Universal Elliptic Functions*

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Let $\wp_{\rm w}(u)$ be the usual $\wp$-function of Weierstrass satisfying

$$\big(\tfrac{d}{du}\,\wp_{\rm w}(u)\big)^2 = 4\wp_{\rm w}(u)^3 - g_2\,\wp_{\rm w}(u) - g_3,$$

where g_2 and g_3 are constant complex numbers such that ${g_2}^3 - 27{g_3}^2 \neq 0$. We shall put suffix "w" to distinguish functions defined by Weierstrass from new functions redefined in this paper. According to Weierstrass paper [7], his sigma function $\sigma_{\rm w}(u)$ is characterized by

$$\sigma_{\rm w}(u) = u + O(u^5), \quad \wp_{\rm w}(u) = -\tfrac{d^2}{du^2}\log\sigma_{\rm w}(u).$$

In [7], Weierstrass gave a recursion relation on the coefficients of power series expansion $\sigma_{\rm w}(u)$ with respect to u as follows:

$$\begin{aligned}
&\sigma_{\rm w}(u) = \sum_{m,n} a_{m,n}\big(\tfrac{1}{2}g_2\big)^m(2g_3)^n\frac{u^{4m+6n+1}}{(4m+6n+1)!},\\
&a_{0,0}=1,\ a_{m,n}=0 \ \text{ if } m<0 \text{ or } n<0,\\
&a_{m,n} = 3(m+1)a_{m+1,n-1} + \tfrac{16}{3}a_{m-2,n+1} - \tfrac{1}{3}(2m+3n-1)(4m+6n-1)a_{m-1,n}.
\end{aligned}$$

This recursion relation implies $a_{m,n}\in\mathbb{Z}[\frac{1}{3}]$ for all (m,n). In other words, the power series expansion of $\sigma_{\rm w}(u)$ with respect to u is of *Hurwitz integral* (see the definition in the next page) over $\mathbb{Z}[\frac{1}{3}, \frac{g_2}{2}, 2g_3]$. Actually, computing first several coefficients by using his recursion, one finds that $\frac{1}{3}$ seems to be unnecessary, namely,

$$a_{m,n}\in\mathbb{Z}.$$

However, it is difficult to remove $\frac{1}{3}$ by using his recursion relation. This was pointed out by V. Buchstaber to the author.

On the other hand, the most general form defining any elliptic curve is

$$y^2 + (\mu_1 x + \mu_3)y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6.$$

Here the coefficients μ_js are assumed to be constant complex numbers, but they can be replaced by indeterminates at the final stage of this paper. We denote the elliptic curve defined by the equation above by $\mathscr{E}$. Since Weierstrass theory seems to ignore on the places at 2 and 3 of base rings, applying his theory on such places becomes often quite complicated calculation. In this paper, we naturally redefine the Weierstrass sigma function, which is rather *directly associated with the elliptic curve $\mathscr{E}$ itself*, and not with the function $\wp_{\rm w}(u)$ for g_2, g_3 of $\mathscr{E}$.

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While exact relation between the two sigma functions is described in the last Section, the new $\sigma(u)$ coincides with $\sigma_{\rm w}(u)$ if and only if ${\mu_1}^2+4\mu_2=0$, where, in that case, the g_2 and g_3 are given by

$$g_2=-(2\mu_3\mu_1+4\mu_4),\quad g_3=-({\mu_3}^2+4\mu_6).$$

The main result of this paper is to show the power series expansion of the new $\sigma(u)$ is of Hurwitz integral. More precisely, we show that the power series expansion of the square $\sigma(u)^2$ with respect to u is of Hurwitz integral over the ring $\mathbb{Z}[\mu_1,\mu_2,\mu_3,\mu_4,\mu_6]$, and that $\sigma(u)$ itself is of Hurwitz integral over $\mathbb{Z}[\frac{\mu_1}{2},\mu_2,\mu_3,\mu_4,\mu_6]$ (see Theorem 2.29). Especially, using our result, we can easily show that the power series expansion of $\sigma_{\rm w}(u)$ is of Hurwitz integral over $\mathbb{Z}[\frac{g_2}{2},2g_3]$ (see Theorem 6.5).

The key of our result is the relation (see Lemma 2.31) between the sigma function and the *fundamental 2-form of Klein* (defined by (1.34)). In the other part of this paper, we treat only formal power series.

The method used in this paper seems not to be applicable for higher genus case. For higher genus sigma functions, we can prove similar result by using Nakayashiki's result [4]. However, since such general method is quite a big tool, the author decided to describe this paper only by elementary method.

As an application of the obtained power series expansion, we give first several terms of n-plication polynomial of $\mathscr{E}$.

Our results might closely relate with the papers [1] and [2] by Mazur, Stein and Tate. In [2], "p-adic sigma function" is defined only over a local field whose residue field is of characteristic p not 2 and is replaced by the square of the sigma function if the defining field has residue field of characteristic 2.

Conventions. As usual we denote by $\mathbb{Z}$ the ring of rational integers, and by $\mathbb{C}$ the field of complex numbers. For an integral domain A and indeterminates z_1, $\cdots$, z_n, we denote by $A[[z_1,\cdots,z_n]]$ the ring of formal power series of z_1, $\cdots$, z_n with coefficients in A. A formal power series of the form

$$\sum_{z_1\geqq 0,\cdots,z_n\geqq 0} a_{k_1,\cdots,k_n}\frac{{z_1}^{k_1}\cdots{z_n}^{k_n}}{k_1!\cdots k_n!}\quad (a_{k_1,\cdots,k_n}\in A) \tag{0.1}$$

is said to be of *Hurwitz integral* over A with respect to z_1, $\cdots$, z_n. We denote by

$$A\langle\langle z_1,\cdots,z_n\rangle\rangle \tag{0.2}$$

the ring of Hurwitz integral series over A with respect to z_1, $\cdots$, z_n.

1 The Fundamental Differential Form

1.1 The most general elliptic curve

Let $\mathscr{E}$ be the elliptic curve defined by

$$y^2 + (\mu_1 x + \mu_3)y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6, \tag{1.1}$$

where the coefficients μ_js are complex numbers such that this equation defines a non-singular curve. Once our main result is established, these coefficients can be regarded as indeterminates. In the sequel, we use the notations

$$\begin{aligned} f(x,y) &= y^2 + (\mu_1 x + \mu_3)y - (x^3 + \mu_2 x^2 + \mu_4 x + \mu_6), \\ f_x(x,y) &= \tfrac{\partial}{\partial x} f(x,y) = \mu_1 y - (3x^2 + 2\mu_2 x + \mu_4), \\ f_y(x,y) &= \tfrac{\partial}{\partial y} f(x,y) = 2y + (\mu_1 x + \mu_3). \end{aligned} \tag{1.2}$$

We use

$$t = -x/y \tag{1.3}$$

as a local parameter at the point ∞ at infinity on $\mathscr{E}$, which is called the *arithmetic parameter* of $\mathscr{E}$.

Since we consider several kinds of variables for a function or formal power series, we must distinguish them clearly. Especially, if we consider a function or a power series with variable t, we denote the value of it at t by writing $\langle t\rangle$. For example, the x-coordinate of $\mathscr{E}$ is written as $x\langle t\rangle$. We regards the coordinate x as a function of another variable u later, which will be denoted by $x(u)$.

Using new function,

$$s = 1/x, \tag{1.4}$$

the equation $f(x,y) = 0$ is changed to

$$s = (1 + \mu_2 s + \mu_4 s^2 + \mu_6 s^3)t^2 + (\mu_1 s + \mu_3 s^2)t. \tag{1.5}$$

Using this recursively, we have

$$\begin{aligned} s = t^2 + \mu_1 t^3 + ({\mu_1}^2 + \mu_2)t^4 + ({\mu_1}^3 + 2\mu_2\mu_1 + \mu_3)t^5 + \\ ({\mu_1}^4 + 3\mu_2{\mu_1}^2 + 3\mu_3\mu_1 + {\mu_2}^2 + \mu_4)t^6 + \cdots . \end{aligned} \tag{1.6}$$

By (1.6), we have

$$\begin{aligned} x\langle t\rangle &= t^{-2} - \mu_1 t^{-1} - \mu_2 - \mu_3 t - (\mu_3\mu_1 + \mu_4)t^2 - (\mu_3{\mu_1}^2 + \mu_4\mu_1 + \mu_2\mu_3)t^3 + \cdots, \\ y\langle t\rangle &= -t^{-3} + \mu_1 t^{-2} + \mu_2 t^{-1} + \mu_3 + (\mu_3\mu_1 + \mu_4)t + (\mu_3{\mu_1}^2 + \mu_4\mu_1 + \mu_2\mu_3)t^2 + \cdots . \end{aligned} \tag{1.7}$$

Here we note that all the coefficients belong to $\mathbb{Z}[\boldsymbol{\mu}]$. Let define a weight that is denoted by wt by setting

$$\mathrm{wt}(x) = -2, \quad \mathrm{wt}(y) = -3, \quad \mathrm{wt}(\mu_j) = -j. \tag{1.8}$$

Then all the equations in this paper are of homogeneous weight.

The space of holomorphic one forms, that is differential forms of 1st kind, on $\mathscr{E}$ is spanned by

$$\omega_1(x,y) = \frac{dx}{f_y(x,y)} = \frac{dx}{2y+\mu_1 x+\mu_3}. \tag{1.9}$$

Then we see

$$\begin{aligned} \omega_1(x,y) &= \frac{dx}{2y+\mu_1 x+\mu_3} = \frac{\frac{dx}{dt}dt}{2y+\mu_1 x+\mu_3} \in (1+t\,\mathbb{Z}[\tfrac{1}{2},\boldsymbol{\mu}][[t]])dt, \\ \omega_1(x,y) &= -\frac{dy}{f_x(x,y)} \in (1+t\,\mathbb{Z}[\tfrac{1}{3},\boldsymbol{\mu}][[t]])dt, \end{aligned} \tag{1.10}$$

so that

$$\begin{aligned} \omega_1(x,y) &= (1+\mu_1 t+(\mu_2+{\mu_1}^2)t^2+(2\mu_1\mu_2+2\mu_3+{\mu_1}^3)t^3+\cdots)dt \\ &\in (1+t\,\mathbb{Z}[\boldsymbol{\mu}][[t]])dt. \end{aligned} \tag{1.11}$$

1.2 The fundamental 2-form

For two variable points (x,y) and (z,w) on $\mathscr{E}$, we define

$$\Omega(x,y,z,w) = \frac{y+w+\mu_1 z+\mu_3}{x-z}\,\omega_1(x,y) = \frac{(y+w+\mu_1 z+\mu_3)dx}{(x-z)(2y+\mu_1 x+\mu_3)}. \tag{1.12}$$

This has a pole of order 1 with residue 1 at (z,w) regarding as a form with variable (x,y) and (z,w) fixed. Indeed, since $(2w+\mu_1 z+\mu_3) = f_y(z,w)$ when $(x,y)=(z,w)$, the residue at (z,w) is 1, and the zeroes of numerator and denominator at $(x,y)=(z,-w-\mu_1 z-\mu_3)$ is canceled. We denote by t', namely by writing $'$, the value such that $x\langle t'\rangle = x\langle t\rangle$ different from t. Then because of $y\langle t\rangle + y\langle t'\rangle = -(\mu_1 x\langle t\rangle+\mu_3)$, we see

$$\begin{aligned} t' &= -\frac{x\langle t'\rangle}{y\langle t'\rangle} = \frac{x\langle t\rangle}{y\langle t\rangle+\mu_1 x\langle t\rangle+\mu_3} \\ &= -t-\mu_1 t^2-{\mu_1}^2 t^3+(-{\mu_1}^3-\mu_3)t^4+(-{\mu_1}^4-3\mu_3\mu_1)t^5+\cdots \in t\,\mathbb{Z}[\mu_1,\mu_3][[t]]. \end{aligned} \tag{1.13}$$

By the first equality, we have

$$\begin{aligned} tt' &= \frac{x\langle t\rangle^2}{(y\langle t\rangle+\mu_1 x\langle t\rangle+\mu_3)y\langle t\rangle} = \frac{x\langle t\rangle^2}{x\langle t\rangle^3+\mu_2 x\langle t\rangle^2+\mu_4 x\langle t\rangle+\mu_6} \\ &= \frac{1}{x\langle t\rangle(1+\mu_2\frac{1}{x\langle t\rangle}+\mu_4\frac{1}{x\langle t\rangle^2}+\mu_6\frac{1}{x\langle t\rangle^3})} \\ &= \tfrac{1}{x\langle t\rangle}(1-\mu_2\tfrac{1}{x\langle t\rangle}+\cdots). \end{aligned} \tag{1.14}$$

Hence,

$$\tfrac{1}{x\langle t\rangle} = tt'+\mu_2(tt')^2+\cdots \in tt'\,\mathbb{Z}[\boldsymbol{\mu}][[(tt')]]. \tag{1.15}$$

If we denote $x_1 = x\langle t_1\rangle$, $y_1 = y\langle t_1\rangle$, et cetera, then Weierstrass preparation theorem (see Corollary 5.7 in the last section) implies that, in $\mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2]]$,

$$x_2{}^{-1} - x_1{}^{-1} = -(t_1 - t_2)(t_1 - t_2{}')\, p(t_1, t_2), \tag{1.16}$$

where (by executing an explicit calculation also)

$$\begin{aligned} p(t_1, t_2) &= 1 + \mu_1 t_1 + \mu_2 t_2{}^2 + (\mu_2 + \mu_1{}^2) t_1{}^2 + \mu_1\mu_2 t_2{}^3 + \cdots \\ &\in x_1{}^{-1}/t_1{}^2 + t_2\mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2]]. \end{aligned} \tag{1.17}$$

The last equality is checked by setting $t_2 = 0$. Moreover, we see

$$\begin{aligned} y_1 + y_2 + \mu_1 x_2 + \mu_3 &= -\frac{x\langle t_1\rangle}{t_1} + \frac{x\langle t_2{}'\rangle}{t_2{}'} \\ &= -\frac{x\langle t_1\rangle}{t_1} + \frac{x\langle t_2\rangle}{t_1} - \frac{x\langle t_2\rangle}{t_1} + \frac{x\langle t_2{}'\rangle}{t_2{}'} \\ &= -\frac{1}{t_1}(x\langle t_1\rangle - x\langle t_2\rangle) - x\langle t_2\rangle\Big(\frac{1}{t_1} - \frac{1}{t_2{}'}\Big) \end{aligned} \tag{1.18}$$

and

$$\begin{aligned} &x_2\Big(\frac{1}{t_1} - \frac{1}{t_2{}'}\Big)\frac{1}{x_2 - x_1} = \frac{-x_1{}^{-1}}{x_2{}^{-1} - x_1{}^{-1}}\Big(\frac{1}{t_1} - \frac{1}{t_2{}'}\Big) \\ &= \frac{-x_1{}^{-1}}{(t_2 - t_1)(t_2{}' - t_1)\, p(t_1, t_2)}\Big(\frac{1}{t_1} - \frac{1}{t_2{}'}\Big) \\ &= \frac{-x_1{}^{-1}}{(t_2 - t_1)(t_2{}' - t_1)\,(x_1{}^{-1}/t_1{}^2 + \text{"a series in } t_2\,\mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2]]\text{"})}\frac{t_2{}' - t_1}{t_1 t_2{}'} \\ &= \frac{-x_1{}^{-1}}{(t_2 - t_1)t_1 t_2{}'\,(x_1{}^{-1}/t_1{}^2 + \text{"a series in } t_2\,\mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2]]\text{"})} \\ &= \frac{t_1}{(t_2 - t_1)t_2}\cdot\frac{t_2}{t_2{}'}\cdot\frac{-x_1{}^{-1}/t_1{}^2}{(x_1{}^{-1}/t_1{}^2 + \text{"a series in } t_2\,\mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2]]\text{"})}. \end{aligned} \tag{1.19}$$

Here we note that

$$t_2{}'/t_2 \in -1 + t_2\,\mathbb{Z}[\mu_1, \mu_3][[t_2]]. \tag{1.20}$$

At the last part in (1.19), since $x_1{}^{-1}/t_1{}^2 \in 1 + t_1\mathbb{Z}[\boldsymbol{\mu}][[t_1]]$, we have

$$\begin{aligned} x_2\Big(\frac{1}{t_1} - \frac{1}{t_2{}'}\Big)\frac{1}{x_2 - x_1} &= -\Big(\frac{1}{t_2 - t_1} - \frac{1}{t_2}\Big)(\text{"a series in } 1 + t_2\,\mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2]]\text{"}) \\ &= \frac{1}{t_2} - \frac{1}{t_2 - t_1}(\text{"a series in } 1 + t_2\,\mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2]]\text{"}) \\ &\qquad + (\text{"a series in } \mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2]]\text{"}). \end{aligned} \tag{1.21}$$

Therefore,

$$\begin{aligned} \frac{y_1 + y_2 + \mu_1 x_2 + \mu_3}{x_2 - x_1} &= \frac{1}{t_1} - \frac{x_2}{x_2 - x_1}\Big(\frac{1}{t_1} - \frac{1}{t_2{}'}\Big) \\ &= \frac{1}{t_1} - \frac{1}{t_2} + (\text{"a series in } \mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2]]\text{"}) \\ &\qquad + \frac{1}{t_2 - t_1}(\text{"a series in } 1 + t_2\mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2]]\text{"}). \end{aligned} \tag{1.22}$$

Now let $b\langle t_1,t_2\rangle \in \mathbb{Z}[\boldsymbol{\mu}][[t_1,t_2]]$ be a series satisfying

$$\begin{aligned}(1.23)\qquad &\Big(\frac{y_1+y_2+\mu_1x_2+\mu_3}{x_1-x_2}-\frac{1}{t_1}+\frac{1}{t_2}\Big)\omega_1\langle t_1\rangle \\ &=\mathbb{Z}[\boldsymbol{\mu}][[t_1,t_2]]\omega_1\langle t_1\rangle-\frac{1}{t_2-t_1}b\langle t_1,t_2\rangle dt_1.\end{aligned}$$

Then

$$\begin{aligned}(1.24)\qquad \lim_{t_2\to t_1}\frac{x_1-x_2}{y_1+y_2+\mu_1x_2+\mu_3}\frac{1}{t_2-t_1}&=\frac{\frac{dx}{dt}\langle t_1\rangle}{2y_1+\mu_1x_1+\mu_3}\\ &=\omega_1\langle t_1\rangle/dt_1.\end{aligned}$$

So that

$$(1.25)\qquad b\langle t_1,t_1\rangle=1.$$

Hence

$$(1.26)\qquad b\langle t_1,t_2\rangle\in 1+(t_2-t_1)\mathbb{Z}[\boldsymbol{\mu}][[t_1,t_2]].$$

We describe this as a theorem:

Theorem 1.27. *One has*

$$(1.28)\qquad \Big(\frac{y_1+y_2+\mu_1x_2+\mu_3}{x_2-x_1}-\frac{1}{t_1}+\frac{1}{t_2}\Big)\,\omega_1\langle t_1\rangle+\frac{dt_1}{t_1-t_2}\in\mathbb{Z}[\boldsymbol{\mu}][[t_1,t_2]]dt_1.$$

An explicit calculation shows that

$$\begin{aligned}(1.29)\qquad &\Big(\frac{y_1+y_2+\mu_1x_2+\mu_3}{x_2-x_1}-\frac{1}{t_1}+\frac{1}{t_2}\Big)\,\omega_1\langle t_1\rangle+\frac{dt_1}{t_1-t_2}\\ =&\Big(-\mu_2t_1-\mu_3t_2t_1\\ &-(\mu_2\mu_1+2\mu_3){t_1}^2\\ &-(2\mu_3\mu_1+\mu_4)t_2{t_1}^2\\ &-(\mu_3\mu_1+\mu_4){t_2}^2t_1\\ &-(\mu_2{\mu_1}^2+4\mu_3\mu_1+{\mu_2}^2+2\mu_4){t_1}^3\\ &-(\mu_3{\mu_1}^2+\mu_4\mu_1+\mu_2\mu_3){t_2}^3t_1\\ &-(2\mu_3{\mu_1}^2+2\mu_4\mu_1+\mu_2\mu_3){t_2}^2{t_1}^2\\ &-(3\mu_3{\mu_1}^2+2\mu_4\mu_1+2\mu_2\mu_3)t_2{t_1}^3\\ &-(\mu_2{\mu_1}^3+6\mu_3{\mu_1}^2+2{\mu_2}^2\mu_1+4\mu_4\mu_1+6\mu_2\mu_3){t_1}^4-\cdots\Big)dt_1.\end{aligned}$$

By using Ω in (1.12), we define

$$(1.30)\qquad \boldsymbol{\xi}(x,y;z,w)=\tfrac{d}{dz}\Omega(x,y;z,w)dz-\omega_1(x,y)\eta_1(z,w).$$

Then, it is easy to check that the differential form of the 3rd kind

$$(1.31)\qquad \eta_1(x,y)=\frac{-xdx}{2y+\mu_1x+\mu_3}$$

is a unique form up to addition of a constant times ω_1 that satisfies

$$\boldsymbol{\xi}(x,y;z,w)=\boldsymbol{\xi}(z,w;x,y). \tag{1.32}$$

Its expansion with respect to t is given by

$$\begin{aligned}\eta_1(x,y) &= -t^{-2}-\mu_3 t-(\mu_4+2\mu_1\mu_3)t^2-(2\mu_1\mu_4+2\mu_3\mu_2+3{\mu_1}^2\mu_3)t^3-\cdots\\ &\in -t^{-2}+t\,\mathbb{Z}[\boldsymbol{\mu}][[t]].\end{aligned} \tag{1.33}$$

Under the situation above, $\boldsymbol{\xi}$ in (1.30) is given by

$$\boldsymbol{\xi}=\frac{F(x,y;z,w)dxdz}{(x-z)^2 f_y(x,y)f_y(z,w)}, \tag{1.34}$$

where

$$\begin{aligned}F(x,y;z,w) &= xz(x+z)+({\mu_1}^2+2\mu_2)xz+\mu_1(zy+xw)\\ &\quad+(\mu_3\mu_1+\mu_4)(x+z)+2yw+\mu_3(y+w)+{\mu_3}^2+2\mu_6.\end{aligned} \tag{1.35}$$

1.3 The lattice of periods and Legendre relation

Let the pair of α and β be a basis of the fundamental group of $\mathscr{E}$, and define

$$\omega'=\int_\alpha \omega_1(x,y),\quad \omega''=\int_\beta \omega_1(x,y), \tag{1.36}$$

We define

$$\Lambda=\mathbb{Z}\omega'+\mathbb{Z}\omega''. \tag{1.37}$$

This is a lattice in $\mathbb{C}$. We define also

$$\eta'=\int_\alpha \eta_1(x,y),\quad \eta''=\int_\beta \eta_1(x,y). \tag{1.38}$$

Then the Legendre relation

$$\omega''\eta'-\omega'\eta''=2\pi\boldsymbol{i} \tag{1.39}$$

holds.

The space of differential forms of the 1st and the 2nd kinds with at most poles only at ∞ modulo the space of exact forms is naturally isomorphic to the 1st cohomology group $H^1(\mathscr{E},\mathbb{C})$. The intersection form on $H_1(\mathscr{E},\mathbb{Z})$ naturally induces an intersection form on $H^1(\mathscr{E},\mathbb{C})$. The pair of two forms ω_1 and η_1 is a symplectic basis of $H^1(\mathscr{E},\mathbb{C})$ with respect to the induced intersection form[1]. The relation (1.39) is direct consequence from this fact.

[1] The author learned this fact from A.Nakayashiki.

2 The sigma function

2.1 Construction of the sigma function

Here we construct $\sigma(u)$ by a similar method as described in [8], pp.447-449.

We start at the following equation on $\mathbb{C}$:

$$u = \int_{\infty}^{(x,y)} \omega_1, \tag{2.1}$$

where (x,y) is a variable point on $\mathscr{E}$. Regarding the coordinates x and y as functions of u on $\mathbb{C}$ and are denoted by

$$u \mapsto x(u), \quad u \mapsto y(u). \tag{2.2}$$

By this definition, we have

$$x(-u) = x(u), \quad y(-u) = y(u) + \mu_1 x(u) + \mu_3 \tag{2.3}$$

for $\ell \in \Lambda$. Both of them has a pole only at $u=0$ and expanded as

$$\begin{aligned} x(u) &= u^{-2} - (\tfrac{1}{12}{\mu_1}^2 + \tfrac{1}{3}\mu_2) + (\tfrac{1}{240}{\mu_1}^4 + \tfrac{1}{30}\mu_2{\mu_1}^2 - \tfrac{1}{10}\mu_3\mu_1 + \tfrac{1}{15}{\mu_2}^2 - \tfrac{1}{5}\mu_4)u^2 + \cdots, \\ y(u) &= -u^{-3} - \tfrac{1}{2}\mu_1 u^{-2} + (\tfrac{1}{24}{\mu_1}^3 + \tfrac{1}{6}\mu_2\mu_1 - \tfrac{1}{2}\mu_3) + \cdots. \end{aligned} \tag{2.4}$$

The variable u is of weight 1 : $\mathrm{wt}(u) = 1$.

For the variable $u \in \mathbb{C}$ and arbitrarily fixed $u^{(0)} \in \mathbb{C} - \Lambda$, there exists a function $u \mapsto \zeta(u)$ satisfying

$$\int_{(x^{(0)},y^{(0)})}^{(x,y)} \eta_1 = \zeta(u) - \zeta(u^{(0)}). \tag{2.5}$$

Indeed, the derivative of the left hand side with respect to u is

$$\eta_1(x(u), y(u))\tfrac{dx(u)}{du} = x(u). \tag{2.6}$$

So that we can take $\zeta(u)$ as an integral of $x(u)$ with respect to u. Note that $\zeta(u)$ is none other than Weierstrass zeta function if $\mu_1 = \mu_2 = \mu_3 = 0$. Here we fix $\zeta(u)$ by the formal integral $u^n \mapsto \frac{1}{n+1}u^{n+1}$ $(n \neq -1)$ without constant term :

$$\zeta(u) = \int_{\text{formal}} x(u)du = u^{-1} - (\tfrac{1}{12}{\mu_1}^2 + \tfrac{1}{3}\mu_2)u + \cdots. \tag{2.7}$$

As a Laurent series on $\mathbb{C}$, this has positive radius of convergence, and has meromorphic continuation to the whole complex plane. For any $u \in \mathbb{C}$, there is unique pair of u' and $u'' \in \mathbb{R}$ satisfying $u = u'\omega' + u''\omega''$. Especially, for any lattice point $\ell \in \Lambda$, we denote as $\ell = \ell'\omega' + \ell''\omega''$. We define

$$L(u,v) = u\,(v'\eta' + v''\eta'') \tag{2.8}$$

for two variables u and $v \in \mathbb{C}$.

The left hand side of (2.5) increases $\ell'\eta'+\ell''\eta''$ that is the period of η_1 when the path of the integral is added by an amount corresponding to $\ell\in\Lambda$, namely,

$$\zeta(u+\ell)=\zeta(u)+\ell'\eta'+\ell''\eta''. \tag{2.9}$$

After integrating $\zeta(u)$, by taking exponential, we get the *sigma function* $\sigma(u)$. In other words, the sigma function is defined by

$$-\frac{d}{du}\log\sigma(u)=\zeta(u) \tag{2.10}$$

up to none zero multiplicative constant. This yields that $\sigma(u)=0$ if and only if $\zeta(u)$ has a pole, namely, $u\in\Lambda$. To fix the multiplicative constant, we suppose that its power series expansion at the origin is of the form

$$\sigma(u)=u+O(u^2). \tag{2.11}$$

By (2.3), the sigma function is an odd function :

$$\sigma(-u)=-\sigma(u). \tag{2.12}$$

From (2.9), we see

$$\sigma(u+\ell)=c(\ell)\sigma(u)\exp\big(u(\ell'\eta'+\ell''\eta'')\big), \tag{2.13}$$

where $c(\ell)$ is a constant depending on ℓ. In this situation, for $\ell\in\Lambda$, $\notin 2\Lambda$, we see that $\sigma(\frac{1}{2}\ell)\neq 0$ and

$$\sigma(\tfrac{1}{2}\ell)=-c(\ell)\sigma(\tfrac{1}{2}\ell)\exp\big(-\tfrac{1}{2}\,\ell(\ell'\eta'+\ell''\eta'')\big). \tag{2.14}$$

Hence, using notation of (2.8)

$$c(\ell)=-\exp\big(\tfrac{1}{2}\ell(\ell'\eta'+\ell''\eta'')\big)=-\exp L(\tfrac{1}{2}\ell,\ell). \tag{2.15}$$

Therefore

$$\sigma(u+\ell)=-\sigma(u)\exp L(u+\tfrac{1}{2}\ell,\ell)\quad(\ell\in\Lambda,\ \notin 2\Lambda). \tag{2.16}$$

If $\ell\in 2\Lambda$, after derivating both sides of (2.13) by u, the similar calculation shows that

$$\sigma(u+\ell)=\sigma(u)\exp L(u+\tfrac{1}{2}\ell,\ell)\quad(\ell\in 2\Lambda) \tag{2.17}$$

because $\sigma(-\frac{1}{2}\ell)=0$. In order to unify these two cases, we define[2]

$$\chi(\ell)=\exp\big[2\pi\boldsymbol{i}\big(\tfrac{1}{2}\ell'+\tfrac{1}{2}\ell''+\tfrac{1}{2}\ell'\ell''\big)\big]. \tag{2.18}$$

Then we have

$$\sigma(u+\ell)=\chi(\ell)\sigma(u)\exp L(u+\tfrac{1}{2}\ell,\ell)\quad(\ell\in\Lambda). \tag{2.19}$$

[2] Note that χ is not a character on Λ.

2.2 Solution to Jacobi's inversion problem

We define[3]

$$\wp(u) = -\tfrac{d^2}{du^2}\log\sigma(u). \tag{2.20}$$

The previous discussion implies that

$$\wp(u) = x(u), \tag{2.21}$$

and this function is expanded as

$$\wp(u) = \frac{1}{u^2} + \sum_{\ell\in\Lambda,\ell\neq 0}\Big(\frac{1}{(u-\ell)^2} - \frac{1}{\ell^2}\Big) - \frac{{\mu_1}^2 + 4\mu_2}{12}. \tag{2.22}$$

This expansion is shown by (2.4) and the situation of zeroes of $\sigma(u)$. If $\mu_1 = \mu_2 = \mu_3 = 0$, this function is none other than Weierstrass $\wp$-function with $g_2 = -4\mu_4$ and $g_3 = -4\mu_6$ in usual notation. Summing up our discussion, we have under the equation (2.1) that

$$\wp(u) = x,\ \ \wp'(u) = 2y + \mu_1 x + \mu_3. \tag{2.23}$$

This is the solution of what we call Jacobi's inversion problem for (2.1).

2.3 Another construction of the sigma function

Now let us construct the sigma function by using a theta series. Firstly, we define

$$R = \gcd\Big(\mathrm{rslt}_x\big(\mathrm{rslt}_y(f, f_y), \mathrm{rslt}_y(f, f_y)\big), \mathrm{rslt}_y\big(\mathrm{rslt}_x(f, f_y), \mathrm{rslt}_x(f, f_y)\big)\Big), \tag{2.24}$$

Here x and y are regarded as indeterminates and rslt means Silvester's resultant with respect to the suffixed variable. Then R is a square element in $\mathbb{Z}[\boldsymbol{\mu}]$. We take $D = R^{1/2}$, a square root of this. D is explicitly written as

$$D = -{b_2}^2 b_8 - 8{b_4}^3 - 27{b_6}^2 + 9b_2b_4b_6, \tag{2.25}$$

where

$$\begin{aligned} b_2 &= {\mu_1}^2 + 4\mu_2, \quad b_4 = 2\mu_4 + \mu_1\mu_3, \quad b_6 = {\mu_3}^2 + 4\mu_6, \\ b_8 &= {\mu_1}^2\mu_6 + 4\mu_2\mu_6 - \mu_1\mu_3\mu_4 + \mu_2{\mu_3}^2 - {\mu_4}^2. \end{aligned} \tag{2.26}$$

Using the standard notation of theta series, the *sigma function* is analytically defined by

$$\sigma(u) = D^{-1/8}\Big(\frac{\pi}{\omega'}\Big)^{1/2}\exp\big(-\tfrac{1}{2}u^2\eta'{\omega'}^{-1}\big)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}({\omega'}^{-1}u\big|{\omega'}^{-1}\omega''). \tag{2.27}$$

Here we fix the 8th root of D and square root of the second factor by the condition that the power series expansion of $\sigma(u)$ is as (2.11).

[3] This is a proposal of redefinition of Weierstrass $\wp$-function from the view point of the theory Abelian functions.

Remark 2.28. (1) Once we have defined the sigma function by (2.27), we should show that it does not depend on the choice of a basis α, β of the fundamental group of $\mathscr{E}$. As is well-known, a proof of this is done by using the transformation property of Dedekind's η function (see Theorem in p.180 of [5]). From this, up to a multiplicative constant $\sigma(u)$ is invariant Jacobi form under standard action of $\mathrm{SL}_2(\mathbb{Z})$.
(2) This definition shows clearly that $\sigma(u)$ is an entire function.
These facts above yield that $\sigma(u)$ has power series whose coefficients depends on only $\{\mu_j\}$.
(3) To fix the first two factors as (2.11) holds, we need Jacobi's derivative formula ([3], p.64) and theta zero values for three even theta series due to Jacobi.
(4) Using (1.39), we can show that $\sigma(u)$ defined as above satisfies (2.19).
(5) The zeroes of the theta series used in (2.27) is well-known ([5], pp.167–168). This shows that $\sigma(u)$ has zeroes of order 1 on Λ and no other zeroes.

The following theorem is one of the main results :

Theorem 2.29. *The power series expansion of $\sigma(u)^2$ at the origin belongs to $\mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]\langle\langle u\rangle\rangle$, and one of $\sigma(u)$ belongs to $\mathbb{Z}[\frac{\mu_1}{2}, \mu_2, \mu_3, \mu_4, \mu_6]\langle\langle u\rangle\rangle$. First several terms of the expansion of $\sigma(u)$ is given by*

$$\begin{aligned}\sigma(u) = u &+ ((\tfrac{\mu_1}{2})^2 + \mu_2)\tfrac{u^3}{3!} + ((\tfrac{\mu_1}{2})^4 + 2\mu_2(\tfrac{\mu_1}{2})^2 + \mu_3\mu_1 + {\mu_2}^2 + 2\mu_4)\tfrac{u^5}{5!} \\ &+ ((\tfrac{\mu_1}{2})^6 + 3\mu_2(\tfrac{\mu_1}{2})^4 + 6\mu_3(\tfrac{\mu_1}{2})^3 + 3{\mu_2}^2(\tfrac{\mu_1}{2})^2 + 6\mu_4(\tfrac{\mu_1}{2})^2 \\ &+ 6\mu_3\mu_2\tfrac{\mu_1}{2} + {\mu_2}^3 + 6\mu_4\mu_2 + 6{\mu_3}^2 + 24\mu_6)\tfrac{u^7}{7!}. + \cdots\end{aligned} \tag{2.30}$$

The following Lemma is the key in the following discussion :

Lemma 2.31. *The sigma function and the 2-form $\boldsymbol{\xi}$ relates by*

$$\frac{\sigma\Big(\int_\infty^{(x,y)}\omega_1 - \int_\infty^{(x_1,y_1)}\omega_1\Big)\,\sigma\Big(\int_\infty^{(z,w)}\omega_1 - \int_\infty^{(z_1,w_1)}\omega_1\Big)}{\sigma\Big(\int_\infty^{(x,y)}\omega_1 - \int_\infty^{(z_1,w_1)}\omega_1\Big)\,\sigma\Big(\int_\infty^{(z,w)}\omega_1 - \int_\infty^{(x_1,y_1)}\omega_1\Big)} = \exp\Big(\int_{(z,w)}^{(x,y)}\int_{(z_1,w_1)}^{(x_1,y_1)}\boldsymbol{\xi}\Big). \tag{2.32}$$

Proof. We increase the path of integrals form ∞ to (x,y) by an amount corresponding to $\ell'{\cdot}\alpha_1 + \ell''{\cdot}\beta_1$ (ℓ', $\ell'' \in \mathbb{Z}$), and denotes the integrals by

$$\widetilde{\int_\infty^{(x,y)}}\omega, \quad \widetilde{\int_{(z,w)}^{(x,y)}}\boldsymbol{\xi}, \quad \text{et cetera.} \tag{2.33}$$

Then the left hand side is multiplied by

$$\exp\Big[L\Big(-\int_\infty^{(x_1,y_1)}\omega + \int_\infty^{(z_1,w_1)}\omega\,\ell'\omega' + \ell''\omega''\Big)\Big] \tag{2.34}$$

because of (2.19), and the part in the exponential is changed to

$$
\begin{aligned}
&\widetilde{\int}_{(z,w)}^{(x,y)}\int_{(z_1,w_1)}^{(x_1,y_1)}\boldsymbol{\xi}(X,Y;Z,W)\\
&\quad=\int_{(z_1,w_1)}^{(x_1,y_1)}\left(\Big[\Omega(X,Y;Z,W)\Big]_{(z,w)}^{(x,y)}-\omega_1(X,Y)\Big(\int_{(z,w)}^{(x,y)}\eta_1(Z,W)+\ell'\eta'+\ell''\eta''\Big)\right)\\
&\quad=\int_{(z_1,w_1)}^{(x_1,y_1)}\int_{(z,w)}^{(x,y)}\boldsymbol{\xi}(X,Y;Z,W)-\int_{(z_1,w_1)}^{(x_1,y_1)}\omega_1(X,Y)\cdot\big(\ell'\eta'+\ell''\eta''\big)\\
&\quad=\int_{(z_1,w_1)}^{(x_1,y_1)}\int_{(z,w)}^{(x,y)}\boldsymbol{\xi}(X,Y;Z,W)-L\Big(\int_{(z_1,w_1)}^{(x_1,y_1)}\omega_1(X,Y),\ \ell'\omega'+\ell''\omega''\Big)
\end{aligned}
\tag{2.35}
$$

because of (1.30) and (1.32). Therefore the transformations of the two sides are exactly the same. We can check this for other variables (z,w), (x_1,y_1), and (z_1,w_1). Moreover, both sides take value 1 when $(x_1,y_1)=(z_1,w_1)$ or $(x,y)=(z,w)$. Hence the two sides must coincide. □

Remark 2.36. Taking double derivative by u of logarithm of (2.32), we see that

$$
\wp\Big(\int_{\infty}^{(x,y)}\omega_1-\int_{\infty}^{(z,w)}\omega_1\Big)=\frac{F(x,y;z,w)}{(x-z)^2}.
\tag{2.37}
$$

This relation would be helpful to understand where the 2-form $\boldsymbol{\xi}$ comes from.

2.4 Frobenius-Stickelberger formula

Lemma 2.38. *One has*

$$
\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma(u)^2\sigma(v)^2}=x(u)-x(v).
\tag{2.39}
$$

Proof. By (2.19), we see both sides are periodic function in u and v with respect to the lattice Λ. On the other hand, since $\sigma(u)$ has poles of order 1 at $u\in\Lambda$, the divisors of the two sides coincide. Because both sides are expanded with respect to u as $1/u^2+\cdots$, the equality holds. □

3 Hurwitz Integrality

Derivating

$$f(x\langle t\rangle, y) = (y - y\langle t\rangle)(y - y\langle t'\rangle), \tag{3.1}$$

by y, we have

$$\begin{aligned} f_y(x\langle t\rangle, y) &= (y - y\langle t\rangle) + (y - y\langle t'\rangle) \\ \therefore\ f_y(x\langle t\rangle, y\langle t\rangle) &= y\langle t\rangle - y\langle t'\rangle. \end{aligned} \tag{3.2}$$

This yields that

$$f_y(x\langle t\rangle, y\langle t\rangle) = \frac{1}{(tt')^3}(t - t')(t^2 + \text{“higher terms in } \mathbb{Z}[\boldsymbol{\mu}][[t]]\text{”}) \tag{3.3}$$

We are going to prove Theorem 2.29. If

$$u = \int_{\infty}^{(x,y)} \omega_1, \tag{3.4}$$

(1.11) shows that

$$u = t + \text{“higher terms”} \in \mathbb{Z}[\boldsymbol{\mu}]\langle\langle t\rangle\rangle, \qquad t = u + \text{“higher terms”} \in \mathbb{Z}[\boldsymbol{\mu}]\langle\langle u\rangle\rangle. \tag{3.5}$$

Using (1.28), (1.30), (1.33), and (1.11), we have

$$\boldsymbol{\xi}\langle t_1, t_2\rangle - \frac{dt_1 dt_2}{(t_1 - t_2)^2} \in \mathbb{Z}[\boldsymbol{\mu}][t_1, t_2]dt_1 dt_2. \tag{3.6}$$

Explicit calculation gives that

$$\begin{aligned} \boldsymbol{\xi}\langle t_1, t_2\rangle = \Big(& \frac{1}{(t_1 - t_2)^2} + \mu_3(t_1 + t_2) + (3\mu_3\mu_1 + 2\mu_4)t_1 t_2 \\ & + (2\mu_3\mu_1 + \mu_4)({t_1}^2 + {t_2}^2) \\ & + (5\mu_3{\mu_1}^2 + 4\mu_4\mu_1 + 3\mu_2\mu_3)({t_1}^2 t_2 + t_1{t_2}^2) \\ & + (3\mu_3{\mu_1}^2 + 2\mu_4\mu_1 + 2\mu_2\mu_3)({t_1}^3 + {t_2}^3) \\ & + (8\mu_3{\mu_1}^3 + 7\mu_4{\mu_1}^2 + 11\mu_2\mu_3\mu_1 + 3{\mu_3}^2 + 4\mu_4\mu_2 + 3\mu_6){t_1}^2{t_2}^2 \\ & + (7\mu_3{\mu_1}^3 + 6\mu_4{\mu_1}^2 + 10\mu_2\mu_3\mu_1 + 4{\mu_3}^2 + 4\mu_4\mu_2 + 4\mu_6)({t_1}^3 t_2 + t_1{t_2}^3) \\ & + (4\mu_3{\mu_1}^3 + 3\mu_4{\mu_1}^2 + 6\mu_2\mu_3\mu_1 + 3{\mu_3}^2 + 2\mu_4\mu_2 + 2\mu_6)({t_1}^4 + {t_2}^4) \\ & + \cdots \Big) dt_1 dt_2. \end{aligned} \tag{3.7}$$

Then we have

$$\begin{aligned} & \int_{{t_1}'}^{{t_2}'} \int_{t_1}^{t_2} \boldsymbol{\xi}\langle T_1, T_2\rangle \\ & = -\log\Big(-\frac{({t_2}' - t_1)(t_2 - {t_1}')}{({t_2}' - t_2)(t_1 - {t_1}')}\Big) + \frac{\mu_3}{2}\big(({t_2}'^2 - {t_1}'^2)(t_2 - t_1) + ({t_2}^2 - {t_1}^2)({t_2}' - {t_1}')\big) \\ & \quad + \text{“a series in } \mathbb{Z}[\boldsymbol{\mu}]\langle\langle t_1, t_2\rangle\rangle \text{ of total degree} \geqq 4\text{”}. \end{aligned} \tag{3.8}$$

Recall (1.16) :

$$(3.9)\qquad \begin{aligned} x\langle t_2\rangle - x\langle t_1\rangle &= \frac{({t_2}' - t_1)(t_2 - t_1)\, p(t_1,t_2)}{x\langle t_1\rangle^{-1} x\langle t_2\rangle^{-1}},\\ p(t_1,t_2) &= 1 + \mu_1 t_1 + \mu_2 {t_2}^2 + (\mu_2 + {\mu_1}^2){t_1}^2 + \mu_1\mu_2 {t_2}^3 + \cdots . \end{aligned}$$

Exchanging t_1 and t_2, we have

$$(3.10)\qquad \begin{aligned} x\langle t_2\rangle - x\langle t_1\rangle &= -\frac{x\langle t_2\rangle^{-1} - x\langle t_1\rangle^{-1}}{x\langle t_1\rangle^{-1} x\langle t_2\rangle^{-1}}\\ &= -\frac{({t_1}' - t_2)(t_1 - t_2)\, p(t_2,t_1)}{x\langle t_1\rangle^{-1} x\langle t_2\rangle^{-1}}. \end{aligned}$$

Let u and v be analytic coordinates corresponding to t_1 and t_2, respectively. Dividing both sides of (2.39) by $u - v$ and $\frac{dx}{du} = 1/f_y(x,y)$ yield that

$$(3.11)\qquad \frac{\sigma(2u)}{\sigma(u)^4} = \frac{d}{du}x(u) = 1\Big/\frac{du}{dx} = f_y(x(u), y(u)).$$

So that

$$(3.12)\qquad \begin{aligned} \frac{\sigma(2u)}{\sigma(u)^4} &= f_y\langle t\rangle = f_y(x\langle t\rangle, y\langle t\rangle)\\ &= y\langle t\rangle - y\langle t'\rangle\\ &= -\frac{y\langle t'\rangle^{-1} - y\langle t\rangle^{-1}}{y\langle t\rangle^{-1} y\langle t'\rangle^{-1}}\\ &= -\frac{x\langle t\rangle}{t} + \frac{x\langle t\rangle}{t'} = (t - t')\,\frac{x\langle t\rangle}{tt'}. \end{aligned}$$

Using these, we see that

$$(3.13)\qquad \begin{aligned} \big(x(u) - x(v)\big)^2 &= \left(\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2}\right)^2 \quad (\because (2.39))\\ &= \frac{\sigma(u+v)^2}{\sigma(2u)\sigma(2v)}\frac{\sigma(2u)}{\sigma(u)^4}\frac{\sigma(2v)}{\sigma(v)^4}\sigma(u-v)^2\\ &= \exp\Big(-\int_{{t_1}'}^{{t_2}'}\int_{t_1}^{t_2} \boldsymbol{\xi}\langle T_1, T_2\rangle\Big) f_y\langle t_1\rangle f_y\langle t_2\rangle \sigma(u-v)^2 \quad (\because (2.32))\\ &= \exp\Big(-\log\frac{({t_2}' - t_2)(t_1 - {t_1}')}{({t_2}' - t_1)(t_2 - {t_1}')}\\ &\qquad + \text{“a series in } \mathbb{Z}[\boldsymbol{\mu}][[t_1,t_2]] \text{ of degree} \geqq 3\text{”}\Big) f_y\langle t_1\rangle f_y\langle t_2\rangle \sigma(u-v)^2\\ &= \left(\frac{({t_2}' - t_1)(t_2 - {t_1}')}{({t_2}' - t_2)(t_1 - {t_1}')} \times \text{“a series of the form } 1 + \cdots \text{ in } \mathbb{Z}[\boldsymbol{\mu}]\langle\langle t_1, t_2\rangle\rangle\text{”}\right)\\ &\qquad \times f_y\langle t_1\rangle f_y\langle t_2\rangle \sigma(u-v)^2. \end{aligned}$$

Summing up, we have arrived the main result as follows :

Theorem 3.14. *Let u and v be analytic coordinates corresponding values t_1 and t_2 of the arithmetic parameter (1.3), respectively. Then the sigma function is expressed as the following product of formal power series:*

$$\sigma(u-v)^2 = (t_2-t_1)^2\, q(t_1)q(t_2)\, p(t_1,t_2)\, p(t_2,t_1)\, r(t_1,t_2), \tag{3.15}$$

where

$$\begin{aligned}
p(t_1,t_2) &= \frac{x\langle t_2\rangle^{-1} - x\langle t_1\rangle^{-1}}{({t_2}'-t_1)(t_2-t_1)} = 1 + \mu_1 t_1 + \mu_2 {t_2}^2 + (\mu_2+{\mu_1}^2){t_1}^2 + \cdots \\
&\quad \in \mathbb{Z}[\boldsymbol{\mu}][[t_1,t_2]],\\
q(t) &= -x\langle t\rangle t t' = 1 - \mu_2 t^2 - \mu_2\mu_1 t^3 - (\mu_2{\mu_1}^2+\mu_4)t^4 \\
&\quad - (\mu_2{\mu_1}^3 + 2\mu_4\mu_1 + \mu_2\mu_3)t^5 + \cdots \qquad \in \mathbb{Z}[\boldsymbol{\mu}][[t]],
\end{aligned}$$

$$\begin{aligned}
r(t_1,t_2) &= \exp\left[\int_{{t_1}'}^{{t_2}'}\int_{t_1}^{t_2}\Big(\boldsymbol{\xi}\langle T_1,T_2\rangle - \frac{dT_1dT_2}{(T_2-T_1)^2}\Big)\right] \\
&= 1 - (\tfrac{1}{12}\mu_1\mu_3 + \tfrac{1}{6}\mu_4)(t_1-t_2)^4 - (\tfrac{1}{6}{\mu_1}^2\mu_3 + \tfrac{1}{3}\mu_4\mu_1)(t_1-t_2)^4(t_1+t_2) \\
&\quad + \big(-(\tfrac{1}{30}{\mu_3}^2 + (\tfrac{43}{180}{\mu_1}^3 + \tfrac{11}{90}\mu_2\mu_1)\mu_3 + \tfrac{43}{90}\mu_4{\mu_1}^2 + \tfrac{11}{45}\mu_2\mu_4 + \tfrac{2}{15}\mu_6)({t_1}^4+{t_2}^4) \\
&\quad + (\tfrac{2}{15}{\mu_3}^2 + (\tfrac{11}{90}{\mu_1}^3 + \tfrac{7}{45}\mu_2\mu_1)\mu_3 + \tfrac{11}{45}\mu_4{\mu_1}^2 + \tfrac{14}{45}\mu_2\mu_4 + \tfrac{8}{15}\mu_6)t_1t_2({t_1}^2+{t_2}^2) \\
&\quad + (-\tfrac{1}{5}{\mu_3}^2 + (\tfrac{7}{30}{\mu_1}^3 - \tfrac{1}{15}\mu_2\mu_1)\mu_3 + \tfrac{7}{15}\mu_4{\mu_1}^2 - \tfrac{2}{15}\mu_2\mu_4 + \tfrac{1}{5}\mu_6){t_1}^2{t_2}^2\big)(t_1-t_2)^2 \\
&\quad + \cdots \quad \in \mathbb{Z}[\boldsymbol{\mu}]\langle\langle t_1,t_2\rangle\rangle.
\end{aligned} \tag{3.16}$$

Since $t_1 = u + \cdots \in \mathbb{Z}[\boldsymbol{\mu}]\langle\langle u\rangle\rangle$ and $t_2 = v + \cdots \in \mathbb{Z}[\boldsymbol{\mu}]\langle\langle v\rangle\rangle$, one has

$$\sigma(u-v)^2 \in (u-v)^2(1 + \text{“higher terms in } \mathbb{Z}[\boldsymbol{\mu}]\langle\langle u,v\rangle\rangle\text{”}). \tag{3.17}$$

First several terms of the expansions above when $t_1 = t$ and $t_2 = 0$ is given as follows :

$$\begin{aligned}
\sigma\langle t\rangle^2 &= t^2 + \mu_1 t^3 + \big({\mu_1}^2+\mu_2\big)t^4 + \big(\mu_3 + {\mu_1}^3 + 2\mu_2\mu_1\big)t^5 \\
&\quad + \big(\tfrac{35}{12}\mu_1\mu_3 + {\mu_1}^4 + 3\mu_2{\mu_1}^2 + \tfrac{5}{6}\mu_4 + {\mu_2}^2\big)t^6 \\
&\quad + \big((\tfrac{23}{4}{\mu_1}^2 + 3\mu_2)\mu_3 + {\mu_1}^5 + 4\mu_2{\mu_1}^3 + (\tfrac{5}{2}\mu_4 + 3{\mu_2}^2)\mu_1\big)t^7 + \cdots,\\
\sigma\langle t\rangle &= t + \mu_1\tfrac{t^2}{2!} + \big(9(\tfrac{\mu_1}{2})^2 + 3\mu_2\big)\tfrac{t^3}{3!} + \big(12\mu_3 + 60(\tfrac{\mu_1}{2})^3 + 18\mu_2\mu_1\big)\tfrac{t^4}{4!} \\
&\quad + \big(145\mu_1\mu_3 + 525(\tfrac{\mu_1}{2})^4 + 450\mu_2(\tfrac{\mu_1}{2})^2 + 50\mu_4 + 45{\mu_2}^2\big)\tfrac{t^5}{5!} + \cdots.
\end{aligned} \tag{3.18}$$

Remark 3.19. When $t_1 = t$ and $t_0 = 0$, because of

$$q(t){\cdot}p(0,t) = -1, \quad q(0) = 1, \quad p(t,0) = -x\langle t\rangle^{-1}/t^2, \tag{3.20}$$

we have

$$\sigma\langle t\rangle^2 = x\langle t\rangle^{-1}\, r(0,t). \tag{3.21}$$

We shall compare with[4]

$$\sigma(u) = u{\cdot}\exp\bigg(\int_0^u\int_0^u\Big(\frac{1}{u^2} - \wp(u)\Big)dudu\bigg) \tag{3.22}$$

[4] This formula is mentioned also in, for example, p.589 of [1] to compute $\sigma\langle t\rangle$. Since we use the definitions (2.10) or (2.27) of $\sigma(u)$ and (2.20) of $\wp(u)$, the p-adic modular form $\boldsymbol{E}_2$ in [1] is unnecessary.

that is immediately shown by (2.20) and (2.23). Although these two formulae (3.21) and (3.22) are resemble each other, it seems quite difficult to show that the expansion of $\sigma(u)$ is of Hurwitz integral despite the case that we know the power series expansion of the function $\wp(u)$ at the beginning.

The following lemma shows Hurwitz integrality of the square root $\sigma(u)$ of $\sigma(u)^2$.

Lemma 3.23. *Let A be a integral domain contains $\mathbb{Z}$ and z be an indeterminate. Let*

$$h(z) = 1 + 2a_1\frac{z}{1!} + 2a_2\frac{z^2}{2!} + 2a_3\frac{z^3}{3!} + \cdots \tag{3.24}$$

be a power series with $a_j \in A$ $(j = 1, \cdots)$. Then a power series $\varphi(z)$ satisfying

$$h(z) = \varphi(z)^2 \tag{3.25}$$

belongs to $A\langle\langle z\rangle\rangle$.

Proof. Expanding $\varphi(z)$ shows

$$\begin{aligned}
&\Big(1 + 2a_1\frac{z}{1!} + 2a_2\frac{z^2}{2!} + \cdots\Big)^{-\frac{1}{2}}\\
&= 1 - \frac{1}{1!}\frac{1}{2}\Big(2a_1z + 2a_2\frac{z^2}{2!} + 2a_3\frac{z^3}{3!} + \cdots\Big) + \frac{1}{2!}\frac{1}{2}\frac{3}{2}\Big(2a_1z + 2a_2\frac{z^2}{2!} + 2a_3\frac{z^3}{3!} + \cdots\Big)^2\\
&\quad - \frac{1}{3!}\frac{1}{2}\frac{3}{2}\frac{5}{2}\Big(2a_1z + 2a_2\frac{z^2}{2!} + 2a_3\frac{z^3}{3!} + \cdots\Big)^3 + \cdots\\
&= 1 - \frac{1}{1!}\Big(a_1z + a_2\frac{z^2}{2!} + a_3\frac{z^3}{3!} + \cdots\Big) + \frac{1}{2!}{\cdot}1{\cdot}3\Big(a_1z + a_2\frac{z^2}{2!} + a_3\frac{z^3}{3!} + \cdots\Big)^2\\
&\quad - \frac{1}{3!}{\cdot}1{\cdot}3{\cdot}5\Big(a_1z + a_2\frac{z^2}{2!} + a_3\frac{z^3}{3!} + \cdots\Big)^3 + \cdots.
\end{aligned} \tag{3.26}$$

The claim follows from this immediately. □

This lemma and Theorem 3.14 yields that

$$\sigma(u) \in \mathbb{Z}[\tfrac{\mu_1}{2}, \mu_2, \mu_3, \mu_4, \mu_6]\langle\langle u\rangle\rangle. \tag{3.27}$$

By an explicit calculation when $v = 0$, namely $t_2 = 0$, the first several terms is seen as follows:

$$\sigma(u) = u + \big((\tfrac{\mu_1}{2})^2 + \mu_2\big)\tfrac{u^3}{3!} + \text{“higher terms in } \mathbb{Z}[\tfrac{\mu_1}{2}, \mu_2, \mu_3, \mu_4, \mu_6]\langle\langle u\rangle\rangle\text{”}. \tag{3.28}$$

This in none other than (2.30).

Remark 3.29. (1) Our result is seen to be suggesting the algebraic behavior of $\sigma(u)$ might go back to investigation of the 2-form $\boldsymbol{\xi}$.
(2) The main result of ours might strongly relate with the result in [2]. Because of the limited knowledge, the author could not explain such relation.

4 n-plication formula

The power series expansion of sigma is useful to get n-plication polynomial for $\mathscr{E}$.

4.1 The case of odd n

For an odd n, the n-plication formula is of the form :

$$\begin{aligned}\psi_n(u) := \frac{\sigma(nu)}{\sigma(u)^{n^2}} &= nx(u)^{\frac{n^2-1}{2}} + C_1\,x(u)^{\frac{n^2-5}{2}}y(u) + C_2\,x(u)^{\frac{n^2-3}{2}} \\ &\quad + C_3\,x(u)^{\frac{n^2-7}{2}}y(u) + C_4\,x(u)^{\frac{n^2-5}{2}} + \cdots + C_{n^2-1}.\end{aligned} \tag{4.1}$$

Namely, the roots $(x(u), y(u))$ of this polynomial, or the roots $u \bmod \Lambda$ is just the set of n-torsion points of $\mathscr{E}$. After expanding both sides in terms of u, comparing the coefficients of the two sides gives C_j as follows :

$$\begin{aligned}
&C_1 = 0, \quad C_2 = \tfrac{1}{24}\,n(n^2-1){\mu_1}^2 + \tfrac{1}{6}\,n(n^2-1)\mu_2, \quad C_3 = 0,\\
&C_4 = \tfrac{1}{1920}\,n(n^2-1)(n^2-9){\mu_1}^4 + \tfrac{1}{240}\,n(n^2-1)(n^2-9)\mu_2{\mu_1}^2\\
&\quad + \tfrac{1}{120}\,n(n^2-1)(n^2+6)\mu_3\mu_1 + \tfrac{1}{120}\,n(n^2-1)(n^2-9){\mu_2}^2 + \tfrac{1}{60}\,n(n^2-1)(n^2+6)\mu_4,\\
&C_5 = 0,\\
&C_6 = \tfrac{1}{322560}\,n(n^2-1)(n^2-3^2)(n^2-5^2){\mu_1}^6 + \tfrac{1}{26880}\,n(n^2-1)(n^2-3^2)(n^2-5^2)\mu_2{\mu_1}^4\\
&\quad + \tfrac{1}{6720}\,n(n^2-1)(n^2-3^2)(n^2+10)\mu_3{\mu_1}^3 + \tfrac{1}{6720}\,n(n^2-1)(n^2-3^2)(n^2-5^2){\mu_2}^2{\mu_1}^2\\
&\quad + \tfrac{1}{3360}\,n(n^2-1)(n^2-3^2)(n^2+10)\mu_4{\mu_1}^2 + \tfrac{1}{1680}n(n^2-1)(n^2-3^2)(n^2+10)\mu_3\mu_2\mu_1\\
&\quad + \tfrac{1}{5040}\,n(n^2-1)(n^2-3^2)(n^2-5^2){\mu_2}^3 + \tfrac{1}{840}\,n(n^2-1)(n^2-3^2)(n^2+10)\mu_4\mu_2\\
&\quad + \tfrac{1}{840}\,n(n^2-1)(n^4+n^2+15){\mu_3}^2 + \tfrac{1}{210}\,n(n^2-1)(n^4+n^2+15)\mu_6.
\end{aligned} \tag{4.2}$$

Here all the fractions are indeed integers for odd n.

4.2 The case of even n

For an even n, the n-plication formula is of the form

$$\begin{aligned}\psi_n(u) := \frac{\sigma(nu)}{\sigma(u)^{n^2}} &= nx(u)^{\frac{n^2-4}{2}}y(u) + C_1\,x(u)^{\frac{n^2-2}{2}} + C_2\,x(u)^{\frac{n^2-6}{2}}y(u) \\ &\quad + C_3\,x(u)^{\frac{n^2-4}{2}} + C_4\,x(u)^{\frac{n^2-8}{2}}y(u) + \cdots + C_{n^2-1}.\end{aligned} \tag{4.3}$$

and by the same method its first several terms are given as follows :

$$\begin{aligned}
&C_1 = -\tfrac{1}{2}\,n\mu_1, \quad C_2 = -\tfrac{1}{24}\,n(n^2-2^2){\mu_1}^2 - \tfrac{1}{6}\,n(n^2-2^2)\mu_2,\\
&C_3 = -\tfrac{1}{48}\,n(n^2-2^2){\mu_1}^3 - \tfrac{1}{12}\,n(n^2-2^2)\mu_2\mu_1 - \tfrac{1}{2}\,n\mu_3,\\
&C_4 = -\tfrac{1}{1920}\,n(n^2-2^2)(n^2-4^2){\mu_1}^4 - \tfrac{1}{240}\,n(n^2-2^2)(n^2-4^2)\mu_2{\mu_1}^2\\
&\qquad - \tfrac{1}{120}\,n(n^2-2^2)(n^2+9)\mu_3\mu_1 - \tfrac{1}{120}\,n(n^2-2^2)(n^2-4^2){\mu_2}^2\\
&\qquad - \tfrac{1}{60}\,n(n^2-2^2)(n^2+9)\mu_4,\\
&C_5 = -\tfrac{1}{3840}\,n(n^2-2^2)(n^2-4^2){\mu_1}^5 - \tfrac{1}{480}\,n(n^2-2^2)(n^2-4^2)\mu_2{\mu_1}^3\\
&\qquad - \tfrac{1}{240}\,n(n^2-2^2)(n^2+14)\mu_3{\mu_1}^2 - \tfrac{1}{240}\,n(n^2-2^2)(n^2-4^2){\mu_2}^2\mu_1\\
&\qquad - \tfrac{1}{120}\,n(n^2-2^2)(n^2+9)\mu_4\mu_1 - \tfrac{1}{12}\,n(n^2-2^2)\mu_3\mu_2.
\end{aligned} \tag{4.4}$$

The fractions in these coefficients are indeed integers for even n.

5 Weierstrass preparation theorem

In this Section, we describe certain type of Weierstrass preparation theorem that is used to show (1.16). Since it was not able to find a reference of this type of Weierstrass preparation theorem, we give a proof which is based on the remark due to H.Serbin [6]. Here let $\mathscr{O}$ be a integral domain (with unity), and z_1, z_2, $\cdots$, z_m be m indeterminates.

Lemma 5.1. *Assume that P and $Q \in \mathscr{O}[[z_1, z_2, \cdots, z_m]]$ are given, and that*

$$P(z_1, 0, 0, \cdots, 0) = c_k\, {z_1}^k + c_{k+1}\, {z_1}^{k-1} + \cdots \quad (c_k \in \mathscr{O}^\times;\ c_{k+1},\ \ c_{k+2},\ \cdots \in \mathscr{O}). \tag{5.2}$$

Then there is unique pair of the 2 polynomials A and $B \in \mathscr{O}[[z_1, z_2, \cdots, z_m]]$ such that

$$Q - PA = B \tag{5.3}$$

and that B does not have higher terms of z_1 greater than $(k-1)$.

Proof. We prove this by induction on m. Let

$$P = \sum_{j=0}^{\infty} p_j\, {z_m}^j,\ \ Q = \sum_{j=0}^{\infty} q_j\, {z_m}^j,\ \ A = \sum_{j=0}^{\infty} a_j\, {z_m}^j,\ \ B = \sum_{j=0}^{\infty} b_j\, {z_m}^j, \quad (p_j,\ q_j,\ a_j,\ b_j \in \mathscr{O}[[z_1, \cdots, z_{m-1}]]). \tag{5.4}$$

Our statement is equivalent to existence of a solution of

$$(q_j - a_0\, p_j - a_1\, p_{j-1} - \cdots - a_{j-1}\, p_1) - a_j\, p_0 = b_j, \qquad (j = 0,\ 1,\ 2, \cdots). \tag{5.5}$$

Because this equation is recursive on j, it reduces to the case $m = 0$. If $P \in \mathscr{O}^\times$ and $Q \in \mathscr{O}$, then

$$A = Q{\cdot}P^{-1} \in \mathscr{O}, \quad B = 0. \tag{5.6}$$

and the our claim has been proved. □

Corollary 5.7. (Weierstass preparation theorem) *For a given $F(w\,; z) \in \mathscr{O}[[w, z_2, \cdots, z_m]]$, assume $F(w\,; 0, \cdots, 0) = w^k + \cdots \in w^k\,(\mathscr{O}[[w]])^\times$. Then there exist uniquely an element $U \in \mathscr{O}[[w, z_2, \cdots, z_m]]$ and a polynomial $G \in \mathscr{O}[w][[z_2, \cdots, z_m]]$ that is monic and of degree k in w such that*

$$F = GU. \tag{5.8}$$

Proof. In the Lemma above, by setting $w = z_1$, $Q = w^k$, and $P = F$, we see there is unique $C \in \mathscr{O}[[w, z_2, \cdots, z_m]]$ such that

$$w^k - FC = -b_1(z)w^{k-1} - b_2(z)w^{k-2} - \cdots - b_{k-1}(z) \quad (z = (z_2, \cdots, z_m)). \tag{5.9}$$

By plugging $z_1 = \cdots = z_m = 0$, we have

$$w^k - (w^k + \cdots)C(w\,; 0, \cdots, 0) = -b_1(0)w^{k-1} - b_2(0)w^{k-2} - \cdots - b_{k-1}(0), \tag{5.10}$$

so that

$$C(w\,;0,\cdots,0) = 1 + O(w). \tag{5.11}$$

The Lemma 5.1 for $Q = 1$ and $P = C$ shows that there is unique $U(w\,;z) \in \mathscr{O}[[w, z_2, \cdots, z_m]]$ such that

$$1 - UC = 0. \tag{5.12}$$

Hence,

$$F(w\,;z) = (w^k + b_1(z)w^{k-1} + b_2(z)w^{k-2} + \cdots + b_{k-1}(z))\, U(w;z), \tag{5.13}$$

and the proof has completed. □

6 Hurwitz integrality of the Weierstrass sigma function

Here we mention datailed relation of the original $\sigma_{\rm w}(u)$ recalled in the Introduction and our new $\sigma(u)$.

Changing the coordinates of $\mathscr{E}$ as $Y = 2y + \mu_1 x + \mu_3$, $X = x - \frac{1}{3}(\mu_2 + \frac{1}{4}{\mu_1}^2)$ transforms the equation of $\mathscr{E}$ to

$$Y^2 = 4X^3 + (-3{\lambda_2}^2 + 2\mu_3\mu_1 + 4\mu_4)X + ({\lambda_2}^3 - \mu_1\mu_3\lambda_2 - \tfrac{1}{3}\mu_4\lambda_2 + {\mu_3}^2 + 4\mu_6), \tag{6.1}$$

where

$$\lambda_2 = \tfrac{1}{6}({\mu_1}^2 + 4\mu_2). \tag{6.2}$$

Therefore the Weierstrass $wp_{\rm w}(u)$ for $\mathscr{E}$ satisfies the differential equation obtained from the above by substitution

$$X = \wp_{\rm w}(u), \quad Y = \tfrac{d}{du}\wp_{\rm w}(u). \tag{6.3}$$

Then we see that the new and Weierstrass sigma functions relate as

$$\sigma(u) = \sigma_{\rm w}(u) \exp\left(\tfrac{1}{24}({\mu_1}^2 + 4\mu_2)u^2\right) \tag{6.4}$$

from (2.22), the first two terms of (2.30), and the condition $\sigma_{\rm w}(u) = u + O(u^5)$ mentioned in the Introduction.

Finally, we show the following theorem.

Theorem 6.5. *The power series expansion of $\sigma_{\rm w}(u)$ with respect to u is of Hurwitz integral over $\mathbb{Z}[\frac{g_2}{2}, 2g_3]$.*

Proof. We assume that $\mu_1 = \mu_2 = \mu_3 = 0$. Then $\sigma(u) = \sigma_{\rm w}(u)$ because ${\mu_1}^2 + 4\mu_2 = 0$ holds. This assumption does not ruin the generality of our claim as follows. For arbitrary pair of algebraically independent complex numbers g_2 and g_3, we set $\mu_4 = -\frac{1}{4}g_2$ and $\mu_6 = -\frac{1}{4}g_3$. Then our result 2.29 states that $\sigma_{\rm w}(u)$ is of Hurwitz integral over $\mathbb{Z}[\mu_4, \mu_6] = \mathbb{Z}[\frac{g_2}{4}, \frac{g_3}{4}]$. However, as is explained in the Introduction, the recursion relation due to Weierstrass in [7] shows that it is of Hurwitz integral over $\mathbb{Z}[\frac{1}{3}, \frac{g_2}{2}, g_3]$, so that over $\mathbb{Z}[\frac{1}{3}, \frac{g_2}{2}, 2g_3] \cap \mathbb{Z}[\frac{g_2}{4}, \frac{g_3}{4}]$ which is $\mathbb{Z}[\frac{g_2}{2}, 2g_3]$. □

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