Weierstraß recursion relation for the σ-function

This is an extract from my old note written by Japanese which explains the detail of Weierstrass paper [1] in faithfully following his argument.

Differentiating

$$\left(\frac{\partial\wp}{\partial u}\right)^2 = 4\wp^3 - g_2\wp - g_3 \tag{0.1}$$

by u, we have

$$\frac{\partial^2\wp}{\partial u^2} = 6\wp^2 - \tfrac{1}{2}g_2.$$

Since $\wp(u)$ is expressed as a power series of u with polynomial coefficients in g_2 and g_3, partial differentiation of (0.1) with respect to g_2 or g_3 gives

$$\begin{cases} 2\dfrac{\partial\wp}{\partial u}\dfrac{\partial}{\partial u}\dfrac{\partial\wp}{\partial g_2} = (12\wp^2 - g_2)\dfrac{\partial\wp}{\partial g_2} - \wp, \\ 2\dfrac{\partial\wp}{\partial u}\dfrac{\partial}{\partial u}\dfrac{\partial\wp}{\partial g_3} = (12\wp^2 - g_2)\dfrac{\partial\wp}{\partial g_3} - 1. \end{cases}$$

Therefore,

$$\begin{cases} 2\dfrac{\partial\wp}{\partial u}\dfrac{\partial}{\partial u}\dfrac{\partial\wp}{\partial g_2} = 2\dfrac{\partial^2\wp}{\partial u^2}\dfrac{\partial\wp}{\partial g_2} - \wp, \\ 2\dfrac{\partial\wp}{\partial u}\dfrac{\partial}{\partial u}\dfrac{\partial\wp}{\partial g_3} = 2\dfrac{\partial^2\wp}{\partial u^2}\dfrac{\partial\wp}{\partial g_3} - 1. \end{cases}$$

Dividing this by $\left(\frac{\partial\wp}{\partial u}\right)^2$, we get

$$\begin{cases} 2\dfrac{\frac{\partial\wp}{\partial u}\frac{\partial}{\partial u}\frac{\partial\wp}{\partial g_2} - \frac{\partial^2\wp}{\partial u^2}\frac{\partial\wp}{\partial g_2}}{\left(\frac{\partial\wp}{\partial u}\right)^2} = -\dfrac{\wp}{\left(\frac{\partial\wp}{\partial u}\right)^2}, \\ 2\dfrac{\frac{\partial\wp}{\partial u}\frac{\partial}{\partial u}\frac{\partial\wp}{\partial g_3} - \frac{\partial^2\wp}{\partial u^2}\frac{\partial\wp}{\partial g_3}}{\left(\frac{\partial\wp}{\partial u}\right)^2} = -\dfrac{1}{\left(\frac{\partial\wp}{\partial u}\right)^2}. \end{cases}$$

Hence,

$$\begin{aligned} 2\frac{\partial}{\partial u}\frac{\frac{\partial\wp}{\partial g_2}}{\frac{\partial\wp}{\partial u}} &= -\frac{\wp}{\left(\frac{\partial\wp}{\partial u}\right)^2}, \\ 2\frac{\partial}{\partial u}\frac{\frac{\partial\wp}{\partial g_3}}{\frac{\partial\wp}{\partial u}} &= -\frac{1}{\left(\frac{\partial\wp}{\partial u}\right)^2}. \end{aligned} \tag{0.2}$$

The LHSs of these are of integrable form. We shall try to transform the RHSs also in integrable form. To do so, we need to calculate

$$\frac{\partial}{\partial u}\frac{1}{\frac{\partial\wp}{\partial u}}, \quad \frac{\partial}{\partial u}\frac{\wp}{\frac{\partial\wp}{\partial u}}, \quad \frac{\partial}{\partial u}\frac{\wp^2}{\frac{\partial\wp}{\partial u}}.$$

Calculation is done as follows:

$$
\begin{aligned}
\frac{\partial}{\partial u}\frac{1}{\frac{\partial\wp}{\partial u}} &= -\frac{\frac{\partial^2\wp}{\partial u^2}}{\left(\frac{\partial\wp}{\partial u}\right)^2} = -\frac{6\wp^2-\frac{1}{2}g_2}{\left(\frac{\partial\wp}{\partial u}\right)^2},\\
\frac{\partial}{\partial u}\frac{\wp}{\frac{\partial\wp}{\partial u}} &= \frac{\left(\frac{\partial\wp}{\partial u}\right)^2-\wp\frac{\partial^2\wp}{\partial u^2}}{\left(\frac{\partial\wp}{\partial u}\right)^2},\\
&= \frac{4\wp^3-g_2\wp-g_3-\wp(6\wp^2-\frac{1}{2}g_2)}{\left(\frac{\partial\wp}{\partial u}\right)^2}\\
&= \frac{-2\wp^3-\frac{1}{2}g_2\wp-g_3}{\left(\frac{\partial\wp}{\partial u}\right)^2}\\
&= \frac{(-2\wp^3+\frac{1}{2}g_2+\frac{1}{2}g_3)-g_2\wp-\frac{3}{2}g_3}{\left(\frac{\partial\wp}{\partial u}\right)^2}\\
&= -\frac{1}{2}+\frac{-g_2\wp-\frac{3}{2}g_3}{\left(\frac{\partial\wp}{\partial u}\right)^2},\\
\frac{\partial}{\partial u}\frac{\wp^2}{\frac{\partial\wp}{\partial u}} &= \frac{2\wp\left(\frac{\partial\wp}{\partial u}\right)^2-\wp^2\frac{\partial^2\wp}{\partial u^2}}{\left(\frac{\partial\wp}{\partial u}\right)^2}\\
&= \frac{2\wp(4\wp^3-g_2\wp-g_3)-\wp^2(6\wp^2-\frac{1}{2}g_2)}{\left(\frac{\partial\wp}{\partial u}\right)^2}\\
&= \frac{2\wp^4-\frac{3}{2}g_2\wp^2-2g_3\wp}{\left(\frac{\partial\wp}{\partial u}\right)^2}\\
&= \frac{(2\wp^4-\frac{1}{2}g_2\wp^2-\frac{1}{2}g_3\wp)-g_2\wp^2-\frac{3}{2}g_3\wp}{\left(\frac{\partial\wp}{\partial u}\right)^2}\\
&= \frac{1}{2}\wp-\frac{g_2\wp^2+\frac{3}{2}g_3\wp}{\left(\frac{\partial\wp}{\partial u}\right)^2}.
\end{aligned}
$$

So, making suitable linear combinations, we get

$$
\begin{aligned}
2\frac{\partial}{\partial u}\left(3g_3g_2\frac{1}{\frac{\partial\wp}{\partial u}}+{g_2}^2\frac{\wp}{\frac{\partial\wp}{\partial u}}+(-18)g_3\frac{\wp^2}{\frac{\partial\wp}{\partial u}}\right) &= -\frac{6\wp^2-\frac{1}{2}g_2}{\left(\frac{\partial\wp}{\partial u}\right)^2}\times 3g_3g_2\\
&\quad+\left(-\frac{1}{2}+\frac{-g_2\wp-\frac{3}{2}g_3}{\left(\frac{\partial\wp}{\partial u}\right)^2}\right)\times {g_2}^2\\
&\quad+\left(\frac{1}{2}\wp-\frac{g_2\wp^2+\frac{3}{2}g_3\wp}{\left(\frac{\partial\wp}{\partial u}\right)^2}\right)\times(-18)g_3\\
&= -\frac{1}{2}{g_2}^2-9g_3\wp-\frac{({g_2}^3-27{g_3}^2)\wp}{\left(\frac{\partial\wp}{\partial u}\right)^2},\\
2\frac{\partial}{\partial u}\left(-{g_2}^2\frac{1}{\frac{\partial\wp}{\partial u}}+9g_3\frac{\wp}{\frac{\partial\wp}{\partial u}}+6g_2\frac{\wp^2}{\frac{\partial\wp}{\partial u}}\right) &= -\frac{9}{2}g_3+3g_2\wp+\frac{1}{2}\frac{{g_2}^3-27{g_3}^2}{\left(\frac{\partial\wp}{\partial u}\right)^2}.
\end{aligned}
$$

Using this and (0.2), we have

$$\begin{cases} -2({g_2}^3-27{g_3}^2)\dfrac{\partial}{\partial u}\dfrac{\frac{\partial\wp}{\partial g_2}}{\frac{\partial\wp}{\partial u}} = \dfrac{1}{2}{g_2}^2+9g_3\wp+\dfrac{\partial}{\partial u}\left(3g_3g_2\dfrac{1}{\frac{\partial\wp}{\partial u}}+{g_2}^2\dfrac{\wp}{\frac{\partial\wp}{\partial u}}+(-18)g_3\dfrac{\wp^2}{\frac{\partial\wp}{\partial u}}\right), \\ -({g_2}^3-27{g_3}^2)\dfrac{\partial}{\partial u}\dfrac{\frac{\partial\wp}{\partial g_3}}{\frac{\partial\wp}{\partial u}} = \dfrac{9}{2}g_3-3g_2\wp+\dfrac{\partial}{\partial u}\left(-{g_2}^2\dfrac{1}{\frac{\partial\wp}{\partial u}}+9g_3\dfrac{\wp}{\frac{\partial\wp}{\partial u}}+6g_2\dfrac{\wp^2}{\frac{\partial\wp}{\partial u}}\right). \end{cases}$$

Because every term has integral that is an odd function, the integral constants must be 0. So that, we have

$$\therefore \begin{cases} -2({g_2}^3-27{g_3}^2)\dfrac{\frac{\partial\wp}{\partial g_2}}{\frac{\partial\wp}{\partial u}} = \dfrac{1}{2}{g_2}^2u-9g_3\dfrac{\partial\log\sigma}{\partial u}+\left(3g_3g_2\dfrac{1}{\frac{\partial\wp}{\partial u}}+{g_2}^2\dfrac{\wp}{\frac{\partial\wp}{\partial u}}+(-18)g_3\dfrac{\wp^2}{\frac{\partial\wp}{\partial u}}\right), \\ -({g_2}^3-27{g_3}^2)\dfrac{\frac{\partial\wp}{\partial g_3}}{\frac{\partial\wp}{\partial u}} = \dfrac{9}{2}g_3u+3g_2\dfrac{\partial\log\sigma}{\partial u}-{g_2}^2\dfrac{1}{\frac{\partial\wp}{\partial u}}+9g_3\dfrac{\wp}{\frac{\partial\wp}{\partial u}}+6g_2\dfrac{\wp^2}{\frac{\partial\wp}{\partial u}}, \end{cases}$$

$$\therefore \begin{cases} -2({g_2}^3-27{g_3}^2)\dfrac{\partial\wp}{\partial g_2} = \dfrac{1}{2}{g_2}^2u\dfrac{\partial\wp}{\partial u}-9g_3\dfrac{\partial\log\sigma}{\partial u}\dfrac{\partial\wp}{\partial u}+3g_3g_2+{g_2}^2\wp-18g_3\wp^2, \\ -({g_2}^3-27{g_3}^2)\dfrac{\partial\wp}{\partial g_3} = \dfrac{9}{2}g_3u\dfrac{\partial\wp}{\partial u}+3g_2\dfrac{\partial\log\sigma}{\partial u}\dfrac{\partial\wp}{\partial u}-{g_2}^2+9g_3\,\wp+6g_2\,\wp^2 \end{cases} \tag{0.3}$$

$$\therefore \begin{cases} -2({g_2}^3-27{g_3}^2)\dfrac{\partial\wp}{\partial g_2} = \dfrac{1}{2}{g_2}^2u\left(\dfrac{\partial\wp}{\partial u}+2\wp\right)-9g_3\dfrac{\partial\log\sigma}{\partial u}\dfrac{\partial\wp}{\partial u}+3g_3g_2-18g_3\wp^2, \\ -({g_2}^3-27{g_3}^2)\dfrac{\partial\wp}{\partial g_3} = \dfrac{9}{2}g_3\left(u\dfrac{\partial\wp}{\partial u}+2\wp\right)+3g_2\dfrac{\partial\log\sigma}{\partial u}\dfrac{\partial\wp}{\partial u}-{g_2}^2+6g_2\,\wp^2. \end{cases}$$

Moreover, we know

$$\begin{aligned} \frac{\partial^2}{\partial u^2}\left(\frac{\frac{\partial\sigma}{\partial g_3}}{\sigma}\right) &= \frac{\partial^2}{\partial u^2}\frac{\partial}{\partial g_3}\log\sigma = \frac{\partial}{\partial g_3}\frac{\partial^2}{\partial u^2}\log\sigma = \frac{\partial\wp}{\partial g_3}, \\ \frac{\partial^2}{\partial u^2}\left(\frac{u}{\sigma}\frac{\partial\sigma}{\partial u}\right) &= -u\frac{\partial\wp}{\partial u}-2\wp, \\ \frac{\partial^2}{\partial u^2}\left(\frac{1}{\sigma}\frac{\partial^2\sigma}{\partial u^2}\right) &= \frac{\partial^2}{\partial u^2}\left(\frac{\partial^2\log\sigma}{\partial u^2}+\left(\frac{\partial\log\sigma}{\partial u}\right)^2\right) \\ &= \frac{\partial}{\partial u}\left(-\frac{\partial\wp}{\partial u}-2\wp\frac{\partial\log\sigma}{\partial u}\right) \\ &= -\frac{\partial^2\wp}{\partial u^2}+2\wp^2-2\frac{\partial\wp}{\partial u}\frac{\partial\log\sigma}{\partial u} \\ &= -4\wp^2+\frac{1}{2}g_2-2\frac{\partial\wp}{\partial u}\frac{\partial\log\sigma}{\partial u}. \end{aligned}$$

Using this, the last of (0.3) implies

$$\begin{cases} -2({g_2}^3-27{g_3}^2)\dfrac{\partial^2}{\partial u^2}\left(\dfrac{\frac{\partial\sigma}{\partial g_2}}{\sigma}\right) = \dfrac{1}{2}{g_2}^2\dfrac{\partial^2}{\partial u^2}\left(\dfrac{u}{\sigma}\dfrac{\partial\sigma}{\partial u}\right)+\dfrac{9}{2}g_3\dfrac{\partial^2}{\partial u^2}\left(\dfrac{1}{\sigma}\dfrac{\partial^2\sigma}{\partial u^2}\right)-\dfrac{9}{4}g_2g_3+3g_2g_3, \\ -({g_2}^3-27{g_3}^2)\dfrac{\partial^2}{\partial u^2}\left(\dfrac{\frac{\partial\sigma}{\partial g_3}}{\sigma}\right) = -\dfrac{9}{2}g_3\dfrac{\partial^2}{\partial u^2}\left(\dfrac{u}{\sigma}\dfrac{\partial\sigma}{\partial u}\right)-\dfrac{3}{2}g_2\dfrac{\partial^2}{\partial u^2}\left(\dfrac{1}{\sigma}\dfrac{\partial^2\sigma}{\partial u^2}\right)+\dfrac{3}{4}{g_2}^2-{g_2}^2. \end{cases}$$

Now we are going to integrate this with respect to u twice. However, as every term is an even function, we see

$$\begin{cases} -2({g_2}^3 - 27{g_3}^2)\left(\dfrac{\frac{\partial\sigma}{\partial g_2}}{\sigma}\right) = \dfrac{1}{2}{g_2}^2\left(\dfrac{u}{\sigma}\dfrac{\partial\sigma}{\partial u}\right) + \dfrac{9}{2}g_3\left(\dfrac{1}{\sigma}\dfrac{\partial^2\sigma}{\partial u^2}\right) + \dfrac{3}{8}g_2g_3u^2 + \dfrac{1}{2}B, \\ \quad -({g_2}^3 - 27{g_3}^2)\left(\dfrac{\frac{\partial\sigma}{\partial g_3}}{\sigma}\right) = -\dfrac{9}{2}g_3\left(\dfrac{u}{\sigma}\dfrac{\partial\sigma}{\partial u}\right) - \dfrac{3}{2}g_2\left(\dfrac{1}{\sigma}\dfrac{\partial^2\sigma}{\partial u^2}\right) - \dfrac{1}{8}{g_2}^2u^2 + \dfrac{1}{2}C, \end{cases}$$

$$\therefore \begin{cases} -4({g_2}^3 - 27{g_3}^2)\left(\dfrac{\frac{\partial\sigma}{\partial g_2}}{\sigma}\right) = {g_2}^2\left(\dfrac{u}{\sigma}\dfrac{\partial\sigma}{\partial u}\right) + 9g_3\left(\dfrac{1}{\sigma}\dfrac{\partial^2\sigma}{\partial u^2}\right) + \dfrac{3}{4}g_2g_3u^2 + B, \\ -2({g_2}^3 - 27{g_3}^2)\left(\dfrac{\frac{\partial\sigma}{\partial g_3}}{\sigma}\right) = -9g_3\left(\dfrac{u}{\sigma}\dfrac{\partial\sigma}{\partial u}\right) - 3g_2\left(\dfrac{1}{\sigma}\dfrac{\partial^2\sigma}{\partial u^2}\right) - \dfrac{1}{4}{g_2}^2u^2 + C, \end{cases} \tag{0.4}$$

$$\therefore \begin{cases} -4({g_2}^3 - 27{g_3}^2)\dfrac{\partial\sigma}{\partial g_2} = {g_2}^2u\dfrac{\partial\sigma}{\partial u} + 9g_3\dfrac{\partial^2\sigma}{\partial u^2} + \dfrac{3}{4}g_2g_3u^2\sigma + B\sigma, \\ -2({g_2}^3 - 27{g_3}^2)\dfrac{\partial\sigma}{\partial g_3} = -9g_3u\dfrac{\partial\sigma}{\partial u} - 3g_2\dfrac{\partial^2\sigma}{\partial u^2} - \dfrac{1}{4}{g_2}^2u^2\sigma + C\sigma. \end{cases}$$

Looking at the terms of the 1st order with respect to u, we have

$$\begin{aligned} 0 &= {g_2}^2 - 0 + 0 + B, \\ 0 &= -9g_2 - 0 + 0 + C. \end{aligned}$$

To eliminate the terms of second derivatives of σ form the last of (0.4), we compute

$$\begin{aligned} &\big\{(\text{The 1st formula}) \times 9g_3 + (\text{The 2nd formula}) \times {g_2}^2\big\} \div ({g_2}^3 - 27{g_3}^2), \\ &\big\{(\text{The 1st formula}) \times g_2 + (\text{The 2nd formula}) \times 3g_3\big\} \div ({g_2}^3 - 27{g_3}^2). \end{aligned}$$

Then we get

$$\frac{\partial^2\sigma}{\partial u^2} - 12g_3\frac{\partial\sigma}{\partial g_2} - \frac{2}{3}{g_2}^2\frac{\partial\sigma}{\partial g_3} + \frac{1}{12}g_2u^2\sigma = 0, \tag{0.5}$$

$$u\frac{\partial\sigma}{\partial u} - 4g_2\frac{\partial\sigma}{\partial g_2} - 6g_3\frac{\partial\sigma}{\partial g_3} - \sigma = 0. \tag{0.6}$$

The formula (0.5) states only that the weight of u is 1 or that the expansion of $\sigma(u)$ is of homogeneous weight. The other formula (0.6) gives the recurrence relation that we wanted to have.

If $\mu_1 = \mu_2 = \mu_3 = 0$, we set

$$\begin{aligned} \sigma(u) &= \sum_{m,n} a_{m,n}\left(\tfrac{1}{2}g_2\right)^m (2g_3)^n \frac{u^{4m+6n+1}}{(4m+6n+1)!} \quad \text{(This is Weierstrass setting)} \\ &= \sum_{m,n} c_{m,n}\,{\mu_4}^m{\mu_6}^n\frac{u^{4m+6n+1}}{(4m+6n+1)!}. \end{aligned}$$

Then (0.6) obviously gives

$$a_{m,n} = 3(m+1)a_{m+1,n-1} - \frac{16}{3}(n+1)a_{m-2,n+1} + \frac{1}{3}(2m+3n-1)(4m+6n-1)a_{m-1,n}.$$

Rewriting this, we have

$$c_{m,n} = 12(m+1)c_{m+1,n-1} - \frac{8}{3}(n+1)c_{m-2,n+1} + \frac{2}{3}(2m+3n-1)(4m+6n-1)c_{m-1,n}.$$

Here we assume $c_{m,n} = 0$ if $m < 0$ or $n < 0$. Therefore, we have arrived

$$\sigma(u) = u + 2\mu_4\frac{u^5}{5!} + 24\mu_6\frac{u^7}{7!} - 36{\mu_4}^2\frac{u^9}{9!} - 288\mu_4\mu_6\frac{u^{11}}{11!} + (-3456{\mu_6}^2 - 552{\mu_4}^3)\frac{u^{13}}{13!} + \cdots.$$

Q1 Can we get similar recursion relation including all μ_j ($j = 1, 3, 2, 4, 6$)?

Q2 In genus two case, J.C.Eilbeck shown that the system of the three equations

$$\begin{aligned}
{\wp_{222}}^2 &= 4\wp_{11} + 4\mu_6 + 4\wp_{12}\wp_{22} + 4\wp_{22}\mu_4 + 4{\wp_{22}}^3 + 4{\wp_{22}}^2\mu_2,\\
{\wp_{122}}^2 &= 4\mu_{10} - 4\wp_{11}\wp_{12} + 4{\wp_{12}}^2\wp_{22} + 4{\wp_{12}}^2\mu_2,\\
\wp_{122}\wp_{222} &= 2\mu_8 + 2{\wp_{12}}^2 + 2\wp_{12}\mu_4 - 2\wp_{22}\wp_{11} + 4\wp_{12}{\wp_{22}}^2 + 4\wp_{22}\wp_{12}\mu_2,
\end{aligned}$$

where we use weight index, is a “generating relation”. Is it possible to apply the argument above for this system to get the 4 recursion relations in [BL08]?

References

[1] K. Weierstrass. Zur theorie der elliptischen functionen. *Mathmatische Werke*, *Bd.2*: 245–255, 1894.