北岡 他 著：『工科系の微分積分学の基礎』 p.136, 例 2.17-2 の詳解

放物線 $y = x^2$ の $0 \leqq x \leqq 1$ の部分の弧の長さ L を求めよ.

解答. 求める長さは

$$\begin{aligned} L &= \int_0^1 \sqrt{1+\left(\tfrac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+(2x)^2}\,dx = \int_0^2 \sqrt{1+t^2}\,\frac{dx}{dt}\,dt \quad (t = \tfrac{x}{2}) \\ &= \int_0^2 \sqrt{1+t^2}\,\frac{1}{2}\,dt = \frac{1}{2}\int_0^2 \sqrt{1+t^2}\,dt. \end{aligned}$$

ここで $\sqrt{1+t^2} = t + z$ とおくと

$$1 + t^2 = t^2 + 2tz + z^2$$

となり,

$$\begin{aligned} t &= \frac{1-z^2}{2z}, \\ \frac{dt}{dz} &= -\frac{1+z^2}{2z^2}, \\ \sqrt{1+t^2} &= t + z = \frac{1-z^2}{2z} + z = \frac{1+z^2}{2z} \end{aligned}$$

なので

$$\begin{aligned} &= \int \sqrt{1+t^2}dt \\ &= -\frac{1}{4}\int \frac{(z^2+1)^2}{z^3}dz \\ &= -\frac{1}{4}\int \frac{z^4+2z^2+1}{z^3}dz \\ &= -\frac{1}{4}\int (z + 2\frac{1}{z} + \frac{1}{z^3})dz \\ &= -\frac{1}{4}\Big(\frac{1}{2}z^2 + 2\log|z| - \frac{1}{2}\frac{1}{z^2}\Big) + C \\ &= -\frac{1}{4}\Big\{\frac{1}{2}\Big(z^2 - \frac{1}{z^2}\Big) + 2\log|z|\Big\} + C \\ &= -\frac{1}{4}\Big\{\frac{1}{2}\Bigg(\big(\sqrt{1+t^2}-t\big)^2 - \frac{1}{\big(\sqrt{1+t^2}-t\big)^2}\Bigg) + 2\log|\sqrt{1+t^2}-t|\Big\} + C \\ &= -\frac{1}{4}\Big\{\frac{1}{2}\big(-4t\sqrt{1+t^2}\big) + 2\log\left|\frac{1}{\sqrt{1+t^2}+t}\right|\Big\} + C \\ &= \frac{1}{2}\big\{\,t\sqrt{1+t^2} + \log\big(\sqrt{1+t^2}+t\big)\,\big\} + C \end{aligned}$$

よって

$$\begin{aligned} L &= \frac{1}{2}\cdot\frac{1}{2}\Big[\,t\sqrt{1+t^2} + \log\big(\sqrt{1+t^2}+t\big)\Big]_0^2 \\ &= \frac{1}{4}\Big(2\sqrt{5} + \log(\sqrt{5}+2)\Big) \\ &= \frac{\sqrt{5}}{2} + \frac{1}{4}\log(\sqrt{5}+2) \ \cdots\cdots\ \text{Ans.} \end{aligned}$$