Frobenius-Stickelberger Formulae for General Curves

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1. The aim of this Talk

To generalize the following :

The original Frobenius-Stickelberger formula is the following equation satisfied by the Weierstrass functions $\sigma(u)$ and $\wp(u) = -\frac{d^2}{du^2}\log\sigma(u)$:

$$\frac{\sigma(u^{(1)}+\cdots+u^{(n)})\prod_{i<j}\sigma(u^{(i)}-u^{(j)})}{\prod_{j=1}^{n}\sigma(u^{(j)})^n}$$
$$= \frac{(-1)^{(n-1)(n-2)/2}}{1!\,2!\,3!\cdots(n-1)!}\cdot\begin{vmatrix} 1 & \wp(u^{(1)}) & \wp'(u^{(1)}) & \wp''(u^{(1)}) & \wp^{(3)}(u^{(1)}) & \cdots \\ 1 & \wp(u^{(2)}) & \wp'(u^{(2)}) & \wp''(u^{(2)}) & \wp^{(3)}(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & \wp(u^{(n)}) & \wp'(u^{(n)}) & \wp''(u^{(n)}) & \wp^{(3)}(u^{(n)}) & \cdots \end{vmatrix},$$

($n\times n$ determinant).

(Modification)

For $\mathscr{C} : y^2 + (\mu_1 x + \mu_3)y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6$, it holds that

$$\frac{\sigma(u^{(1)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} - u^{(j)})}{\prod_{j=1}^{n} \sigma(u^{(j)})^n}$$
$$= (-1)^n \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & x^3(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & x^3(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & xy(u^{(n)}) & x^3(u^{(n)}) & \cdots \end{vmatrix},$$

($n \times n$ determinant), where $x(u)$, $y(u)$ is just x, y determined by

$$u = \int_{\infty}^{(x,y)} \frac{\mathrm{d}x}{2y + \mu_1 x + \mu_3}.$$

2. Pair of integers

Through out this talk, we use

$$\mathbf{0} < d < q, \quad \mathbf{gcd}(d, q) = \mathbf{1}$$

and

$$g = \frac{(d-\mathbf{1})(q-\mathbf{1})}{\mathbf{2}}$$

(this is the genus of the curve we consider)

We denote in decreasing order

$$\{w_g,\ w_{g-1},\ \cdots,\ w_1\} = \mathbb{Z}_{\geq 0} \setminus \{ad + bq \mid a \geq \mathbf{0},\ b \geq \mathbf{0}\}$$

(the Weierstrass gap sequence w.r.t. (d, q)).

3. The curves which we consider

$$\mathscr{C} : y^d + p_1(x)y^{d-1} + \cdots + p_{d-1}(x)y = x^q + \cdots$$

where $p_j(x)$ is a polynom. of x of degree $\leq \lceil jq/d \rceil$

Example 1. $(d,q) = (2,3)$

$$y^2 + (\mu_1 x + \mu_3)y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6$$

Example 2. $(d,q) = (2,5)$

$$y^2 + (\mu_1 x^2 + \mu_3 x + \mu_5)y = x^5 + \mu_2 x^2 + \cdots + \mu_{10}$$

Example 3. $(d,q) = (3,4)$

$$y^3 + (\mu_1 x + \mu_5)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y$$
$$= x^4 + \mu_3 x^2 + \mu_6 x + \cdots + \mu_{12}$$

1bis. Main Result

(Anyway, we look at the main result!)

Conjecture. Let $n \geqq 2$ be an integer.

For $u^{(i)} \bmod \Lambda \in W^{[1]} \ \ (1 \leqq i \leqq n)$, the following euqality holds:

$$\frac{\sigma_{\natural^n}(u^{(1)} + u^{(2)} + \cdots + u^{(n)}) \displaystyle\prod_{i<j} \prod_{\substack{\gamma \in \mathrm{Gal}(\mathscr{C}/\mathbb{P}^1) \\ \gamma \neq \mathrm{id}}} \sigma_\flat(u^{(i)} + [\gamma]u^{(j)})}{\displaystyle\prod_{j=1}^{n} \Bigg(\sigma_\sharp(u^{(j)})^{(d-1)(n-j)+1} \prod_{\substack{\gamma \in \mathrm{Gal}(\mathscr{C}/\mathbb{P}^1) \\ \gamma \neq \mathrm{id}}} \sigma_\sharp([\gamma]u^{(j)})^{j-1} \Bigg)}$$
$$= \pm \left| (x^{a_j} y^{b_j})(u^{(i)}) \right|_{1 \leqq i,j \leqq n} \cdot \left| (x^{j-1})(u^{(i)}) \right|_{1 \leqq i,j \leqq n}^{d-2},$$

where $\{da_j + qb_j\}$ is the Weierstrass non-gap sequence at ∞.

Theorem. This is OK for $(d, q) = (2,$ "any"$)$, $(3, 4)$, $(3, 5)$, $(4, 5)$, $(5, 6)$.

4. Weight

$$\mathrm{wt}(\mu_j) = -j, \quad \mathrm{wt}(x) = -d, \quad \mathrm{wt}(y) = -q$$

Then, all the equations in this talk are of homogeneous weight.

For example, the equation of $\mathscr{C}$ in case of $(d,q) = (3,4)$,

$$\begin{aligned} y^3 + (\mu_1 x + \mu_5)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y \\ = x^4 + \mu_3 x^2 + \mu_6 x + \cdots + \mu_{12} \end{aligned}$$

is homogeneous of weight $d \cdot q = 3 \times 4 = 12$.

5. Canonical Base of $\Gamma(\mathscr{C}, \Omega^1)$

Let $f(x,y) = y^d + p_1(x)y^{d-1} + \cdots + p_{d-1}(x)y$
$$-(x^q + \mu_d x^{q-1} + \cdots)$$

Example 1. $y^2 + \cdots = x^3 + \cdots$

$\Gamma(\mathscr{C}, \Omega^1)$ is spanned by

$$\omega_1 = \frac{\mathbf{d}x}{f_y(x,y)} = \frac{\mathbf{d}x}{2y + \mu_1 x + \mu_3} \in (1 + t\,\mathbb{Z}[\mu][[t]])dt$$

where $t = -x/y$ (*the arithmetic parameter*)

Example 2. $\quad y^2 + \cdots = x^5 + \cdots$

$\Gamma(\mathscr{C}, \Omega^1)$ is spanned by

$$\omega_1 = \frac{\mathrm{d}x}{f_y(x,y)} = \frac{\mathrm{d}x}{2y + \mu_1 x^2 + \mu_3 x + \mu_5} \in t^2 + t^3\,\mathbb{Z}[\mu][[t]]$$

$$\omega_2 = \frac{x\mathrm{d}x}{f_y(x,y)} = \frac{x\mathrm{d}x}{2y + \mu_1 x^2 + \mu_3 x + \mu_5} \in 1 + t\,\mathbb{Z}[\mu][[t]]$$

where $t = -x^2/y$ (*the arithmetic parameter*)

Example 3. $\quad y^3 + \cdots = x^4 + \cdots$

$\Gamma(\mathscr{C}, \Omega^1)$ is spanned by

$$\omega_1 = \frac{\mathrm{d}x}{f_y(x,y)} \in 1 + t\,\mathbb{Z}[\mu][[t]]$$

$$\omega_2 = \frac{x\mathrm{d}x}{f_y(x,y)} \in t + t^2\,\mathbb{Z}[\mu][[t]]$$

$$\omega_3 = \frac{y\mathrm{d}x}{f_y(x,y)} \in t^4 + t^5\,\mathbb{Z}[\mu][[t]]$$

where $t = x/y$ (*the arithmetic parameter*)

6. Differentials of the 2nd kind

$$H^1(\mathscr{C}, \mathbb{C}) \cong \frac{H^0\big(\mathscr{C}, \mathrm{d}\mathscr{O}(*\infty)\big)}{\mathrm{d}H^0\big(\mathscr{C}, \mathscr{O}(*\infty)\big)} \quad \text{(by Serre duality, etc.)}$$

For any ω and η in this space, we define

$$\omega \star \eta = \frac{1}{2\pi i}\int_{\partial\mathscr{C}_{\mathrm{r.p.}}}\Big(\int_\infty^{\mathrm{P}}\omega\Big)\eta(\mathrm{P}) = \sum_{\mathrm{P}\in\mathscr{C}}\operatorname*{Res}_{\mathrm{P}}\Big(\int_\infty^{\mathrm{P}}\omega\Big)\eta(\mathrm{P})$$

$$= \frac{1}{2\pi i}\sum_{j=1}^{g}\Big(\int_{\alpha_j}\omega\int_{\beta_j}\eta - \int_{\alpha_j}\eta\int_{\beta_j}\omega\Big).$$

where $\{\alpha_j, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\}$ is a symplectic base of $H_1(\mathscr{C}, \mathbb{Z})$.

This product is just the transported one from the usual symplectic structure on $H_1(\mathscr{C}, \mathbb{Z}) \otimes \mathbb{C}$ under $H^1(\mathscr{C}, \mathbb{C}) \cong H^1(\mathscr{C}, \mathbb{C})^\vee \cong H_1(\mathscr{C}, \mathbb{Z}) \otimes \mathbb{C}$.

Note that $\omega_i \star \omega_j = 0$. We extend $\{\,\omega_1, \cdots, \omega_g\,\}$ to a symplectic base

$$\{\,\omega_1, \cdots, \omega_g, \eta_1, \cdots, \eta_g\,\}$$

of $H^1(\mathscr{C}, \mathbb{C})$, namely $\omega_i \star \eta_j = \delta_{ij}$, $\eta_i \star \eta_j = 0$, by requiring that the 2-form (*Klein's fundamental 2-form*) defined by

$$\xi(x, y; z, w) =$$

$$\omega_1(x, y)\,\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{(x-z)}\frac{f(Z, y) - f(Z, w)}{y - w}\Big|_{Z=z}\mathrm{d}z - \sum_{j=1}^{g}\omega_j(x, y)\,\eta_j(z, w),$$

is symmetric, i.e. $\xi(x, y; z, w) = \xi(z, w; x, y)$, and

$$\xi(x, y; z, w) \in \frac{1}{(t_2 - t_1)^2} + \mathbb{Z}[\mu][[t_1, t_2]],$$

where t_1 and t_2 are the arithmetic local parameter of (x, y) and (z, w) on $\mathscr{C}$, respectively.

Choice of $\{\eta_j\}$ is not unique, but we chose the "simplest" one.

Example 1. $y^2 + (\mu_1 x + \mu_3)y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6, \quad (g = 1)$.

$$\omega_1 = \frac{\mathbf{d}x}{f_y(x,y)} \in (1 + t\,\mathbb{Z}[\mu][[t]])\mathbf{d}t, \qquad .$$

$$\xi = \frac{F(x,y;z,w)dxdz}{(x-z)^2 f_y(x,y) f_y(z,w)},$$

where

$$\begin{aligned} F(x,y;z,w) &= xz(x+z) + ({\mu_1}^2 + 2\mu_2)xz + \mu_1(zy + xw) \\ &\quad + (\mu_3\mu_1 + \mu_4)(x+z) + 2yw + \mu_3(y+w) + {\mu_3}^2 + 2\mu_6 \end{aligned}$$

Then

$$\eta_1 = \frac{x\mathbf{d}x}{f_y(x,y)} \in (t^2 + t^3\,\mathbb{Z}[\mu][[t]])\mathbf{d}t.$$

Example 2. For $y^2 + (\mu_1 x + \mu_3)y = x^5 + \mu_2 x^4 + \cdots + \mu_{10}$, $(g = 2)$,

$$\omega_1 = \frac{\mathbf{d}x}{f_y(x,y)}, \quad \omega_2 = \frac{x\mathbf{d}x}{f_y(x,y)},$$

we have $\xi(x,y;z,w) = \dfrac{F(x,y;z,w)dx_1dz}{(z-x)^2 f_y(x_1,y_1) f_w(z,w)}$, where

$$\begin{aligned} F(x,y;z,w) = {} & (x^2z^3 + x^3z^2) + ({\mu_1}^2 + 2\mu_2)x^2z^2 + (\mu_3\mu_1 + \mu_4)(xz^2 + x^2z) \\ & + \mu_1(yz^2 + wx^2) + (2\mu_5\mu_1 + {\mu_3}^2 + 2\mu_6)xz + \mu_3(yz + wx) \\ & + (\mu_5\mu_3 + \mu_8)(z + x) + 2yw + \mu_5(y + w) + ({\mu_5}^2 + 2\mu_{10}) \end{aligned}$$

and

$$\eta_1 = \frac{x^2\mathbf{d}x}{f_y(x,y)}, \quad \eta_2 = \frac{(3x^3 + ({\mu_1}^2 + 2\mu_2)x^2 + (\mu_3\mu_1 + \mu_4)x + \mu_1 y)\mathbf{d}x}{f_y(x,y)}.$$

7. Periods

We define matrices of periods

$$\omega' = \left[\int_{\alpha_i} \omega_j\right],\ \omega'' = \left[\int_{\beta_i} \omega_j\right],\ \eta' = \left[\int_{\alpha_i} \eta_j\right],\ \eta'' = \left[\int_{\beta_i} \eta_j\right],$$

and the period lattice

$$\Lambda = \mathbb{Z}^g\,\omega' + \mathbb{Z}^g\,\omega'' \ \in \ \mathbb{C}^g.$$

8. The Sigma Function

Let $\zeta_j(u)$ be the function without constant term in power series expansion w.r.t. u such that

$$\zeta_j(u+\ell'\omega'+\ell''\omega'')-\zeta_j(u)=\big[\ell'\eta'+\ell''\eta''\big]_{j\text{-th entry}}.$$

Then the differential equation for $u=(u_1,\cdots,u_g)$

$$\boxed{\mathrm{d}\log\sigma(u)=\zeta_1(u)\mathrm{d}u_1+\cdots+\zeta_g(u)\mathrm{d}u_g}$$

has a solution $\sigma(u)$. Namely, $\zeta_j(u)=\frac{\partial}{\partial u_j}\log\sigma(u)$.

Let $\delta'\omega+\delta''\omega''\in\frac{1}{2}\Lambda$ be the Riemann constant for $\mathscr{C}$ w.r.t. base point ∞, $\chi(\ell)=\chi(\ell'\omega'+\ell''\omega'')=\exp(2\pi i(\ell'\delta''+\ell''\delta'+\frac{1}{2}{}^t\ell'\ell''))$, and $L(u,v)=L(u,v'\omega'+v''\omega'')={}^tu(v'\omega'+v''\omega'')$. It is well-known that

(1) $\sigma(u+\ell)=\chi(\ell)\sigma(u)\exp L(u+\frac{1}{2}\ell,\ell)$,

(2) $\sigma(u)=0 \iff u\ (\mathrm{mod}\,\Lambda)\in\Theta$, the "standard" theta divisor.

8bis. Analytic construction of $\sigma(u)$

Such function is realized by

$$\sigma(u) = c \,\exp\left(-\tfrac{1}{2}\,{}^t u \eta' {\omega'}^{-1} u\right) \vartheta\begin{bmatrix}\delta'' \\ \delta'\end{bmatrix}({\omega'}^{-1}u \,|\, {\omega'}^{-1}\omega''),$$

where the theta series is usual one, $c = \frac{1}{D^{1/8}}\left(\frac{\det(\omega')}{(2\pi)^g}\right)^{1/2}$ with discriminant D, $\pi = 3.141592\cdots$,
and δ', $\delta'' \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$ corresponds the Rieman constant.

Note that this function is independent of choice of $\{\alpha_j, \beta_j\}$.

Derivatives. We denote $\sigma_{ij\ldots k}(u) = \dfrac{\partial}{\partial u_i}\dfrac{\partial}{\partial u_j}\cdots\dfrac{\partial}{\partial u_k}\sigma(u)$.

9. Precise Vanishing of $\sigma(u)$

We will define special higher derivatives

$$\sigma_{\natural^1}(u),\ \ \sigma_{\natural^2}(u),\ \ \cdots,\ \ \sigma_{\natural^{g-1}}(u),\quad (\sigma_{\natural^n}(u){=}\sigma(u)\ \text{ for }\ n\geq g).$$

which have the following nice properties: (Slightly rough description)

We change suffix as $(u_1,\cdots,u_g)=(u_{\langle w_g\rangle},\cdots,u_{\langle w_1\rangle})$.

For each $g>n>0$, let $W^{[n]}$ be the image of $\mathbf{Sym}^n(\mathscr{C})$ via Abelian map sending ∞ to the origin and $[-1]W^{[n]}$ is the set given by $[-1]$-plication of it,

and let $W^{[0]}=\{(0,0,\cdots,0)\}$.

Let $\Theta^{[n]}=W^{[n]}\cup[-1]W^{[n]}$.

Note the stratification : $\{0\}=\Theta^{[0]}\subset\Theta^{[1]}\subset\cdots\subset\Theta^{[g-1]}\subset\Theta^{[g]}=J$.

Then, for $g-1>n\geq 0$ and $u\ (\mathrm{mod}\,\Lambda)\in\Theta^{[n+1]}$

$$\sigma_{\natural^{n+1}}(u)=0\iff u\ (\mathrm{mod}\,\Lambda)\in\Theta^{[n]}\qquad.$$

10. Table of $\sigma_{\natural^n}$

(Each number in $\langle\ \rangle$ indicates $\mathbf{wt}\,(u_j)$ for $j \in \natural^n$.)

(d,p)	g	$\sharp = \natural^1$	$\flat = \natural^2$	$\natural^3$	$\natural^4$	$\natural^5$	$\natural^6$	$\natural^7$	$\natural^8$	$\cdots$
$(2,3)$	1	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(2,5)$	2	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(2,7)$	3	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(2,9)$	4	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(2,11)$	5	$\langle 3,7\rangle$	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(2,13)$	6	$\langle 1,5,9\rangle$	$\langle 3,7\rangle$	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(2,15)$	7	$\langle 3,7,11\rangle$	$\langle 1,5,9\rangle$	$\langle 3,7\rangle$	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$(3,4)$	3	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(3,5)$	4	$\langle 4\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(3,7)$	6	$\langle 1,6\rangle$	$\langle 1,5\rangle$	$\langle 4\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(3,9)$	7	$\langle 4,10\rangle$	$\langle 2,7\rangle$	$\langle 1,5\rangle$	$\langle 4\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

11a. Definition of $\natural^n$

We explain by an example : $(d, q) = (3, 7)$, $g = 6$.

Write a $g \times g = 6 \times 6$ table as follows.

We first write the Weierstrass gap sequence with respect to (d, q) on the last column, namely,

					11
					8
					5
					4
					2
					1

11b. Definition of $\natural^2$ for $(d,q) = (3,7)$, $g = 6$. (continuation)

Then, put into other boxes naturally increasing non-negative integers as follows:

6	7	8	9	10	11
3	4	5	6	7	8
0	1	2	3	4	5
	0	1	2	3	4
			0	1	2
				0	1

11c. Definition of $\natural^2$ for $(d,q) = (3,7)$, $g = 6$. (Continuation)

If we wish to get $\natural^n = \natural^2$, extract $(g-n)\times(g-n) = 4\times 4$ minor on the lower right corner. and Remove all rows and columns including $\mathbf{0}$.

6	7	8	9	10	11
3	4	5	6	7	8
0	1	2	3	4	5
	0	1	2	3	4
			0	1	2
				0	1

$\longrightarrow$

2	3	4	5
1	2	3	4
	0	1	2
		0	1

$\longrightarrow$

2	5
1	4

11d. Definition of $\natural^2$ for $(d,q)=(3,7)$, $g=6$. (continuation)

6	7	8	9	10	11
3	4	5	6	7	8
0	1	2	3	4	5
	0	1	2	3	4
			0	1	2
				0	1

2	3	4	5
1	2	3	4
	0	1	2
		0	1

$\longrightarrow$

2	$\boxed{5}$
$\boxed{1}$	4

Finally, by reading the numbers on the off-diagonal, we have

$$\natural^2 = \langle 1,5\rangle \quad \text{and} \quad \sigma_{\natural^2}(u) = \sigma_{\langle 1,5\rangle}(u) = \frac{\partial^2}{\partial u_{\langle 1\rangle}\partial u_{\langle 5\rangle}}\sigma(u).$$

Note that $u_{\langle 1\rangle} = u_6$ and $u_{\langle 5\rangle} = u_3$ in old notation.

12. Galois group $\mathbf{Gal}(\mathscr{C}/\mathbb{P}^1)$ and Its action

Consider the projection to the x-coordinate $\mathscr{C} \longrightarrow \mathbb{P}^1$, $(x, y) \mapsto x$.

Denote by $\mathbf{Gal}(\mathscr{C}/\mathbb{P}^1)$ the Galois group of this covering.

Take a generic point (x, y) on the curve $\mathscr{C}$.

Then all the points with the same x-coordinate (thers are d such points) are given by $\{(x, \gamma(y)) \mid \gamma \in \mathbf{Gal}(\mathscr{C}/\mathbb{P}^1)\}$.

Since the whole space $\mathbb{C}^g$ is the pull back of $\mathbf{Sym}^g(\mathscr{C})$ with respect to mod Λ, this action of $\mathbf{Gal}(\mathscr{C}/\mathbb{P}^1)$ extends to the whole space. We denote this action by

$$\gamma : u \mapsto [\gamma]u \quad \text{for } \gamma \in \mathbf{Gal}(\mathscr{C}/\mathbb{P}^1).$$

Then we see

$$\sum_{\gamma \in \mathbf{Gal}(\mathscr{C}/\mathbb{P}^1)} [\gamma]u = 0.$$

13. Properties of higher derivatives of $\sigma(u)$

Notation (revisited):

$\Theta^{[n]} = W^{[n]} \cup [-1]W^{[n]}$, $W^{[n]}$ is image of $\mathbf{Sym}^n(\mathscr{C})$ via Abelian map.

Let I be a multi-index.

(1) If $\mathbf{wt}(I) < \mathbf{wt}(\natural^n)$, then $\sigma_I(u) = 0$ for $u \bmod \Lambda \in \Theta^{[n]}$.

(2) If $\mathbf{wt}(I) = \mathbf{wt}(\natural^n)$, then $\sigma_I(u)$ is an integer times $\sigma_{\natural^n}(u)$ on $\Theta^{[n]}$.

(3) If $u \bmod \Lambda \in \Theta^{[n]}$, then

$$\sigma_{\natural^n}(u+\ell) = \chi(\ell)\,\sigma_{\natural^n}(u) \exp L(u+\tfrac{1}{2}\ell, \ell) \quad (\ell \in \Lambda).$$

(4) Let v, $u^{(j)} (\bmod \Lambda) \in W^{[1]}$. For $u = u^{(1)} + \cdots + u^{(n)} \ (\bmod \Lambda) \in W^{[n]}$

$$\left.\begin{array}{l} v \mapsto \sigma_{\natural^{n+1}}(u+v) \\ \quad \text{vanishes} \end{array}\right\} \iff \left\{\begin{array}{l} v \equiv [\gamma]u^{(j)} \ (\bmod \Lambda) \\ \text{for some } j \text{ and } \gamma \in \mathbf{Gal}(\mathscr{C}/\mathbb{P}^1), \neq \mathbf{id}. \end{array}\right.$$

(5) $$\sigma_{\natural^{n+1}}(u+v) = \sigma_{\natural^n}(u) v_{\langle 1\rangle}{}^{w_{g-n}-g+n+1} + O(v_{\langle 1\rangle}{}^{w_{g-n}+(g-n)+2})$$

for $u \bmod \Lambda \in W^{[n]}$ and $v \bmod \Lambda \in W^{[1]}$.

14. Original Frobenius-Stickelberger Formula

Example. $y^2 + (\mu_1 x + \mu_3)y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6$.

Original Frobenius-Stickelberger formula:

$$\frac{\sigma(u^{(1)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} - u^{(j)})}{\prod_{j=1}^{n} \sigma(u^{(j)})^n} =$$

$$(-1)^n \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & x^3(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & x^3(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & xy(u^{(n)}) & x^3(u^{(n)}) & \cdots \end{vmatrix},$$

($n \times n$ determinant).

15. Frob.-Stickel.-Type Formula for a Hyperell.-Curve ([Ô,2005])

Example. $y^2 = x^{2g+1} + \mu_2 x^{2g} + \cdots$.

Let $u^{(j)} (\operatorname{mod} \Lambda) \in \Theta^{[1]}$. Then (assuming $n \geqq g$ for simplicity)

$$\frac{\sigma(u^{(1)} + \cdots + u^{(n)}) \prod_{i<j} \sigma_\flat(u^{(i)} - u^{(j)})}{\prod_{j=1}^n \sigma_\sharp(u^{(j)})^n}$$

$$= \begin{vmatrix} 1 & x(u^{(1)}) & \cdots & x^g(u^{(1)}) & y(u^{(1)}) & x^{g+1}(u^{(1)}) & xy(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & \cdots & x^g(u^{(2)}) & y(u^{(2)}) & x^{g+1}(u^{(2)}) & xy(u^{(2)}) & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & \cdots & x^g(u^{(n)}) & y(u^{(n)}) & x^{g+1}(u^{(n)}) & xy(u^{(n)}) & \cdots \end{vmatrix}$$

($n \times n$ determinants)

16. New Frobenius-Stickelberger-Type Formula ([Ô,2009])

Example. $y^3 = x^4 + \mu_3 x^3 + \cdots$ (purely trigonal case).

$\zeta = \exp(2\pi i/3)$ acts as $[\zeta](u_1, u_2, u_3) = (\zeta u_1, \zeta u_2, \zeta^2 u_3)$.

New Frobenius-Stickelberger-type formula:

Let $u^{(j)} (\bmod \Lambda) \in W^{[1]}$. Then (for simplicity, assuming $n \geqq 3$)

$$\frac{\sigma(u^{(1)} + \cdots + u^{(n)}) \prod_{i<j} \sigma_\flat(u^{(i)} + [\zeta]u^{(j)})\, \sigma_\flat(u^{(i)} + [\zeta^2]u^{(j)})}{\prod_{j=1}^{n} \sigma_\sharp(u^{(j)})^{2n-1}}$$

$$= \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & y^2(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & y^2(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & xy(u^{(n)}) & y^2(u^{(n)}) & \cdots \end{vmatrix} \begin{vmatrix} 1 & x(u^{(1)}) & x^2(u^{(1)}) & \cdots & x^{n-1}(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & \cdots & x^{n-1}(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^{n-1}(u^{(n)}) \end{vmatrix}$$

($n \times n$ determinants)

We can give similar formula for any weightable plane curve.

17. Most general Frobenius-Stickelberger-Type Formula ([Ô])

Conjecture (General case)

Let $n \geqq 2$ be an integer. For $u^{(i)} \bmod \Lambda \in W^{[1]}$ $(1 \leqq i \leqq n)$, the following euqality holds:

$$\frac{\sigma_{\natural^n}(u^{(1)} + u^{(2)} + \cdots + u^{(n)}) \displaystyle\prod_{i<j} \prod_{\substack{\gamma \in \mathbf{Gal}(\mathscr{C}/\mathbb{P}^1) \\ \gamma \neq \mathbf{id}}} \sigma_\flat(u^{(i)} + [\gamma]u^{(j)})}{\displaystyle\prod_{j=1}^{n} \left(\sigma_\sharp(u^{(j)})^{(d-1)(n-j)+1} \prod_{\substack{\gamma \in \mathbf{Gal}(\mathscr{C}/\mathbb{P}^1) \\ \gamma \neq \mathbf{id}}} \sigma_\sharp([\gamma]u^{(j)})^{j-1} \right)}$$

$$= \pm \left| (x^{a_j} y^{b_j})(u^{(i)}) \right|_{1 \leqq i,j \leqq n} \cdot \left| (x^{j-1})(u^{(i)}) \right|_{1 \leqq i,j \leqq n}^{d-2},$$

where $\{da_j + qb_j\}$ is the Weierstrass non-gap sequence.

Theorem. This is OK for $(d, q) = (2,$ "any"$)$, $(3, 4)$, $(3, 5)$, $(4, 5)$, $(5, 6)$.

18. Idea of proof of vanishing properties

If $v \bmod \Lambda \in \Theta^{[1]}$ is a variable,

$v_{\langle 1\rangle} = v_g$ is a local parameter at $(0, 0, \cdots, 0)$ along $\Theta^{[1]} = \mathscr{C}$.

Example: $(d, q) = (3, 7)$, $g = 6$. For $u \bmod \Lambda \in W^{[g-2]}$, $v \bmod \Lambda \in W^{[1]}$,

$$\begin{aligned} 0 &= \sigma(u+v) \\ &= \sigma_{\langle 1\rangle}(u) v_{\langle 1\rangle} + \sigma_{\langle 11\rangle}(u) \tfrac{1}{2!} {v_{\langle 1\rangle}}^2 + \sigma_{\langle 2\rangle}(u) (\tfrac{1}{2} {v_{\langle 1\rangle}}^2 + \cdots) + \cdots. \end{aligned}$$

Hence, $\boxed{\sigma_{\langle 1\rangle}(u) = 0, \ \sigma_{\langle 11\rangle}(u) = -\sigma_{\langle 2\rangle}(u) \ \text{ on } \ W^{[g-2]}}$.

For $u \bmod \Lambda \in W^{[g-3]}$, $v \bmod \Lambda \in W^{[1]}$,

$$\begin{aligned} 0 &= \sigma_{\langle 1\rangle}(u+v) \\ &= \sigma_{\langle 11\rangle}(u) v_{\langle 1\rangle} + \sigma_{\langle 111\rangle}(u) \tfrac{1}{2!} {v_{\langle 1\rangle}}^2 + \sigma_{\langle 12\rangle}(u) (\tfrac{1}{2} {v_{\langle 1\rangle}}^2 + \cdots) + \cdots \end{aligned}$$

Therefore, we see $\boxed{\sigma_{\langle 11\rangle}(u) = 0, \ \sigma_{\langle 111\rangle}(u) = -\sigma_{\langle 12\rangle}(u) \ \text{ on } \ W^{[g-3]}}$.

Repeating such argument, we arrive the desired properties for $\sigma_{\natural^n}(u)$.

Remarks.

I do not have some unified proof for all cases.

Note that we never used Riemann's singularity theorem and similar results.

19. Proof of the first step of Frob.-Stick. Formula

Exmple $(d,q)=(3,5)$, $g=4$; $\mathbf{Gal}(\mathscr{C}/\mathbb{P}^1)=\{\mathbf{id},\ \gamma,\ \gamma^2\}$.

Let $v'=[\gamma]v$ and $v''=[\gamma^2]v$

For u, $v \bmod \Lambda \in W^{[1]}$, our claim is

$$\frac{\sigma_\flat(u+v)\,\sigma_\flat(u+v')\,\sigma_\flat(u+v'')}{\sigma_\sharp(u)^3\sigma_\sharp(v)\sigma_\sharp(v')\sigma_\sharp(v'')}=\begin{vmatrix}1 & x(u)\\ 1 & x(v)\end{vmatrix}^2 \quad (\sharp=\langle 4\rangle,\ \flat=\langle 1\rangle).$$

We regard the two sides as functions of u. They are periodic with respect to Λ.

—. The LHS has zeroes at v, v', v'' of order 2 because $v+v'+v''=0$,

and it has poles only at $u=0$ of order $4\times 3-6=6$.

—. The RHS has the same zeroes and poles.

Comparing the two sides on the coefficient of leading terms of power series expansions w. r. t. $u_{\langle 1\rangle}$ show the desired formula.

Thank you very much!