Frobenius-Stickelberger-Type Formulae for General Curves

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Abstract. We generalize classical formula of Frobenius-Stickelberger (see (0) below) to the most general curve of type of Buchstaber-Enolskii-Leykin.

(0) **The original Frobenius-Stickelberger.** For the curve $y^2+(\mu_1x+\mu_3)y = x^3+\mu_2x^2+\mu_4x+\mu_6$,

$$\frac{\sigma(u^{(1)}+\cdots+u^{(n)})\prod_{i<j}\sigma(u^{(i)}-u^{(j)})}{\prod_{j=1}^{n}\sigma(u^{(j)})^n} = \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & xy(u^{(n)}) & \cdots \end{vmatrix} \quad (n \times n \text{ determinant}),$$

where $u = \displaystyle\int_\infty^{(x(u),y(u))} \frac{dx}{2y+\mu_1x+\mu_3}$.

(1) **The curve.** (d,q) is a given pair of integers with $0<d<q$, $\gcd(d,q)=1$, and let

$$f(x,y) = y^d + p_1(x)y^{d-1} + p_2(x)y^{d-2} + \cdots + p_{d-1}(x)y - p_d(x).$$

where $p_j(x)$ is a polynomial of x of degree $\lceil \frac{jq}{d} \rceil$ and $p_d(x)$ is monic. Coefficients are suitably suffixed and denoted by μ_js. We consider the non-singular complete curve defined by

$$\mathscr{C} : f(x,y) = 0.$$

This has unique point ∞ at infinity and of genus $g = \frac{(d-1)(q-1)}{2}$.
Example : $(d,q)=(3,4)$, $y^3 + (\mu_1x+\mu_4)y^2 + (\mu_2x^2+\mu_5x+\mu_8)y = x^4+\mu_3x^3+\mu_6x^2+\mu_9x+\mu_{12}$.

(2) **Arithmetic parameter at ∞.** There exists a local parameter t of the form $t = x^ay^b$ with integers a, b such that x and y are expanded in terms of t with the coefficients in $\mathbb{Z}[\{\mu_j\}]$.
Example : If $(d,q)=(3,4)$, then $t = x/y$.

(3) **Symplectic homology base.** $\{\alpha_j, \beta_j\}_{j=1,\dots g}$ is a symplectic basis of $H_1(\mathscr{C},\mathbb{Z})$.

(4) **Weierstrass gap sequence at ∞.** Let $\{ w_g\,(=2g-1),\ w_{g-1},\ \cdots,\ w_2,\ w_1\,(=1)\}$ be the sequence of Weierstrass gaps at ∞ in descending order.

(5) **Canonical base of the space of differential forms of 1st kind.** Let

$$\{\, a_1d+b_1q\,(=0),\ a_2d+b_21\,(=d),\ a_3d+b_3q,\ \cdots,\ a_jd+b_jq,\ \cdots\cdots\}$$

be the increasing sequence of Weierstrass non-gaps at ∞ on $\mathscr{C}$. Then[1]

$$\{\omega_1,\ \omega_2,\ \cdots,\omega_g\} = \left\{ \frac{x^{a_1}y^{b_1}dx}{f_y(x,y)}\Big(=\frac{dx}{f_y(x,y)}\Big),\ \frac{x^{a_2}y^{b_2}dx}{f_y(x,y)}\Big(=\frac{x\,dx}{f_y(x,y)}\Big),\ \cdots,\ \frac{x^{a_j}y^{b_j}dx}{f_y(x,y)},\ \cdots,\ \frac{x^{a_g}y^{b_g}dx}{f_y(x,y)} \right\}$$

form a basis of the space $\Gamma(\mathscr{C},\Omega^1)$ of holomorphic 1-forms on $\mathscr{C}$. We simply denote $\omega = (\omega_1,\cdots,\omega_g)$.

(6) **Differentials of the 2nd kind.**
Using canonical isomorphism $H^1(\mathscr{C},\mathbb{C}) \cong H^0\big(\mathscr{C}, d\varinjlim_n \mathscr{O}(n\cdot\infty)\big)\big/ d\varinjlim_n H^0\big(\mathscr{C},\mathscr{O}(n\cdot\infty)\big)$, we define intersection form $\star$ on this space as follows. For any ω and η in this space,

$$\omega\star\eta = \frac{1}{2\pi \boldsymbol{i}}\int_{\partial\mathscr{C}_{\mathrm{r.p.}}}\Big(\int_\infty^{\mathrm{P}}\omega\Big)\eta(\mathrm{P}) = \sum_{\mathrm{P}\in\mathscr{C}}\mathrm{Res}_{\mathrm{P}}\Big(\int_\infty^{\mathrm{P}}\omega\Big)\eta(\mathrm{P}) = \frac{1}{2\pi \boldsymbol{i}}\sum_{j=1}^{g}\Big(\int_{\alpha_j}\omega\int_{\beta_j}\eta - \int_{\alpha_j}\eta\int_{\beta_j}\omega\Big),$$

where $\mathscr{C}_{\mathrm{r.p.}}$ is a regular polygon of the Riemann surface associated to $\mathscr{C}$. This product is just the transported one from usual symplectic structure on $H_1(\mathscr{C},\mathbb{Z})\otimes\mathbb{C}$ under $H^1(\mathscr{C},\mathbb{C}) \cong H^1(\mathscr{C},\mathbb{C})^\vee \cong H_1(\mathscr{C},\mathbb{Z})\otimes\mathbb{C}$. Note that $\omega_i\star\omega_j = 0$.
We extend $\{\omega_1,\cdots,\omega_g\}$ to a symplectic base

$$\{\omega_1,\ \cdots,\ \omega_g,\ \eta_1,\ \cdots,\ \eta_g\}$$

of $H^1(\mathscr{C},\mathbb{C})$ (i.e. $\omega_i\star\eta_j = \delta_{ij}$, $\eta_i\star\eta_j = 0$) by requiring the following two conditions:

(7) **The conditions (Klein's fundamental 2-form).** The required conditions are

① The 2-form $\xi(x,y;z,w) = \omega_1(x,y)\dfrac{d}{dz}\dfrac{1}{(x-z)}\dfrac{f(Z,y)-f(Z,w)}{y-w}\Big|_{Z=z}dz - \displaystyle\sum_{j=1}^{g}\omega_j(x,y)\,\eta_j(z,w)$
(Klein's fundamental 2-form) on $\mathscr{C}\times\mathscr{C}$, with (x,y) and $(z,w)\in\mathscr{C}$, is symmetric, i.e. $\xi(x,y;z,w) = \xi(z,w;x,y)$; and

② $\xi(x,y;z,w) \in \dfrac{1}{(t_2-t_1)^2} + \mathbb{Z}[\mu][[t_1,t_2]]$, where t_1 and t_2 are the arithmetic local parameter of (x,y) and (z,w) on $\mathscr{C}$, respectively.

Though such choice of $\{\eta_j\}$ is not unique, we chose the "simplest" one.

(8) **Period matrices.** We set

$$\omega' = \Big[\oint_{\alpha_j}\omega_i\Big],\quad \omega'' = \Big[\oint_{\beta_j}\omega_i\Big],\quad \eta' = \Big[\oint_{\alpha_j}\eta_i\Big],\quad \eta'' = \Big[\oint_{\beta_j}\eta_i\Big].$$

(9) **Period lattice.** Let $\Lambda = \omega'\mathbb{Z}^g + \omega''\mathbb{Z}^g$ be the lattice of the periods with respect to $\{\omega_j\}$.

(10) **The Jacobian variety and standard embedding of the curve.**
Let $J = \mathbb{C}^g/\Lambda$ be the Jacobian variety of $\mathscr{C}$, $\iota : \mathscr{C} \hookrightarrow J$ is the canonical embedding sending ∞ to the origin of J, and κ is the modulo Λ mapping $\mathbb{C}^g \to J = \mathbb{C}^g/\Lambda$. Then $\kappa^{-1}\iota(\mathscr{C})$ is a universal Abelian covering of $\mathscr{C}$.

(11) **The stratification.** Let $W^{[n]}$ is the image of canonical map $\mathrm{Sym}^n(\mathscr{C}) \to J$ sending the n-tuple of the point ∞ to the origin of J. Let $\Theta^{[n]} = W^{[n]} \cup [-1]W^{[n]}$. Then we have the following stratification :

$$\begin{array}{ccccccccccc}
\infty\in\mathscr{C} = \mathrm{Sym}^1\mathscr{C} & \subset & \mathrm{Sym}^2\mathscr{C} & \subset & \cdots\cdots\cdot & \subset & \mathrm{Sym}^{g-1}\mathscr{C} & \subset & \mathrm{Sym}^g\mathscr{C} \\
\downarrow & & \downarrow & & \cdots\cdot & & \downarrow & & \downarrow \\
O\in \iota(\mathscr{C}) = W^{[1]} & \subset & W^{[2]} & \subset & \cdots\cdots\cdot & \subset & W^{[g-1]} & \subset & W^{[g]} \\
\| & & \cap & & \cdots\cdot & & \| & & \| \\
O\in \quad \Theta^{[1]} & \subset & \Theta^{[2]} & \subset & \cdots\cdots\cdot & \subset & \Theta^{[g-1]} & \subset & \Theta^{[g]} = J = \mathbb{C}^g/\Lambda.
\end{array}$$

(12) **Discriminant.** Let

$$\left.\begin{array}{l}
R_1 = \mathrm{rslt}_x\Big(\mathrm{rslt}_y\big(f(x,y), \frac{\partial}{\partial x}f(x,y)\big), \mathrm{rslt}_y\big(f(x,y), \frac{\partial}{\partial y}f(x,y)\big)\Big) \\
R_2 = \mathrm{rslt}_y\Big(\mathrm{rslt}_x\big(f(x,y), \frac{\partial}{\partial x}f(x,y)\big), \mathrm{rslt}_x\big(f(x,y), \frac{\partial}{\partial y}f(x,y)\big)\Big)
\end{array}\right\},\quad R = \gcd(R_1,R_2) \quad \text{in } \mathbb{Z}[\mu]$$

where rslt_z is Sylvester's resultant with respect to z. Then R is a perfect d-th power[2] in $\mathbb{Z}[\mu]$. We define $D\in\mathbb{Z}[\mu]$ a d-th power root of R. D is called the *discriminant* of $\mathscr{C}$.

(13) **Riemann constant.** Regarding ∞ on $\mathscr{C}$ the base point of $\mathscr{C}$, the Riemann constant $K = \omega'^{-1}\delta$ is written by δ', $\delta'' \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$ as $\omega'K = \delta = \omega'\delta' + \omega''\delta'' \in \frac{1}{2}\Lambda$. Note that, in this case, the canonical class is $K_{\mathscr{C}} = 2(g-1)\infty$ because we can choose $\omega_1 = \frac{dx}{f_y}$ as a representative of the class.

(14) **Coordinate of the whole space.** $\mathbb{C}^g = \{u = (u_{\langle w_g\rangle}, u_{\langle w_{g-1}\rangle}, \cdots, u_{\langle w_1\rangle})\}$.

(15) **Weight.** We define $\mathrm{wt}(\,\cdot\,)$ by taking $\mathrm{wt}(u_{\langle w_j\rangle}) = w_j$, $\mathrm{wt}(\mu_j) = -j$, $\mathrm{wt}(x) = -d$, $\mathrm{wt}(y) = -q$.

(16) **Definition of the sigma function.**
Definition. The *sigma function* of $\mathscr{C}$ is an entire function on the space $\mathbb{C}^g = \{u = (u_{\langle w_g\rangle}, u_{\langle w_{g-1}\rangle}, \cdots, u_{\langle w_1\rangle})\}$ defined by

$$\sigma(u) = c\,\exp\big(-\tfrac{1}{2}\,{}^t u\eta'\omega'^{-1}u\big)\,\vartheta\begin{bmatrix}\delta''\\ \delta'\end{bmatrix}(\omega'^{-1}u\,;\,\omega'^{-1}\omega''),$$

where the theta series is usual one, $c = \frac{1}{D^{1/8}}\Big(\frac{\det(\omega')}{(2\pi)^g}\Big)^{1/2}$, and $\pi = 3.141592\cdots$.
Note that $\sigma(u)$ is independent of choice of $\{\alpha_j,\ \beta_j\}$.

(17) **Properties of the sigma function**.
Lemma.
① *The function $\sigma(u)$ is an odd or even function according to $\frac{(d^2-1)(q^2-1)}{24}$ is odd or even integer, and has poles of order 1 along $\kappa^{-1}(\Theta^{[g-1]})$.*
② *It satisfies the following translational relation :*

$$\sigma(u+\ell) = \chi(\ell)\,\sigma(u)\exp L(u+\tfrac{1}{2}\ell,\ell) \quad \textit{for all} \quad \ell\in\Lambda,$$

where $L(u,\ v'\omega'+v''\omega'') = {}^t u\,(v'\eta'+v''\eta'')$ *with* $u\in\mathbb{C}^g$, $v', v''\in\mathbb{R}^g$,
and $\chi(\ell) = \exp\big(2\pi\boldsymbol{i}\,({}^t\ell'\delta'' + {}^t\ell''\delta' + \frac{1}{2}{}^t\ell'\ell'')\big)$.
③ *For* $\mathrm{P}, \mathrm{Q}, \mathrm{P}_k, \mathrm{Q}_k$, $(1 \leqq k \leqq g)$ *on* $\mathscr{C}$,

$$\frac{\sigma\Big(\int_\infty^{\mathrm{P}}\omega - \sum_{r=1}^{g}\int_\infty^{\mathrm{P}_r}\omega\Big)\sigma\Big(\int_\infty^{\mathrm{Q}}\omega - \sum_{r=1}^{g}\int_\infty^{\mathrm{Q}_r}\omega\Big)}{\sigma\Big(\int_\infty^{\mathrm{Q}}\omega - \sum_{r=1}^{g}\int_\infty^{\mathrm{P}_r}\omega\Big)\sigma\Big(\int_\infty^{\mathrm{P}}\omega - \sum_{r=1}^{g}\int_\infty^{\mathrm{Q}_r}\omega\Big)} = \exp\Big(\sum_{r=1}^{g}\int_{\mathrm{Q}_r}^{\mathrm{Q}}\int_{\mathrm{P}_r}^{\mathrm{P}}\xi\Big).$$

(18) **The Galois action.** Associating to the covering $\mathscr{C}\to\mathbb{P}^1$ given by $(x,y)\mapsto x$, we consider the Galois group $\mathrm{Gal}(\mathscr{C}/\mathbb{P}^1)$. We denote by $[\gamma]$ the action of $\gamma\in\mathrm{Gal}(\mathscr{C}/\mathbb{P}^1)$ on the space $\mathbb{C}^g$ induced from γ. Then $\displaystyle\sum_{\gamma\in\mathrm{Gal}(\mathscr{C}/\mathbb{P}^1)}[\gamma]u = (0,0,\cdots,0)$ for $u\in\mathbb{C}^g$.

(19) **Special derivative $\sigma_{\natural^n}(u)$ of $\sigma(u)$.**
We define special multi-indices $\natural^n$ with respect to $\{w_g,\cdots,w_2,w_1\}$. We shall explain the definition of this by an example.
For example, to get $\natural^2$ for $(d,q)=(3,5)$, $g=6$,
① write a $g\times g$ table as follows,
② line up the Weierstrass gaps $\{w_g,\cdots,w_2,w_1\}$ in the last column,
③ put into other boxes naturally increasing non-negative integers as follows,
④ extract $(g-n)\times(g-n) = 4\times4$ minor matrix in the lower right corner,
⑤ remove all the rows and columns including 0.
⑥ Finally, read the numbers along off-diagonal.

$$\begin{array}{|c|c|c|c|c|c|}\hline 6&7&8&9&10&11\\\hline 3&4&5&6&7&8\\\hline 0&1&2&3&4&5\\\hline &0&1&2&3&4\\\hline &&&0&1&2\\\hline &&&&0&1\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 2&3&4&5\\\hline 1&2&3&4\\\hline &0&1&2\\\hline &&0&1\\\hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 2&\boxed{5}\\\hline \boxed{1}&4\\\hline\end{array}$$

Then, we have

$$\natural^2 = \langle 1,5\rangle \quad\text{and}\quad \sigma_{\natural^2}(u) = \sigma_{\langle 1,5\rangle}(u) = \frac{\partial^2}{\partial u_{\langle 1\rangle}\partial u_{\langle 5\rangle}}\sigma(u).$$

Moreover, let $\sharp = \natural^1$ and $\flat = \natural^2$.

(20) **Functions $x(u)$ and $y(u)$.** We define $x(u)$, $y(u)$ for $u\in\kappa^{-1}(\Theta^{[1]}) = \kappa^{-1}\iota(\mathscr{C})$ as the coordinates (x,y) determined by $u = \displaystyle\int_\infty^{(x,y)}(\omega_1,\omega_2,\cdots,\omega_g)$.

(21) **Key Conjecture.** (almostly a theorem)

Let I be a multi-index with respect to $\{w_g,\cdots,w_2,w_1\}$.
① *If $\mathrm{wt}(I) < \mathrm{wt}(\natural^n)$ then $\sigma_I(u) = 0$ identically on $\kappa^{-1}(\Theta^{[n]})$.*
② *If $\mathrm{wt}(I) = \mathrm{wt}(\natural^n)$ then the translational formula holds :*

$$\sigma_I(u+\ell) = \chi(\ell)\,\sigma_I(u)\exp L(u+\tfrac{1}{2}\ell,\ell) \quad \textit{for } u\in\kappa^{-1}(\Theta^{[n]}) \textit{ and } \ell\in\Lambda,$$

and $\sigma_I(u)$ is equal to an integer times $\sigma_{\natural^n}(u)$ on $\kappa^{-1}(\Theta^{[n]})$.
③ *For $u\in\kappa^{-1}(\Theta^{[n+1]})$,*

$$\sigma_{\natural^{n+1}}(u) = 0 \iff u\in\kappa^{-1}(\Theta^{[n]}).$$

④ *For $u = u^{(1)}+\cdots+u^{(n)}\in\kappa^{-1}(\Theta^{[n]})$ with $u^{(j)}\in\kappa^{-1}(\Theta^{[1]})$ and $v\in\kappa^{-1}(\Theta^{[1]})$, we have*

$$\left.\begin{array}{c} v\mapsto\sigma_{\natural^{n+1}}(u+v) \\ \textit{vanishes}\end{array}\right\} \iff \left\{\begin{array}{l} v\equiv[\gamma]u^{(j)}\ (\mathrm{mod}\,\Lambda) \\ \textit{for some } 1\leqq j\leqq n \textit{ and } \gamma\in\mathrm{Gal}(\mathscr{C}/\mathbb{P}^1),\neq\mathrm{id}.\end{array}\right.$$

⑤ *We have the expansion*

$$\sigma_{\natural^{n+1}}(u+v) = \sigma_{\natural^n}(u)v_{\langle 1\rangle}{}^{w_{g-n}-(g-n)+1} + O(v_{\langle 1\rangle}{}^{w_{g-n}-(g-n)+2})$$

with respect to $v_{\langle 1\rangle}$ for $u\in\kappa^{-1}(\Theta^{[n]})$ and $v\in\kappa^{-1}(\Theta^{[1]})$.

(22) **Main results.**
Theorem 1. *Let $n\geqq 2$ be an integer. For $u^{(i)}\in\kappa^{-1}\iota(\mathscr{C})$ $(1\leqq i\leqq n)$, the following equality holds if the "Key Conjecture" is valid :*

$$\frac{\sigma_{\natural^n}(u^{(1)}+u^{(2)}+\cdots+u^{(n)})\displaystyle\prod_{i<j}\prod_{\substack{\gamma\in\mathrm{Gal}(\mathscr{C}/\mathbb{P}^1)\\ \gamma\neq\mathrm{id}}}\sigma_\flat(u^{(i)}+[\gamma]u^{(j)})}{\displaystyle\prod_{j=1}^{n}\Big(\sigma_\sharp(u^{(j)})^{(d-1)(n-j)+1}\prod_{\substack{\gamma\in\mathrm{Gal}(\mathscr{C}/\mathbb{P}^1)\\ \gamma\neq\mathrm{id}}}\sigma_\sharp([\gamma]u^{(j)})^{j-1}\Big)} = \pm\Big|(x^{a_j}y^{b_j})(u^{(i)})\Big|_{1\leqq i,j\leqq n}\cdot\Big|(x^{j-1})(u^{(i)})\Big|_{1\leqq i,j\leqq n}^{d-2}.$$

Theorem 2. *The "Key Conjecture" is proved for $(d,s) = (2,\text{"any"}), (3,4), (3,5), (4,5), (5,6)$, etcetera.*

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[1] Please do not confuse italic d with Roman d.
[2] Although this is very plausible, I do not have any proof.