Some New Addition Formulae for Weierstrass Elliptic Functions

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Abstract. In a previous paper (Eilbeck, Matsutani and Ônishi, Phil. Trans. R. Soc. A 2011 369, 1245-1263), we introduced new 2- and 3-variable addition formulae for the Weierstrass elliptic functions in the special case of an equianharmonic curve. In the present paper we remove the restriction to the equianharmonic curve, extending the ideas to the general elliptic curve. We present explicit new 2-variable and 3-variable addition formulae for this curve, and prove the structure of the formulae for the n-variable case.

1. Introduction

Consider the Weierstrass equation

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3, \tag{1.1}$$

where g_2 and g_3 are constants. The functions $\sigma(u)$ and $\wp(u) = -\frac{d^2}{du^2}\log\sigma(u)$ from Weierstrass's theory of elliptic functions are well studied. There is an especially well-known addition formula

$$-\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(u) - \wp(v), \tag{1.2}$$

(see p.451 of [11], for instance). In general, for n variables $u^{(j)}$ ($j = 1$, ..., n), it is known that

$$\begin{aligned}&\frac{\sigma(u^{(1)}+u^{(2)}+\cdots+u^{(n)})\prod_{i<j}\sigma(u^{(i)}-u^{(j)})}{\prod_j \sigma(u^{(j)})^n}\\ &= \frac{1}{\prod\limits_j j!}\begin{vmatrix} 1 & \wp(u^{(1)}) & \wp'(u^{(1)}) & \cdots & \wp^{(n-2)}(u^{(1)}) \\ 1 & \wp(u^{(2)}) & \wp'(u^{(2)}) & \cdots & \wp^{(n-2)}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(u^{(n)}) & \wp'(u^{(n)}) & \cdots & \wp^{(n-2)}(u^{(n)}) \end{vmatrix}.\end{aligned} \tag{1.3}$$

These are a reflection of the involution of the elliptic curve associated to (1.1). Some of the related addition formulae may be found on p.458 of [11], for example.

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There is also a three-term analogy of equation (1.2), which reflects the cyclic automorphism group of order three, in the so-called equianharmonic case where the Weierstrass invariant $g_2 = 0$ (and g_3 is assumed non-zero). Let ζ be a primitive cube root of unity, (without loss of generality we may take $\zeta = (-1+\sqrt{-3})/2$). Then the functions specialised to this case satisfy

$$-\frac{\sigma(u+v)\sigma(u+\zeta v)\sigma(u+\zeta^2 v)}{\sigma(u)^3\sigma(v)^3} = \frac{1}{2}(\wp'(u)+\wp'(v)). \tag{1.4}$$

This was given as Proposition 5.1 of [4].

We have similar, but more complicated, generalizations for certain specialized trigonal curves of genus three, (Theorem 10.1 in [3] and Theorem 5.4 in [2]), and genus four, (Theorem 8 in [5]). The aim of this paper is to introduce generalisations of (1.3) similar to (1.4), but for the most general elliptic curve, (2.1).

Our new results are based on the following observation. The curve defined by (1.1):

$$\mathscr{C}\ :\ y^2 = x^3 - \tfrac{g_2}{4}x - \tfrac{g_3}{4} \qquad (y = \tfrac{1}{2}\wp'(u),\ x = \wp(u)) \tag{1.5}$$

has two natural maps

$$\begin{array}{ccccc} \mathbb{P}^1 & \longleftarrow & \mathscr{C} & \longrightarrow & \mathbb{P}^1 \\ x & \longleftarrow\!\shortmid & (x,y) & \longmapsto & y, \end{array} \tag{1.6}$$

where both $\mathbb{P}^1$ denote projective lines. The map to the left is a double covering and the map to the right is a triple covering. The formulae (1.2) and (1.3) come from the former and so it is natural to ask what would formulae would follow from the latter.

We proceed by considering the most general elliptic curve, first describing the structure of a new class of n-variable addition formulae in Theorem 4.1 and then giving explicit expressions in the cases $n = 2$ (Theorem 5.1) and $n = 3$ (Theorem 6.1).

When we consider the 2-variable case with the general Weierstrass curve (Remark 5.3[2]) we find the new generalisation of (1.4)

$$-\frac{\sigma(u+v)\sigma(u+v^{\star})\sigma(u+v^{\star\star})}{\sigma(u)^3\sigma(v)\sigma(v^{\star})\sigma(v^{\star\star})} = \frac{1}{2}(\wp'(u)+\wp'(v)), \tag{1.7}$$

where $v, v^{\star}, v^{\star\star}$ are so-called conjugate variables satisfying $\wp'(v) = \wp'(v^{\star}) = \wp'(v^{\star\star})$, $v + v^{\star} + v^{\star\star} = 0$. These conjugate variables are the generalisations of $v, \zeta v, \zeta^2 v$ in the equianharmonic case.

2. Preliminaries

The reader is referred to [8] for the details of the material in this section. Define

$$f(x,y) = y^2 + (\mu_1 x + \mu_3)y - (x^3 + \mu_2 x^2 + \mu_4 x + \mu_6). \tag{2.1}$$

We consider the curve $\mathscr{C}$ defined by $f(x,y) = 0$ with the unique point ∞ at infinity. Although we assume $\mathscr{C}$ is non-singular, the formulae in our theorems are valid even if $\mathscr{C}$ is singular. It is known that any elliptic curve over any perfect field is written in this form (see [1], Chapter 8, or [10], Section 3 of Chapter 3). The results for this curve are valid as identities on power series over quite general base rings and are not restricted to the case of the complex numbers.

We may define weights, denoted wt, by

$$\mathrm{wt}(x) = -3, \quad \mathrm{wt}(y) = -2, \quad \mathrm{wt}(\mu_j) = -j.$$

From this definition, it is easy to see that any formula in this paper is of homogeneous weight. In general a numerical subscript throughout this paper will refer to the corresponding (negative) weight, except for the classical constants g_2 and g_3, which have weight -4 and -6 respectively.

Any differential of the first kind is a constant multiple of

$$\omega = \omega(x,y) = \frac{dx}{f_y(x,y)} = \frac{dx}{2y + (\mu_1 x + \mu_3)} = -\frac{dy}{f_x(x,y)},$$

where f_y and f_x denote $\frac{\partial}{\partial y}f$ and $\frac{\partial}{\partial x}f$, respectively. Let Λ denote the lattice consisting of the integrals of this differential along any closed paths:

$$\Lambda = \left\{ \oint \omega \right\}.$$

We define two meromorphic functions $x(u)$ and $y(u)$ by the set of equalities

$$u = \int_{\infty}^{(x(u),y(u))} \omega, \quad f\big(x(u), y(u)\big) = 0. \tag{2.2}$$

Clearly, these are periodic with respect to Λ and have poles only at the points in Λ. Note that it follows from these definitions that the variable u is of weight $1 : \mathrm{wt}(u) = 1$.

From the definitions in (2.2) we have

$$x(-u) = x(u), \quad y(-u) = y(u) + \mu_1 x(u) + \mu_3.$$

Both $x(u)$ and $y(u)$ have a pole only at $u = 0$, of order 2 and 3, respectively.

Let us take a local parameter t around the point ∞ satisfying

$$y = \frac{1}{t^3}. \tag{2.3}$$

This choice of a local parameter is different from the usual one : $t = -x/y$. Using (2.3) and (2.2), we can obtain the power series expansions of $x(u)$ and $y(u)$ beginning with

$$\begin{aligned} x(u) &= u^{-2} - (\tfrac{1}{12}{\mu_1}^2 + \tfrac{1}{3}\mu_2) \\ &\quad + (\tfrac{1}{240}{\mu_1}^4 + \tfrac{1}{30}\mu_2{\mu_1}^2 - \tfrac{1}{10}\mu_3\mu_1 + \tfrac{1}{15}{\mu_2}^2 - \tfrac{1}{5}\mu_4)u^2 + \cdots, \\ y(u) &= -u^{-3} - \tfrac{1}{2}\mu_1 u^{-2} + (\tfrac{1}{24}{\mu_1}^3 + \tfrac{1}{6}\mu_2\mu_1 - \tfrac{1}{2}\mu_3) + \cdots. \end{aligned} \tag{2.4}$$

For two variable points (x, y) and (z, w) on $\mathscr{C}$, we define

$$\Omega(x, y, z, w) = \frac{y + w + \mu_1 z + \mu_3}{x - z},$$
$$\omega(x, y) = \frac{(y + w + \mu_1 z + \mu_3)dx}{(x - z)(2y + \mu_1 x + \mu_3)}.$$

These have a pole of order 1 with residue 1 at (z, w) when regarded as a form with variable (x, y) and (z, w) fixed. Indeed, since $(2w+\mu_1 z+\mu_3) = f_y(z, w)$ when $(x, y) = (z, w)$, the residue at (z, w) is 1, and the zeroes of numerator and denominator at $(x, y) = (z, -w - \mu_1 z - \mu_3)$ is cancelled.

For a differential η of the 2nd kind with pole only at ∞, we define

$$\boldsymbol{\xi}(x, y; z, w) = \tfrac{d}{dz}\Omega(x, y; z, w)dz - \omega(x, y)\eta(z, w),$$

where (x, y), $(z, w) \in \mathscr{C}$. Then, the differential of the second kind

$$\eta(x, y) = \frac{-x dx}{2y + \mu_1 x + \mu_3}, \tag{2.5}$$

chosen as in [8], satisfies

$$\boldsymbol{\xi}(x, y; z, w) = \boldsymbol{\xi}(z, w; x, y).$$

We fix the notation η for the form (2.5) from now on. Let α and β be a pair of two closed paths on $\mathscr{C}$ which is a representative of a symplectic base of the homology group $H_1(\mathscr{C}, \mathbb{Z})$. We let η' and η'' be periods of η with respect to the closed paths α and β. In general, for a given $v \in \mathbb{C}$, we denote by v' and v'' the real numbers such that

$$v = v'\omega' + v''\omega''.$$

Let

$$L(u, v) = u(v'\eta' + v''\eta'')$$

for u and $v \in \mathbb{C}$. We define the *sigma function* of $\mathbb{C}$ by

$$\sigma(u) = \eta_{\mathrm{Ded}}({\omega'}^{-1}\omega'')^{-3} \cdot \frac{\omega'}{2\pi} \cdot \exp\big(-\tfrac{1}{2}u^2\eta'{\omega'}^{-1}\big)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}({\omega'}^{-1}u \big| {\omega'}^{-1}\omega''), \tag{2.6}$$

where $\eta_{\mathrm{Ded}}({\omega'}^{-1}\omega'')$ is the value of Dedekind's eta function at ${\omega'}^{-1}\omega''$. It is easily checked that

$$\sigma(-u) = -\sigma(u). \tag{2.7}$$

It is known that the σ-function does not depend on the choice of symplectic base α and β of $H_1(\mathscr{C},\mathbb{Z})$, and that it has the following quasi-periodicity property:

Lemma 2.8. *The σ-function satisfies*

$$\sigma(u+\ell)=\chi(\ell)\sigma(u)\exp L(u+\tfrac{1}{2}\ell,\ell)\quad(\ell\in\Lambda). \tag{2.9}$$

The σ-function may be represented by a series expansion starting with

$$\begin{aligned}\sigma(u)=u&+({\overline{\mu}_1}^2+\mu_2)(\tfrac{1}{3!})u^3+({\overline{\mu}_1}^4+2\mu_2{\overline{\mu}_1}^2+\mu_3\mu_1+{\mu_2}^2+2\mu_4)(\tfrac{1}{5!})u^5\\&+({\overline{\mu}_1}^6+3\mu_2{\overline{\mu}_1}^4+6\mu_3{\overline{\mu}_1}^3+3{\mu_2}^2{\overline{\mu}_1}^2+6\mu_4{\overline{\mu}_1}^2\\&\quad+6\mu_3\mu_2\overline{\mu}_1+{\mu_2}^3+6\mu_4\mu_2+6{\mu_3}^2+24\mu_6)(\tfrac{1}{7!})u^7+\cdots,\end{aligned}$$

where $\overline{\mu}_1=\mu_1/2$.

Throughout this paper, for simplicity, we use $\mathbb{Z}[\mu_1,\mu_2,\mu_3,\mu_4,\mu_6]=\mathbb{Z}[\boldsymbol{\mu}]$ and $\mathbb{Q}[\mu_1,\mu_2,\mu_3,\mu_4,\mu_6]=\mathbb{Q}[\boldsymbol{\mu}]$. We remark here that (Hurwitz integrality) this expansion is of the form

$$\sigma(u)=\sum_{n=1}^{\infty}A_n\frac{u^n}{n!}\quad\text{with } A_n\in\mathbb{Z}[\overline{\mu}_1,\mu_3,\mu_2,\mu_4,\mu_6]. \tag{2.10}$$

However, it is known that

$$\sigma(u)^2=\sum_{n=2}^{\infty}A'_n\frac{u^n}{n!}\quad\text{with } A'_n\in\mathbb{Z}[\boldsymbol{\mu}]. \tag{2.11}$$

The reader is referred to the discussion in [8]. This integrality of the coefficients of this expansion is taken up in Remark 4.7 below. Logically, in this paper, we need only the fact that $A_n\in\mathbb{Q}[\boldsymbol{\mu}]$.

We now define as usual the elliptic functions

$$\wp(u)=-\frac{d^2}{du^2}\log\sigma(u),\qquad \wp'(u)=\frac{d}{du}\wp(u). \tag{2.12}$$

Then we have

$$\wp(-u)=\wp(u)\quad\text{and}\quad \wp'(-u)=-\wp'(u). \tag{2.13}$$

by (2.7). Our $\wp(u)$ for the general curve is slightly different from Weierstrass's. Our $\wp(u)$ has the expansion,

$$\wp(u)=\frac{1}{u^2}+\sum_{\ell\in\Lambda,\ell\neq0}\Big(\frac{1}{(u-\ell)^2}-\frac{1}{\ell^2}\Big)-\frac{{\mu_1}^2+4\mu_2}{12},$$

which is shown by the situation of zeroes of $\sigma(u)$.

Comparing the power series expansions in (2.4) and the essential part of the expansion of $\wp(u)$ with respect to u obtained by (2), we have

$$\begin{aligned}\wp(u)&=x(u),\\ \wp'(u)&=2y(u)+\mu_1x(u)+\mu_3.\end{aligned} \tag{2.14}$$

If the parameters in (2.1) take values $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = -\frac{1}{4}g_2$, $\mu_6 = -\frac{1}{4}g_3$, then the function $\wp(u)$ defined by (2.12) satisfies the classical equation (1.1). Moreover the function $\sigma(u)$ defined in (2.6) is exactly the same as the Weierstrass σ-function. Under such a transformation the results of this section map to the well-known results for the Weierstrass functions.

3. Conjugate points

For a variable point (x, y) on $\mathscr{C}$, we have three points (up to multiplicity) with the same second coordinate y. We will denote them by

$$(x, y), \quad (x^\star, y), \ \text{and} \ (x^{\star\star}, y).$$

Moreover, for

$$v = \int_\infty^{(x,y)} \omega, \tag{3.1}$$

we define

$$v^\star = \int_\infty^{(x^\star,y)} \omega, \ \text{and} \ v^{\star\star} = \int_\infty^{(x^{\star\star},y)} \omega.$$

Here the paths of integration are defined as the continuous transformations by taking * or ** for all points on the path in (3.1).

Lemma 3.2. *In the above notation we have*

$$v + v^\star + v^{\star\star} = 0. \tag{3.3}$$

Proof. Summing up the holomorphic differential ω on all three sheets of the covering we get a holomorphic differential on the base $\mathbb{P}^1$ (the right hand side of (1.6)), which must vanish, implying (3.3).

Alternatively, an algebraic proof is as follows. Since, for a given y, the x, $x^\star$, $x^{\star\star}$ are the solution of the equation $f(X, y) = 0$ of X, we see $f(X, y) = -(X - x)(X - x^\star)(X - x^{\star\star})$. So

$$\begin{aligned} f_x(x, y) &= -(x - x^\star)(x - x^{\star\star}), \\ f_x(x^\star, y) &= -(x^\star - x)(x^\star - x^{\star\star}), \\ f_x(x^{\star\star}, y) &= -(x^{\star\star} - x)(x^{\star\star} - x^\star). \end{aligned}$$

Since

$$\frac{1}{(x - x^\star)(x - x^{\star\star})} + \frac{1}{(x^\star - x)(x^\star - x^{\star\star})} + \frac{1}{(x^{\star\star} - x)(x^{\star\star} - x^\star)} = 0,$$

we find

$$-\frac{dy}{f_x(x, y)} - \frac{dy}{f_x(x^\star, y)} - \frac{dy}{f_x(x^{\star\star}, y)} = 0$$

and hence the desired equality. □

Note that if $\ell \in \Lambda$ then $\ell^\star$, $\ell^{\star\star} \in \Lambda$, and that $\ell + \ell^\star + \ell^{\star\star} = 0$ by Lemma 3.2.

In the Weierstrass case when the parameters in (2.1) take values $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = -\frac{1}{4}g_2$, $\mu_6 = -\frac{1}{4}g_3$, then we have $\wp'(v) = \wp'(v^\star) = \wp'(v^{\star\star})$.

Using the curve equation (2.1) and a local parameter (2.3) we may obtain an expansion beginning

$$
\begin{aligned}
x = t^{-2} &+ \frac{1}{3}\mu_1 t^{-1} - \frac{1}{3}\mu_2 + \left(-\frac{1}{3^4}{\mu_1}^3 - \frac{1}{3^2}\mu_2\mu_1 + \frac{1}{3}\mu_3\right) t \\
&+ \left(\frac{1}{3^5}{\mu_1}^4 + \frac{1}{3^3}\mu_2{\mu_1}^2 - \frac{1}{3^2}\mu_3\mu_1 + \frac{1}{3}{\mu_2}^2 - \mu_4\right) t^2 \\
&+ \left(-\frac{4}{3^8}{\mu_1}^6 - \frac{5}{3^6}\mu_2{\mu_1}^4 + \frac{5}{3^5}\mu_3{\mu_1}^3 + \Big(-\frac{2}{3^4}{\mu_2}^2 + \frac{1}{3^3}\mu_4\Big){\mu_1}^2\right. \\
&\quad \left. + \frac{2}{3^3}\mu_2\mu_3\mu_1 - \frac{1}{3^2}{\mu_3}^2 - \frac{2}{3^4}{\mu_2}^3 + \frac{1}{3^2}\mu_4\mu_2 - \frac{1}{3}\mu_6\right) t^4 + O(t^5).
\end{aligned}
\tag{3.4}
$$

By looking at the recursion relation giving this expansion, we see this expansion belongs to $\mathbb{Z}[\boldsymbol{\mu}][[\frac{1}{3}t]]$.

Throughout this paper, ζ is a fixed primitive cube root of unity. Transforming $t \to \zeta t$ and $t \to \zeta^2 t$ gives rise to similar expansions of $x^\star$ and $x^{\star\star}$ in terms of t. Using the definition of ω and a formal reversing of the function $t \mapsto v$, we expand the function $v \mapsto t$. Substituting this into the expansions of $t \mapsto x^\star$ and $t \mapsto x^\star$ gives expansions

$$
\begin{aligned}
v^\star &= \zeta v + \cdots \in \mathbb{Z}[\boldsymbol{\mu}, \zeta][[\tfrac{1}{3}v]], \\
v^{\star\star} &= \zeta^2 v + \cdots \in \mathbb{Z}[\boldsymbol{\mu}, \zeta][[\tfrac{1}{3}v]],
\end{aligned}
\tag{3.5}
$$

This implies that $\sigma(v^\star)/\sigma(v)$ and $\sigma(v^{\star\star})/\sigma(v)$ are power series of v with coefficients in $\mathbb{Q}[\boldsymbol{\mu}]$.

4. New Addition formula (General form)

First, we describe the general structure of our new class of addition formula, before constructing explicit examples in the following sections.

Theorem 4.1. *Let $u^{(1)}$, $u^{(2)}$, $\cdots$, $u^{(n)}$ be n-variables. Then*

$$
\frac{\sigma(u^{(1)} + u^{(2)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} + u^{(j)\star})\, \sigma(u^{(i)} + u^{(j)\star\star})}{\prod_{j=1}^{n} \sigma(u^{(j)})^{2n+1-2j} \sigma(u^{(j)\star})^j \sigma(u^{(j)\star\star})^j}
\tag{4.2}
$$

may be expressed as a polynomial in the $x(u^{(j)})$ and $y(u^{(j)})$ for $j = 1, \ldots,$ n of weight $-(n^2-1)$ over the ring $\mathbb{Q}[\boldsymbol{\mu}]$. Moreover, if $\mu_1 = \mu_2 = \mu_4 = 0$, it is symmetric with respect to any exchange

$$
\big(x(u^{(i)}), y(u^{(i)})\big) \longleftrightarrow \big(x(u^{(j)}), y(u^{(j)})\big).
$$

Proof. Regarding (4.2) as a function of each $u^{(j)}$, we can check that it is meromorphic and periodic with respect to Λ, (see the proof of Theorem 5.1 for details on such checks). Hence, it must have a rational expression in terms of $x(u^{(j)})$, $y(u^{(j)})$ for $j = 1, \ldots, n$. For arbitrarily fixed j, let $v = u^{(j)}$. Then as a function of v, (4.2) has its only pole at $v = 0$ (of order $2n-1$). By counting, it is of weight $1 + n(n-1) - n(2n-1) = -(n^2-1)$. Therefore (4.2) is a polynomial of the $x(u^{(j)})$ and $y(u^{(j)})$ of weight $-(n^2-1)$.

Hence an addition formula may be derived by taking (4.2) as the left hand side and constructing the described polynomial for the right hand side To find the right hand side we use the method of undetermined coefficients as follows. Firstly, we prepare the monomials

$$\prod_{j=1}^{n} x(u^{(j)})^{p_j} y(u^{(j)})^{\varepsilon_j} \tag{4.3}$$

of weight $-(n^2-1)$ or larger, where p_j are non-negative and ε_j are 0 or 1. By looking at the leading terms of these monomials, we see that they are linearly independent over $\mathbb{Q}(\boldsymbol{\mu})$. Of course, there are only finitely many such monomials. Secondly, set the right hand side as

$$\sum_{\{p_j,\varepsilon_j\}} C_{\{p_j,\varepsilon_j\}} \prod_{j=1}^{n} x(u^{(j)})^{p_j} y(u^{(j)})^{\varepsilon_j} \tag{4.4}$$

with undetermined coefficients $C_{\{p_j,\varepsilon_j\}} \in \mathbb{Q}(\boldsymbol{\mu})$. Then, after rewriting the right hand side by using (2.14) as a rational function of $\sigma(u^{(j)})$, $\sigma'(u^{(j)})$, $\sigma''(u^{(j)})$, $\sigma'''(u^{(j)})$ for $j = 1, \cdots, n$, where $\sigma'(u) = \frac{d}{du}\sigma(u)$, $\sigma''(u) = \frac{d^2}{du^2}\sigma(u)$, and $\sigma'''(u) = \frac{d^3}{du^3}\sigma(u)$, we multiply

$$\prod_{j=1}^{n} \sigma(u^{(j)})^{2n-1}$$

to the both sides. Then we get the following equality:

$$\begin{aligned}
&\sigma(u^{(1)} + u^{(2)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} + u^{(j)\star})\, \sigma(u^{(i)} + u^{(j)\star\star}) \\
&\qquad \times (\text{a power series of } u^{(j)}\text{s with coefficients in } \mathbb{Q}[\boldsymbol{\mu}]) \\
&\qquad\quad = \text{a polynomial of } \sigma(u^{(j)}),\ \sigma'(u^{(j)}),\ \sigma''(u^{(j)}) \text{ for } j = 1, \cdots, n.
\end{aligned}$$

Here, we used that $\sigma(u^{\star})/\sigma(u)$ and $\sigma(u^{\star\star})/\sigma(u)$ are power series of u with coefficients in the ring $\mathbb{Q}[\boldsymbol{\mu}]$. Now, we focus to each term of the form

$$\prod_{j=1}^{n} \sigma'(u^{(j)})^{s_j}\, \sigma(u^{(j)})^{k_j} \tag{4.5}$$

for some set $\{s_j > 0,\, k_j > 0\}$ in the right hand side. Since

$$x(u) = \frac{\sigma''(u)\sigma(u) - \boxed{\sigma'(u)^2}}{\sigma(u)^2} \quad \text{and}$$

$$y(u) = \frac{-\frac{1}{2}\sigma'''(u)\sigma(u)^2 + \frac{3}{2}\sigma''(u)\sigma'(u)\sigma(u) - \boxed{\sigma'(u)^3}}{\sigma(u)^3} - \frac{\mu_1}{2}\cdot\frac{\sigma''(u)\sigma(u) - \sigma'(u)^2}{\sigma(u)^2} + \frac{\mu_3}{2},$$

it is clear that the term (4.5) comes from a unique term of (4.3). So that, if we expand the right hand side as a power series with respect to $\{u^{(j)}\}$, the leading terms

$$\prod_{j=1}^{n} u^{(j)\,k_j} \tag{4.6}$$

from (4.5) comes from a unique term of (4.4), say

$$C_{\{p_j,\varepsilon_j\}} \prod_{j=1}^{n} x(u^{(j)})^{p_j} y(u^{(j)})^{\varepsilon_j},$$

and has the coefficient $\pm C_{\{p_j,\varepsilon_j\}}$. Now, by comparing the two sides with respect to the term (4.6), we see the coefficient $C_{\{p_j,\varepsilon_j\}}$ must belong to $\mathbb{Q}[\boldsymbol{\mu}]$. The last assertion is proved by the following formula: if $\mu_1 = \mu_2 = \mu_4 = 0$, we have

$$\sigma(\zeta u) = \zeta\sigma(u)$$

(See [4], Lemma 4.1), and that $u^\star = \zeta u$, $u^{\star\star} = \zeta^2 u$. Namely, the left hand side in this case is symmetric with respect to any exchange $u^{(i)} \longleftrightarrow u^{(j)}$. □

Remark 4.7. Our computations suggest that the expression of (4.2) in terms of $x(u^{(j)})$s and $y(u^{(j)})$s has coefficients in $\mathbb{Z}[\boldsymbol{\mu}]$. This phenomenon may follow from (2.10) and (2.11). Because of this, Theorem 4.1 may be valid as a power series identity over quite general base rings and is not restricted to the case of the complex numbers.

5. New Addition formula (2-variable case)

The first explicit main result of this paper now follows.

Theorem 5.1. *Using the notation of the previous sections, we have*

$$
\begin{aligned}
-\frac{\sigma(u+v)\sigma(u+v^{\star})\sigma(u+v^{\star\star})}{\sigma(u)^3\sigma(v)\sigma(v^{\star})\sigma(v^{\star\star})} &= y(u) - y(-v) \\
&= y(u) + y(v) + \mu_1 x(v) + \mu_3 \\
&= \frac{1}{2}\big(\wp'(u) + \wp'(v)\big) + \frac{\mu_1}{2}\big(\wp(u) - \wp(v)\big).
\end{aligned}
\tag{5.2}
$$

Remark 5.3. We comment on how our formula is modified when specializing the curve.

1. For a fixed x, we have two points on the curve. If one point is denoted say (x, y), then the other point is $(x, -y - \mu_1 x - \mu_3)$. In this situation, if $u = \int_{\infty}^{(x,y)} \omega$ then $-u = \int_{\infty}^{(x,-y-\mu_1 x-\mu_3)} \omega$. So, if we replace the sigma function in (1.2) in the Introduction by the most general sigma function (2.6), the left hand side of (1.2) has the same form as equation (1.2). The right hand side for the fully general curve $\mathscr{C}$ has exactly the same form as the right hand side of (1.2), which can be easily checked.

2. As we mentioned just after (2.14), when the parameters in (2.1) take values $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = -\frac{1}{4}g_2$, $\mu_6 = -\frac{1}{4}g_3$, the function $\wp(u)$ defined by (2.12) satisfies the classical equation (1.1). In this case, by (2.13) and (2.14), the right hand side of the formula reduces to the addition formula (1.7) given in the introduction.

3. If we do consider the equianharmonic case (by setting further $\mu_4 = g_2 = 0$) then Theorem 5.1 reduces to Proposition 5.1 of [4], with equation (5.2) becoming (1.4) from the Introduction.

Proof. (of Theorem 5.1.) The left hand side of (5.2) is a meromorphic function of both u and v. Using (3.2) and (2.8), we see the left hand side is invariant with respect to the transformations $u \mapsto u + \ell$, $v \mapsto v + \ell$ for $\ell \in \Lambda$. Indeed, for the transformation $u \mapsto u + \ell$, the exponent of the exponential factor becomes

$$
\begin{aligned}
&L(u+v+\tfrac{1}{2}\ell,\ell) + L(u+v^{\star}+\tfrac{1}{2}\ell,\ell) + L(u+v^{\star\star}+\tfrac{1}{2}\ell,\ell) - 3\,L(u+\tfrac{1}{2}\ell,\ell) \\
&= L(v+v^{\star}+v^{\star\star},\ell) \\
&= L(0,\ell) \\
&= 0,
\end{aligned}
$$

While for $v \mapsto v + \ell$, it becomes

$$\begin{aligned} &L(u+v+\tfrac{1}{2}\ell,\ell)+L(u+v^\star+\tfrac{1}{2}\ell^\star,\ell^\star)+L(u+v^{\star\star}+\tfrac{1}{2}\ell^{\star\star},\ell^{\star\star})\\ &\quad -L(v+\tfrac{1}{2}\ell,\ell)-L(v^\star+\tfrac{1}{2}\ell^\star,\ell^\star)-L(v^{\star\star}+\tfrac{1}{2}\ell^{\star\star},\ell^{\star\star})\\ &=L(u,\ell)+L(u,\ell^\star)+L(u,\ell^{\star\star})\\ &=L(u,\ell+\ell^\star+\ell^{\star\star})\\ &=L(u,0)\\ &=0. \end{aligned}$$

Therefore, the left hand side is a function of u modulo Λ. It also has a unique pole at $u = 0$. It is well-known that such a function is a polynomial of $\wp(u)$ and its higher order derivatives. In this case the poles are of order 3, so we need only use $\wp$ and $\wp'$. Since the equation must be of homogeneous weight (weight -3 on both sides), we know that the left hand side must be of the form

$$a_1\wp'(u)+a_2\wp'(v)+b_1\mu_1\wp(u)+b_2\mu_1\wp(v)+c_1{\mu_1}^3+c_2\mu_1\mu_2+c_3\mu_3$$

with absolute constants a_1, a_2, b_1, b_2, c_1, c_2 and c_3. However, for arbitrary fixed v, as a function of u, the left hand side has zeroes at $u = -v$, $u = -v^\star$, $u = -v^{\star\star}$ (of order 1 each), and no other zeros. Using the fact that the $\wp(u)$ is an even function we have that

$$a_2 = a_1 \ (= a \text{ say}), \quad -b_2 = b_1 \ (= b \text{ say}), \quad c_1 = c_2 = c_3 = 0.$$

Substituting the truncated expansion (3.4) up to the constant term and (3.5) into (5.2) gives

$$-\frac{1}{u^3}-\frac{1}{v^3}+\frac{1}{2}\mu_1\left(\frac{1}{u^2}-\frac{1}{v^2}\right)+\cdots.$$

Since

$$\wp(u)=\frac{1}{u^2}+\cdots,$$

we find the coefficients are as stated

$$a=\tfrac{1}{2},\quad b=\tfrac{1}{2}, \tag{5.4}$$

concluding the proof. □

We finish the section with some further remarks on the formula (1.7). It could be argued that this formula lacks symmetry as the variables u and v are treated differently. We can replace u by $u^\star$ and $u^{\star\star}$ in turn, remembering that $\wp(u) = \wp(u^\star) = \wp(u^{\star\star})$, then add the three to get

$$\sum_{i=1}^{3}\left[\frac{\prod_{j=1}^{3}\sigma(u_i+v_j)}{\sigma(u_i)^3\prod_{j=1}^{3}\sigma(v_j)}\right]=\frac{3}{2}\left(\wp'(u)+\wp'(v)\right)$$

where for typographical convenience we use $u_i, i = 1, 2, 3$, to represent $u, u^\star$, and $u^{\star\star}$ respectively. However in producing such a formula we are throwing away information, in particular by subtracting two of the three relations described above we can get

$$\frac{\sigma(u+v)\sigma(u+v^\star)\sigma(u+v^{\star\star})}{\sigma(u)^3} = \frac{\sigma(u^\star+v)\sigma(u^\star+v^\star)\sigma(u^\star+v^{\star\star})}{\sigma(u^\star)^3},$$

and similarly for $(u, u^\star)$ and $(u^\star, u^{\star\star})$. A similar equation is seen in Corollary 12.2 of [9].

6. New Addition formula (3-variable case)

The second main result below, is a natural three variable extension of Theorem 5.1, (see also [7] and [9]).

Theorem 6.1. *Let u, v, and w be variables. Denote, for brevity, $(x_u, y_u) = (x(u), y(u))$ and similarly for v and w. With the notation of the previous sections we have a new addition formula with left hand side*

$$\frac{\sigma(u+v+w)\sigma(u+v^\star)\sigma(u+v^{\star\star})\sigma(u+w^\star)\sigma(u+w^{\star\star})\sigma(v+w^\star)\sigma(v+w^{\star\star})}{\sigma(u)^5\sigma(v)^3\sigma(v^\star)\sigma(v^{\star\star})\sigma(w)\sigma(w^\star)^2\sigma(w^{\star\star})^2}$$

and right hand side given by $\sum_{i=0}^{8} r_i$ with the r_i as below. Each r_i is a polynomial in x_u, x_v, x_w, y_u, y_v, y_w, and the $\{\mu_j\}$ (of combined weight i).

$$\begin{aligned}
r_0 &= (y_u y_v + y_u y_w + y_v y_w - x_u x_v x_w)(x_u + x_v + x_w) \\
&\quad - x_u^2 x_v^2 - x_u^2 x_w^2 - x_v^2 x_w^2, \\
r_1 &= \mu_1(x_v x_u y_v + 2x_v x_u y_w + 2y_w x_u^2 + x_w x_u y_w - x_w^2 y_u + x_v x_u y_u \\
&\quad + x_w y_v x_u + y_v x_u^2 + y_w x_v^2), \\
r_2 &= (x_u^2 x_v - x_u x_w^2 + y_w y_u)\mu_1^2 - (x_v^2 x_w - y_v y_u + x_u^2 x_v + x_u x_w^2 \\
&\quad + 2x_v x_w x_u - y_w y_u - y_w y_v + x_v x_w^2 + x_u^2 x_w + x_u x_v^2)\mu_2, \\
r_3 &= \mu_1^3 y_w x_u + (x_u y_v + 2y_w x_u + x_v y_w - x_w y_u)\mu_2\mu_1 \\
&\quad + (y_v + y_w + y_u)(x_u + x_v + x_w)\mu_3, \\
r_4 &= -\mu_1^2 x_u \mu_2 x_w + (x_u^2 - x_w^2 + 2x_u x_v + x_u x_w)\mu_3\mu_1 \\
&\quad - (x_u x_v + x_v x_w + x_u x_w)\mu_2^2 - (x_u^2 + x_v^2 + x_w^2)\mu_4, \\
r_5 &= \mu_1^2 y_w \mu_3 - (y_u - y_w)\mu_4\mu_1 + (y_v + y_w + y_u)\mu_3\mu_2, \\
r_6 &= -\mu_1^2 x_u \mu_4 + (x_u - x_w)\mu_3\mu_2\mu_1 - (x_u + x_v + x_w)(\mu_2\mu_4 - \mu_6 - \mu_3^2), \\
r_7 &= 0, \\
r_8 &= -\mu_1\mu_3\mu_4 + (\mu_6 + \mu_3^2)\mu_2 - \mu_4^2.
\end{aligned}$$

Proof. The left hand side of the new formula is meromorphic in u, v, and w. Moreover, we can check easily that it is periodic with respect to Λ. Hence it may be expressed in terms of elliptic functions. Further, we can check that the left hand side has poles of order five each in u, v and w and so the right hand side must have an expression in $\wp(u), \wp(v)$ and $\wp(w)$ and their derivatives up to third order. More specifically, the right hand side will be a sum of terms, each a product of three functions, one in each of the variables and with all functions taken from the set $\{1, \wp, \wp', \wp'', \wp'''\}$. Such an expression is clear from the linear algebra when considering the space of elliptic functions graded by pole order, (for more details on such spaces see for example [2, 6]). This also clarifies why $r_7 = 0$: since there is no elliptic function of weight 1 to include in the right hand side.

The coefficients of this right hand side may then be determined using the series expansions of the functions discussed earlier. Since the left hand side is of weight -8 the expansions used need to contain terms with monomials in μ_i up to weight -8. MAPLE was used to implement this calculation (with details on similar calculations given in [2]). The right hand side presented above was then obtained by making the substitutions implied by (2.14). □

Remark 6.2. Using the mappings in (2.14) we can rewrite the right hand side of the formula in Theorem (6.1) in terms of $\wp$ and its first derivative.

Remark 6.3. Let

$$f_2 = x_u + x_v + x_w + \mu_2,$$
$$f_4 = x_u x_v + x_v x_w + x_u x_w - \mu_4 + \mu_1 y_w.$$

where the suffices of f are chosen to denote the weight. Each of these vanishes when $v = u^\star$ and $w = u^{\star\star}$ at the same time since then $y(u) = y(u^\star) = y(u^{\star\star})$ and $x(u)$, $x(u^\star)$, $x(u^{\star\star})$ are the three solutions of the cubic equation

$$X^3 + \mu_2 X^2 + (\mu_4 - \mu_1 y(u))X + \mu_6 - y(u)^2 - \mu_3 y(u) = 0.$$

A calculation with Gröbner bases implemented with MAPLE shows that the right hand side of the formula presented in Theorem 6.1 lies in the ideal generated by f_2 and f_4. Specifically, we have that

$$\sum_{i=0}^{8} r_i = Q_6 f_2 + Q_4 f_4,$$

where

$$\begin{aligned} Q_6 = &\, y_w\mu_1^3 - (\mu_4 - x_vx_w - x_ux_v)\mu_1^2 \\ &+ (x_u\mu_3 - \mu_3x_w - x_wy_w - x_wy_u + 2y_wx_u + x_uy_v)\mu_1 \\ &- (x_ux_v + x_vx_w + x_ux_w)\mu_2 + \mu_3^2 + (y_v + y_w + y_u)\mu_3 - x_ux_v^2 + \mu_6 \\ &- x_ux_w^2 + y_wy_u - x_v^2x_w + y_vy_u - x_vx_w^2 + y_wy_v - x_vx_wx_u - x_u\mu_4, \\ Q_4 = &\, (y_u + \mu_3)\mu_1 - (\mu_2 + x_v + x_w)\mu_1^2 + (x_v + x_w)\mu_2 \\ &+ \mu_4 + x_w^2 + x_vx_w + x_v^2. \end{aligned}$$

This expression, along with (3.3) shows that both sides of the equation in Theorem (6.1) vanish when $v = u^\star$ and $w = u^{\star\star}$.

Remark 6.4. In Remark 5.3 we discussed how the 2-variable formula collapsed to known results when restricting the curve. We note here some similar restrictions for the 3-variable result.

1. If $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = -\frac{1}{4}g_2$, $\mu_6 = -\frac{1}{4}g_3$ in (2.1), then the right hand side of the formula in Theorem 6.1 becomes

$$\begin{aligned} &-\tfrac{1}{16}g_2^2 + \tfrac{1}{4}g_2(\wp(v)^2 + \wp(w)^2 + \wp(u)^2) \\ &\qquad - \wp(u)^2\wp(w)^2 - \wp(v)^2\wp(w)^2 - \wp(u)^2\wp(v)^2 \\ &\qquad - \tfrac{1}{4}(\wp(u) + \wp(v) + \wp(w))\big(4\wp(u)\wp(v)\wp(w) + g_3 \\ &\qquad\quad - \wp'(u)\wp'(v) - \wp'(v)\wp'(w) - \wp'(u)\wp'(w)\big). \end{aligned} \tag{6.5}$$

2. If instead one simplifies by setting $\mu_1 = \mu_2 = \mu_4 = 0$ then we of course get another simplification of the right hand side, but also a simplification of the left hand side. In this case x^3 is the only term in the curve equation with x and so the starred variables can all be described using roots of unity acting on the non-starred variables. So in this case we have

$$\begin{aligned} &\frac{\sigma(u+v+w)\sigma(u+\zeta v)\sigma(u+\zeta^2 v)\sigma(u+\zeta w)\sigma(u+\zeta^2 w)\sigma(v+\zeta w)\sigma(v+\zeta^2 w)}{\sigma(u)^5\sigma(v)^3\sigma(\zeta v)\sigma(\zeta^2 v)\sigma(w)\sigma(\zeta w)^2\sigma(\zeta^2 w)^2} \\ &= (x_v + x_u + x_w)\mu_6 + (x_v + x_u + x_w)\mu_3^2 \\ &\quad + (y_u + y_v + y_w)(x_v + x_u + x_w)\mu_3 - x_u^2x_w^2 - x_v^2x_w^2 - x_u^2x_v^2 \\ &\quad - (x_v + x_u + x_w)(x_vx_wx_u - y_vy_w - y_uy_w - y_vy_u). \end{aligned} \tag{6.6}$$

3. The equianharmonic case is a sub-case of the both the previous cases. Here we will have the simplified left hand side from (6.6) and a further reduced right hand side which may be obtained by setting $g_2 = 0$ in (6.5). Using $\wp$-coordinates analogously to (1.4), we have the right hand side

$$\begin{aligned} &\tfrac{1}{4}(\wp(u) + \wp(v) + \wp(w))(\wp'(u)\wp'(v) + \wp'(v)\wp'(w) + \wp'(u)\wp'(w) \\ &\quad - g_3 - 4\wp(v)\wp(w)\wp(u)) - \wp(u)^2\wp(v)^2 - \wp(u)^2\wp(w)^2 - \wp(v)^2\wp(w)^2. \end{aligned}$$

7. Further remarks

1. In the rational case, $\mu_i = 0, (g_i = 0)$, all the equations collapse to simple algebraic identities.

2. For the equianharmonic curve $y^2 = x^3 + \mu_6$, there is an action of the group of the sixth roots of unity acts on this curve, and on the coordinate space $\mathbb{C}$ of $\wp(u)$ and $\sigma(u)$. Let $\zeta = \exp(2\pi \boldsymbol{i}/3)$, a third root of unity. In [4], we gave a 3-variable formula of the form

$$\frac{\sigma(u+v+w)\sigma(u+\zeta v+\zeta^2 w)\sigma(u+\zeta^2 v+\zeta w)}{\sigma(u)^3\sigma(v)^3\sigma(w)^3}$$

as a polynomial of $\wp(u)$, $\wp(v)$, $\wp(w)$, and their first order derivatives. If we consider a naive generalization of this in our setting, namely,

$$\frac{\sigma(u+v+w)\sigma(u+v^\star+w^{\star\star})\sigma(u+v^{\star\star}+w^\star)}{\sigma(u)^3\sigma(v)\sigma(v^\star)\sigma(v^{\star\star})\sigma(w)\sigma(w^\star)\sigma(w^{\star\star})},$$

we find this is no longer a periodic function with respect to Λ, as may be checked by the translational formula (2.9). This means that, if we increase v to $v+\ell$ (and similarly for w), the factors which appear in (2.9) do not cancel out.

3. As described in Theorem 4.1, there will be generalisations of such formulae to cases with n variables. However, we find that trying to derive the expanded form of the right hand side in the 4-variable case using naive series expansions greatly exceeds the memory limits of the current machines available to us. We expect that progress would follow from the discovery of a more compact expression for these right hand sides, for example, as a determinant.

4. Our result might be generalized to higher genus curves. For example, the natural analogue for Theorem 5.1 for the curve

$$y^2 + (\mu_1 x^2 + \mu_3 x + \mu_5)y = x^5 + \mu_2 x^4 + \mu_4 x^3 + \mu_6 x^2 + \mu_8 x + \mu_{10}$$

could be obtained by considering five roots of x for a fixed y.

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