \Delta(a_1, c_1, \cdots, a_g, c_g, c_{g+1}).$$

Let

$$\omega_j = \frac{x^{j-1} dx}{y}, \quad (j = 1, \cdots, g),$$

be a basis of the space $\Gamma(C, \Omega^1_C)$ of holomorphic 1-forms, and

$$\eta_j = \frac{1}{4y} \sum_{k=j}^{2g-j} (k + 1 - j) \lambda_{k+1+j} x^k dx, \quad (j = 1, \cdots, g),$$

be a basis of the space $\Gamma(C, \Omega^1_C(2\infty)) - \Gamma(C, \Omega^1_C)$, the differentials of second kind whose member has a pole of second order at the infinity and has no other poles (see [**B1**, p.195, Ex. i] and [**B2**, p.314]). We fix generators α_i, β_i (i=1, $\cdots$, g) of the fundamental group $\pi_1(C)$ of C as indicated by Fig. 1. Then

$$\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0, \quad \alpha_i \cdot \beta_j = \delta_{ij} \quad \text{for } i, j = 1, \cdots g.$$

As usual we let

$$\omega' = \begin{bmatrix} \int_{\alpha_1} \omega_1 & \cdots & \int_{\alpha_g} \omega_1 \\ \vdots & \ddots & \vdots \\ \int_{\alpha_1} \omega_g & \cdots & \int_{\alpha_g} \omega_g \end{bmatrix}, \quad \omega'' = \begin{bmatrix} \int_{\beta_1} \omega_1 & \cdots & \int_{\beta_g} \omega_1 \\ \vdots & \ddots & \vdots \\ \int_{\beta_1} \omega_g & \cdots & \int_{\beta_g} \omega_g \end{bmatrix} \tag{1.2}$$

be the period matrices. Then the modulus Z of C given by

$$Z = {\omega'}^{-1} \omega'',$$

which belongs to g-dimensional Siegel upper half space. Furthermore let

$$H = \begin{bmatrix} \int_{\alpha_1} \eta_1 & \cdots & \int_{\alpha_g} \eta_1 \\ \vdots & \ddots & \vdots \\ \int_{\alpha_1} \eta_g & \cdots & \int_{\alpha_g} \eta_g \end{bmatrix}.$$

The lattice of periods is denoted by

$$\Lambda := [\,\mathbf{Z} \quad \mathbf{Z} \quad \cdots \quad \mathbf{Z}\,]\,\omega' + [\,\mathbf{Z} \quad \mathbf{Z} \quad \cdots \quad \mathbf{Z}\,]\,\omega'' \ (\subset \mathbf{C}^g),$$

Where $\mathbf{Z}$ is the ring of integers. We introduce some theta characteristics followed by [**M**, p.3.88]. Let

$$\eta_{2i-1} = \begin{bmatrix} {}^t(0 & \cdots & 0 & \overset{i\text{-th place}}{\overbrace{\tfrac{1}{2}}} & 0 & \cdots & 0) \\ {}^t(\tfrac{1}{2} & \cdots & \tfrac{1}{2} & 0 & 0 & \cdots & 0) \end{bmatrix}$$

$$\eta_{2i} = \begin{bmatrix} {}^t(0 & \cdots & 0 & \overset{i\text{-th place}}{\overbrace{\tfrac{1}{2}}} & 0 & \cdots & 0) \\ {}^t(\tfrac{1}{2} & \cdots & \tfrac{1}{2} & \tfrac{1}{2} & 0 & \cdots & 0) \end{bmatrix}.$$

Let $B=\{a_1,a_2,\cdots,a_g,c_1,c_2,\cdots,c_g,c_{g+1}\}$ be the set of x-coordinates of brunch points of C. Then we denote, for every subset $T\subset B$,

$$\eta_T:=\sum_{c_i\in T}\eta_{2i-1}+\sum_{a_i\in T}\eta_{2i}.$$

Especially we set

$$\begin{aligned}\delta:&=\eta_A \quad\text{with}\quad A:=\{a_1,\cdots,a_g\}\\ &=\begin{bmatrix} {}^t(1/2 & 1/2 & \cdots & 1/2)\\ {}^t(g/2 & (g-1)/2 & \cdots & 1/2)\end{bmatrix}.\end{aligned}$$

For any two subsets S and T of B, we let $S\circ T$ denote the symmetric difference of S and T: that is $S\circ T=S\cup T-S\cap T$.

For a, b in $\left(\frac{1}{2}\mathbf{Z}\right)^g$, let

$$\begin{aligned}&\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(z)=\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(z,Z)\\ &=\sum_{n\in\mathbf{Z}^g}\exp\left[2\pi\sqrt{-1}\left\{\frac{1}{2}\ {}^t(n+a)Z(n+a)+\ {}^t(n+a)(z+b)\right\}\right].\end{aligned}$$

Then the hyperelliptic sigma-function on $\mathbf{C}^g$ with respect to Λ defined in [**B1**, p. 283] or [**B2**, p. 336] is

$$\sigma(u):=\sigma(u;C)=\ \exp\Big(-\frac{1}{2}uH\omega'^{-1}\ {}^tu\Big)\vartheta[\delta]\left(\omega'^{-1}\ {}^tu,\ Z\right), \tag{1.3}$$

here

$$u=(u_1,\ \cdots,\ u_g).$$

Proposition 1.4. (Thomae) *Let $D=\{c_1,\cdots,c_g,c_{g+1}\}$. Then the following formula holds for all $S\subset B$ with $\#S$ even,*

$$\begin{aligned}&\vartheta\left[\eta_S\right](0)^4\\ &=\begin{cases}0 & \text{if}\quad \#S\circ D\neq g+1,\\ \pm\det\left(\dfrac{\omega'}{2\pi}\right)^2\displaystyle\prod_{\substack{s,t\in S\circ D\\ s\prec t}}(s-t)\cdot\prod_{\substack{s,t\notin S\circ D\\ s\prec t}}(s-t) & \text{if}\quad \#S\circ D=g+1,\end{cases}\end{aligned}$$

where the ordering $\prec$ is the one defined in (1.1). The sign $\pm$ is independent to S.

Proof. See [**Th**] or [**M**, p.3.120]. □

§2. Leading terms of the sigma function

Let $F(\xi_1,\xi_2,\cdots,\xi_n)$ be a polynomial of ξ_i's whose partial degree with respect to ξ_i is at most $g+1$ for each ξ_i. Then we let

$$F(\xi_1,\xi_2,\cdots,\xi_n)\big|_{\xi^{r-1}=u_r}$$

denote the homogeneous polynomial of u_1, u_2, $\cdots$, u_g which is given by, after plugging formally $\xi_j^{r-1}=u_r$ for any j, homogenizing by u_1. For instance, if $F(\xi_1,\xi_2,\xi_3)=\xi_1\xi_2^2+\xi_2^3$ and $g\geq 4$, then, by plugging $\xi_1=u_2$, $\xi_2^2=u_3$ and $\xi_2^3=u_4$, we have $F(\xi_1,\xi_2,\xi_3)\big|_{\xi^{r-1}=u_r}=u_2u_3+u_4u_1$ by homogenizing $u_2u_3+u_4$ by u_1.

Proposition 2.1. *On the Taylor expansion of $\sigma(u) = \sigma(u; C)$ at the origin, the following statements hold.*
(i) *Assume that the genus g is odd and put $g = 2n - 1$. Then the lowest terms of $\sigma(u)$ is given by*

$$\left[\frac{\partial^{(g+1)/2}}{\partial u_1 \partial u_3 \cdots \partial u_g}\sigma(u)\Big|_{u=0}\right] \cdot \left[\frac{1}{n!}\Delta(\xi_1, \cdots \xi_n)|_{\xi^{r-1}=u_r}\right].$$

(ii) *Assume that the genus $g = 2n$ is even. Then the lowest terms of $\sigma(u)$ is given by*

$$\left[\frac{\partial^{g/2}}{\partial u_1 \partial u_3 \cdots \partial u_{g-1}}\sigma(u)\Big|_{u=0}\right] \cdot \left[\frac{1}{n!}\Delta(\xi_1, \cdots \xi_n)|_{\xi^{r-1}=u_r}\right].$$

Proof. see [**B2**, p.360]. □

We know that $u_1 u_3 \cdots u_g$, if g odd, or $u_1 u_3 \cdots u_{g-1}$, if g even, is one of the terms of lowest degree. We are interested in the following constant:

Definition 2.2.

$$\gamma(C) := \begin{cases} \dfrac{\partial^{(g+1)/2}}{\partial u_1 \partial u_3 \cdots \partial u_g}\sigma(u; C)\Big|_{u=0} & \text{if } g \text{ is odd },\\ \dfrac{\partial^{g/2}}{\partial u_1 \partial u_3 \cdots \partial u_{g-1}}\sigma(u; C)\Big|_{u=0} & \text{if } g \text{ is even }. \end{cases}$$

Proposition 2.3. *We have*

$$\gamma(C)^8 = \Delta(C)\det\left(\frac{\omega'}{2\pi}\right)^4.$$

Proof.
In (1.4), if we take the empty set $\emptyset$ as S, then

$$\vartheta[0](0)^4 = \pm\det\left(\frac{\omega'}{2\pi}\right)^2 \prod_{1\le i<j}(a_i - a_j)\prod_{1\le i<j}(c_i - c_j). \tag{2.4}$$

Let

$$P(x) = (x - a_1)\cdots(x - a_g).$$

Let $\ell_r := \lambda_{2g+1}\dfrac{P'(a_r)^2}{\sqrt{-1}\cdot f'(a_r)}$, where $P'(X) = \frac{d}{dX}P(X)$. Then, by [**B2**, p.358], we have

$$\gamma(C)^4 = \frac{\vartheta[0](0)^4\prod_{i<j}(a_i - a_j)^2}{\ell_1 \cdots \ell_g} \tag{2.5}$$

From the formulae (2.4) and (2.5), the proof completes by an easy calculation. □

§3. Description by even Thetanullwerten

From now on we assume that $g \geq 2$. Let

$$E := \{S \subset B \,|\, \#S \circ D = g+1\}.$$

If $\#S \circ D = g+1$ then $\#S \cap D = \#S \cap \complement D$. So, in this case, $\#S$ must be even. Then

$$\begin{aligned} \#E &= \sum_{m=0}^{g} \#\{S \,|\, S \in E, \#S = 2m\} \\ &= \sum_{m=0}^{g} \#\{S \,|\, S \in E, \#S \cap D = m\} \\ &= \sum_{m=0}^{g} \binom{g+1}{m}\binom{g}{m}. \end{aligned} \tag{3.1}$$

Here D is the set defined in (1.4). For example, if $g = 2$ or $g = 3$ then $\#E$ is equal to 10 or 25, respectively.

Proposition 3.2. *Assume the genus $g \geq 2$. Let*

$$\mu = \frac{\#E \cdot g}{2(2g+1)} (\in \mathbf{N}).$$

Then

$$\Delta(C)^{\mu} \cdot \det\left(\frac{\omega'}{2\pi}\right)^{2\#E} = \prod_{S \in E} \vartheta[\eta_S](0)^4.$$

Note that if g=2 then μ=2. So, in this case, the result of [**G1**] is stronger than our result (see [**M**, p.3.104]).

Proof of 3.2. We rewrite the factors of

$$\prod_{S \in E} \vartheta[\eta_S](0)^4.$$

by (1.4). Then we can easily verify that each $(s-t)$ with s, $t \in B$ appears in the same proportion in this product. Thus, we have

$$\prod_{S \in E} \vartheta[\eta_S](0)^4 = \left(\pm\det\left(\frac{\omega'}{2\pi}\right)^2\right)^{\#E} \Delta(C)^M$$

for some natural number M. We let compute M. The number of factors of the form $(s-t)$ with s, $t \in B$ appearing as factors of $\prod_{S \in E} \vartheta[\eta_S](0)^4$ is $\#E \cdot$

$\left\{\binom{g+1}{2}+\binom{g}{2}\right\}$. The number of such factors appearing in $\Delta(C)$ is $2\binom{2g+1}{2}$. Hence

$$\begin{aligned} M &= \#E\cdot\left\{\binom{g+1}{2}+\binom{g}{2}\right\}\Big/2\binom{2g+1}{2} \\ &= \frac{\#E\cdot g}{2(2g+1)} \\ &= \mu, \end{aligned}$$

and the statement has benn proved. □

Now, (2.3) and (3.2) imply:

Theorem 3.3. *Let $\gamma(C)$ be as in* (2.2)*. Let μ be as in* (3.2)*. Then*

$$\gamma(C)^{8\mu} = \left(\pm\det\left(\frac{\omega'}{2\pi}\right)\right)^{-2\#E+4\mu} \prod_{S\in E} \vartheta[\eta_S]\,(0)^4,$$

where the sign $\pm$ is that of (1.4)*and $\#E$ is given by* (3.1)*.*

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