記号 $\underset{\substack{1\leqq i\leqq m\\1\leqq j\leqq n}}{[\boldsymbol{a}_{ij}]}$ の使用法の練習

一般に 2 つの行列 $[g_{ij}]_{m\times n}$ と $[h_{jk}]_{n\times r}$ の積をこの記号で書けば

$$\underset{\substack{1\leqq i\leqq m\\1\leqq j\leqq n}}{[g_{ij}]}\ \underset{\substack{1\leqq\square\leqq\square\\1\leqq\square\leqq r}}{[h_{\square k}]} = \underset{\substack{1\leqq\square\leqq\square\\1\leqq\square\leqq\square}}{\left[\sum_{\square=1}^{\square} g_{ij}h_{\square\square}\right]}$$

例題 1. $A(B+C)=AB+AC$ を証明せよ.

証明. いま

$$A=\underset{\substack{1\leqq i\leqq m\\1\leqq j\leqq n}}{[a_{\square j}]},\quad B=\underset{\substack{1\leqq\square\leqq\square\\1\leqq\square\leqq r}}{[b_{\square k}]},\quad C=\underset{\substack{1\leqq\square\leqq\square\\1\leqq\square\leqq\square}}{[c_{\square\square}]}$$

とおく. これらについて,

$$\begin{aligned}
A(B+C) &= A\underset{\substack{1\leqq\square\leqq\square\\1\leqq\square\leqq\square}}{[b_{\square\square}+c_{\square\square}]}\\
&= \underset{\substack{1\leqq\square\leqq\square\\1\leqq\square\leqq\square}}{\left[\sum_{j=1}^{\square} a_{\square j}(b_{\square\square}+c_{\square\square})\right]},\\
&= \underset{\substack{1\leqq\square\leqq\square\\1\leqq\square\leqq\square}}{\left[\sum_{j=1}^{\square} \left(a_{\square j}b_{\square\square}+a_{\square j}c_{\square\square}\right)\right]},\\
&= \underset{\substack{1\leqq\square\leqq\square\\1\leqq\square\leqq\square}}{\left[\sum_{j=1}^{\square} a_{\square j}b_{\square\square}+\sum_{j=1}^{\square} a_{\square j}c_{\square\square}\right]},\\
&= \underset{\substack{1\leqq\square\leqq\square\\1\leqq\square\leqq\square}}{\left[\sum_{j=1}^{\square} a_{\square j}b_{\square\square}\right]}+\underset{\substack{1\leqq\square\leqq\square\\1\leqq\square\leqq\square}}{\left[\sum_{j=1}^{\square} a_{\square j}c_{\square\square}\right]},\\
&= AB+AC.
\end{aligned}$$

例題 2. $(AB)C = A(BC)$ を証明せよ.

証明. 先ほどと同様に,

$$A = \left[a_{ij}\right]_{\substack{1\leqq i\leqq m\\ 1\leqq j\leqq n}},\ B = \left[b_{\square k}\right]_{\substack{1\leqq \square\leqq \square\\ 1\leqq k\leqq r}},\ C = \left[c_{\square t}\right]_{\substack{1\leqq \square\leqq \square\\ 1\leqq t\leqq \ell}}$$

とおく. このとき

$$\begin{aligned}
(AB)C &= \left(\left[a_{ij}\right]_{\substack{1\leqq i\leqq m\\ 1\leqq j\leqq n}} \left[b_{\square k}\right]_{\substack{1\leqq \square\leqq \square\\ 1\leqq \square\leqq r}}\right)\left[c_{\square t}\right]_{\substack{1\leqq \square\leqq \square\\ 1\leqq \square\leqq \ell}} = \left[\sum_{\square=1}^{\square} a_{ij} b_{\square\square}\right]_{\substack{1\leqq \square\leqq \square\\ 1\leqq \square\leqq \square}} \left[c_{\square t}\right]_{\substack{1\leqq \square\leqq \square\\ 1\leqq \square\leqq \ell}} \\
&= \left[\sum_{\square=1}^{\square}\left(\sum_{\square=1}^{\square} a_{ij} b_{\square\square}\right) c_{\square\square}\right]_{\substack{1\leqq \square\leqq \square\\ 1\leqq \square\leqq \square}} = \left[\sum_{\square=1}^{\square}\left(\sum_{\square=1}^{\square} a_{ij} b_{\square\square} c_{\square\square}\right)\right]_{\substack{1\leqq \square\leqq \square\\ 1\leqq \square\leqq \square}}
\end{aligned}$$

同様に

$$\begin{aligned}
A(BC) &= \left[a_{ij}\right]_{\substack{1\leqq i\leqq m\\ 1\leqq j\leqq n}}\left(\left[b_{\square k}\right]_{\substack{1\leqq \square\leqq \square\\ 1\leqq \square\leqq r}} \left[c_{\square t}\right]_{\substack{1\leqq \square\leqq \square\\ 1\leqq \square\leqq \ell}}\right) = \left[a_{ij}\right]_{\substack{1\leqq i\leqq m\\ 1\leqq j\leqq n}} \left[\sum_{\square=1}^{\square} b_{\square\square} c_{\square\square}\right]_{\substack{1\leqq \square\leqq \square\\ 1\leqq \square\leqq \square}} \\
&= \left[\sum_{\square=1}^{\square} a_{ij}\left(\sum_{\square=1}^{\square} b_{\square\square} c_{\square\square}\right)\right]_{\substack{1\leqq \square\leqq \square\\ 1\leqq \square\leqq \square}} = \left[\sum_{\square=1}^{\square}\left(\sum_{\square=1}^{\square} a_{ij} b_{\square\square} c_{\square\square}\right)\right]_{\substack{1\leqq \square\leqq \square\\ 1\leqq \square\leqq \square}}
\end{aligned}$$

ここで

$$x_{\square\square} = a_{ij} b_{\square\square} c_{\square\square}$$

とおけば等式（別紙参照）

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{r} x_{\square\square}\right) = \sum_{k=1}^{\square}\left(\sum_{j=1}^{\square} x_{\square\square}\right)$$

から

$$(AB)C = A(BC)$$

がわかり, 所望の等式が成り立つことがわかる.