Chapter 2 Affine Plane of Finite Order [1]

Abstructing essence of the notion of “point" and “line" from the *classical plane geometry*, we define and study *affine planes* in this chapter. While in the classical plane geometry, points and lines are concretely exist, we define them quite axiomatically. If we are standing on completely theoretical view point, we need not to use the word “point(s)" and “plane(s)". However, it is natural for us to use these words.

2.1 Axioms of an affine plane

Let us start with defining an affine plane. We consider a plane that is a *set* consists of *points*. Each *line* is a certain subset of the plane satisfying additional condition(s). If the following axioms are satisfied, we call the system of those points and lines to be an *affine plane*.

The axioms we use here are those of Hartshorne [12]. While there are many kinds of sets of axioms which define affine planes, the axioms of Hartshorne is so smart to describe our theory. So we use his axioms and follow his arguments.

The axioms of affine plane:

Axiom A1. For any two different point p and q, there exists only one line that passing through both of p and q.

Axiom A2. For a given line l and a given point p which is not on l, there exists unique line passing through p and parallel to l.

Axiom A3. There are three points on the plane such that they are not on any line at once.

Here we used the word “parallel" in A2. The meaning of this word is as follows. We say two lines are *parallel*, if and only if they do not meet(intersect each other) or coincide. By A1, we may use term “the line pq", namely the line pq means the unique line passing through both of p and q.

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Proposition 2.1.1 *Any two lines do not intersect or intersect each other at only one point.*

Proof. If the given different two lines intersect each other at two different points, we have different two lines passing through the same two points. This contradicts to A1. □

Proposition 2.1.2 *We have the following properties* :

(1) $l||l$ (Reflection),

(2) *If* $l||m$ *then* $m||l$ (Symmetry),

(3) *If* $l||m$ *and* $m||n$ *then* $l||n$ (Transitivity).

Proof. (1) and (2) are obvious from the definition of the notion “parallel". We prove (3) nearly by *reduction ad absurdum*. (3) If l and n meet (intersect) each other, then l and n coincide because of the assumption and A2. □

[1] This is an Engish translation of Chapter 2 of “新版 幾何の魔術"

Proposition 2.1.3 *There exists certain four points such that any three of the four do not on any line at once.*

Proof. There are three points such that no line contains them at once. Let such the point be p, q, and r. Let l be the unique line passing through p and being parallel to qr. Let m be the unique line passing through r and being parallel to pq. Then l and m are not parallel, since if they are parallel then we see pq and qr are parallel by using transitivity twice and have contradiction to the assumption on p, q, and r. So we have the point, say s, at which l intersects with m.

Now we claim that
(a) there is no line on which p, q, and r are;
(b) there is no line on which p, q, and s are;
(c) there is no line on which p, r, and s are;
(d) there is no line on which q, r, and s are.
If we have proved these four claims, the proof will be completed.

(a) is obvious. For (b): If p, q, and s are on a line, we see that l intersects with qr at q. This is a contradiction because l and qr are parallel. (c) is shown similarly to (b). To prove (d) is the exercise 2.3 of this lecture. □

Proposition 2.1.4 *For each point, there are at least two lines which pass through the point.*

Proof. Let p, q, r, and s are the four points just we got in Proposition 2.1.3. Let a be a point. Though the point a may coincide to one of the four points, a can not coincide the other three points. So we may assume that a is different from p, q, and r. (For the other cases, the arguments are done similarly to this case by renaming the four points.) Suppose there is only one line that passes through a. Then the lines ap, aq, and ar coincide and are the same line. This is a contradiction because no line contains all three points p, q, and r. □

We will prove later that there are at least three points which pass through a point (see Theorem 2.2.1).

Proposition 2.1.5 *Each line contains at least two points.*

Proof. Let l be a given line. For the points p, q, r, and s in Proposition 2.1.3, if l contains any two of these points then we have nothing to prove. Therefore we may assume l does not contain at least three points of these four points. We suppose l does not contain p, q, and r, since the other cases are similarly proved. As any line does not contain all three p, q, and r at the same time, any two of pq, qr, and rp are not parallel with each other. Hence there are at most one line (in these three lines) which is parallel with l. Let m and n are any two lines of the three which are not parallel with l. Let a and b be the points at which m and n intersect with l, respectively. If a and b are coincide, then m and n contains different two points (namely

$a(= b)$ and r). Then m and n must coincide because of A1. This contradicts to our choice of m and n. Thus we see that l contains “two points" a and b. □

2.2 Counting the number of points and lines in an affine plane

Proposition 2.2.1 *On an affine plane, any lines contain points of the same number.*

Proof. We compare the number of points on given two lines l and m. Firstly, we consider the case that l and m are not parallel. Let q be the intersecting point of l and m. There are points on l and m respectively different from q. We call them p and r, respectively. In the proof of Proposition 2.1.3, the firstly given points are merely supposed such that any line contains them at the same time. So we can find a points s such that any three of the four points p, q, r, and s are not on any line at the same time.

Let x_1 be a point different from q. Let m' be the point which passes through x_1 and parallel with qr. Since qs and qr are not parallel, m' and qs are not parallel. Let x'_1 be the intersecting point of m' and qs. The line l' that passes through x'_1 parallel to l is intersect with m at a point, say y_1. Let x_2 be another point on l different from x_1. By the similar argument gives a point x_2 (on qs). Since the parallel lines do not intersect each other, x'_2 is different from x'_1. (We get the point y_2 on m as we have gotten y_1 from x'_1) We see also y_2 is different from y_1. By the correspondence x_i to y_i above maps any new point on l to a new point on m. This shows the number of the points on l is less than or equal to that of m. Repeating the argument above by exchanging l and m at the beginning, we see the number of the points on m is less than or equal to that of l. Consequently, we see that the number of the points of l is equal to that of m.

Secondly, we consider the other case, namely, the case that l and m are parallel. Let p and q are points on l and m, respectively. The line pq is not parallel with either l nor with m. Let x_1 be a point different from p (if exists). Let y_1 be the intersecting point of m and the line passing through x_1 and parallel with pq. The similar argument with the first case shows the desired statement. □

While the facts we have shown hold for any plane which contains infinitely many points too, from now on we treat only the plane with finitely many points.

We call an affine plane in which any line has exactly n points an *affine plane of order n*.

Theorem 2.2.1 *On an affine plane of order n, there are exactly* $(n + 1)$ *lines that pass through a given point.*

Proof. Let p be a point. Let q be a point different from p. If all the points are on the line pq, it contradicts to A3. Hence, we find a point r that is not on pq. Let l be the line which passes through r and is parallel with pq. By the assumption (namely, that we consider an affine plane of order n), we have n points on l. We name these points as $p_1(= r), p_2, \cdots, p_n$. We claim that pq does not coincide with any of $pp_1, \cdots, pp_n$. Indeed, if coincides with any one of the above, we see pq coincides with l because $pq \| l$. Since l does not meet with pq, any two lines pp_i and pp_j $(i \neq j)$ are different. We have $(n + 1)$ lines $pq, pp_1, pp_2, \cdots, pp_n$ which pass through p. If there exists another line that passes through p, then such a line is different from pq and is not parallel to l. Hence such a line meet with l. Their intersection point is different from any p_j, so that we have more number of points than n. (This is a contradiction.) □

Theorem 2.2.2 *On an affine plane of order n, any family of parallel lines consists of exactly n lines.*

Proof. Let l be a given line. Let p be a point on l and q be a point not on l. Since we consider an affine plane of order n, there exists $(n-2)$ points other than p and q. We draw the lines which are parallel to l and pass through these $(n-2)$ points or q. We then have $(n-1)$ lines with parallel to l. Adding l itself to these, we have a family of parallel n lines. Since any line that is parallel to l meet with pq, it is impossible to exist other lines parallel to l. □

Theorem 2.2.3 *On an affine plane of order n, there are exactly n^2 points.*

Proof. Theorem 2.2.1 shows that there are exactly $(n+1)$ lines passing through a given point p. Each of these $(n+1)$ lines contains $(n-1)$ points other than p. Any points on different lines are also different. (If not, we have lines which meet p and another point. Hence such the lines coincide.) Therefore we have

$$1 + (n+1)(n-1) = 1 + n^2 - 1 = n^2$$

points. (Here the 1 at the beginning counts the point p.) If there exists another point, we have a new line passing through this point and p. Hence we never have another point. □

Theorem 2.2.4 *On an affine plane of order n, there are exactly $n(n+1)$ lines.*

Proof. For each point, we have $(n+1)$ lines passing through it. There are n^2 points on the plane as we have seen. So we seems to have $n^2(n+1)$ lines. But we counted any line n times, because any line contains n points. So the correct number of the lines is, by dividing n,

$$\frac{n^2(n+1)}{n} = n(n+1).$$

(Thus the proof has completed.) □

Theorem 2.2.5 *On an affine plane of order n, there are exactly $(n+1)$ families of parallel lines.*

Proof. For each point, we have $(n+1)$ lines passing through it. Any two of them are not parallel. For each line in these $(n+1)$ lines, we can find n lines (including itself) parallel to it. We then have $n(n+1)$ lines. There does not exist any more line from Theorem 2.2.4. □